Some results for a f(R) truncation for a QG action using the exponential parametrization of the metric

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- We have already heard about some features of one metric truncations for effective average actions with exponential parametrization.

Talks: Percacci, Knorr
Posters: Falls, Labus

- We heard about an f(R) truncation in the conventional linear param.

Talk: Litim

- Here is discussed the f(R) case in the non linear “exp” representation.

Outline

- Framework
- Parameterization and gauge fixing
- Flow equations.
- An example of global solution
- Conclusions and outlook

Framework

What is the fundamental nature of gravitational interaction? Of spacetime? And matter?
Will an answer ever come?

- Single fundamental theory? QFT Stringy Discrete models …
- Sequence of theories with trasmutation/generation of (some) degrees of freedom?

Criteria? Experimental input (hard) Simplicity/Beauty/Unification …

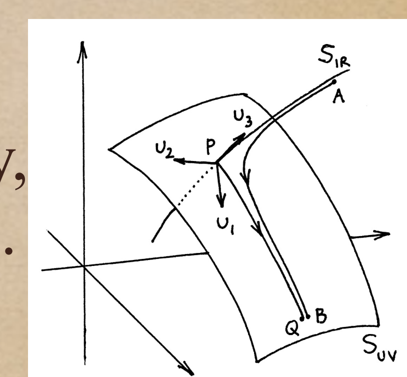
RG paradigm

Useful since able to unify the fundamental and effective theory point of view.

Bottom-up approach. (Donoghue)

At least classical field theory and effective field theory are good descriptions for gravity.

Simplest attempt:
A gravitational QFT described by a metric and diffeomorphism symmetry, whose dynamics reveals a UV fixed point with finite dim. critical surface.



If bare action has no irrelevant operators, it is asymptotically safe.
Otherwise description is effective, originating from a more fundamental theory.

Approach

Asymptotic safety paradigm using FRG techniques for the effective action $\Gamma[g_{\mu\nu}, \phi]$ (Reuter)

This approach is clearly in constrast to the stringy ideas of UV/IR duality,
related to assuming black-hole formation in transplanckian scattering. (Lust)
(a fact not necessarily true for a quantum gravitational theory in the UV)

In this framework we cannot avoid the use of a background field formalism.
In a metric formulation (euclidean): $g_{\mu\nu}\left(\bar{g}_{\mu\nu}, h_{\mu\nu}\right)$

- Issue of background independence for double metric description / modified splitting Ward Identities.
- Issue of choosing truncations as well as coarse-graining schemes. Simple enough, but able to keep the most important features of the full theory.

Many degrees of approximations in the covariant description:
Single metric (field) descriptions can be still non local and complicated:

On maximally symmetric background (e.g. a sphere), for a local "LPA" truncation,
pure gravity still not so trivial!

$$\Gamma[g_{\mu\nu}] = \int \mathrm{d}^d x \sqrt{g} f\left(R\right)$$

Gravity sector: the metric

We use an exponential parameterization of the metric with euclidean signature:

$$g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu \qquad \sqrt{g} = e^{h/2}\sqrt{\bar{g}} = \sqrt{\bar{g}}\left(1+\frac{h}{2}+\frac{h^2}{8}+\cdots\right) \qquad trh = h = 2d\omega$$

As a change of variables the Jacobian is well defined.

We take the attitude that the metric has a non linear nature, preferring the exponential parameterization.

Think about frames and vielbeins… coset space $GL(d)/O(p,q)$

(See also Nink)

Remark: at quantum level the off shell effective action is equivalent to other parameterizations if

- a Jacobian is taken into account
- the geometric formulation a la Vilkowisky-De Witt in considered, indeed e.g. expectations values are not trivially related, …

Remark: non linear transformation —> momentum coarse-graining qualitative different!

Gauge transformations

Gravity is a gauge theory: physics does not change under diffeomorphisms.

$$\delta_\epsilon g_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu} \equiv \epsilon^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \epsilon^\rho + g_{\nu\rho}\partial_\mu \epsilon^\rho$$

The quantum gauge transformation for the fluctuations defined in the exponential parametrization:

$$\delta_\epsilon^{(Q)} h^\mu{}_\nu = (\mathcal{L}_\epsilon \bar{g})^\mu{}_\nu + \mathcal{L}_\epsilon h^\mu{}_\nu + [\mathcal{L}_\epsilon \bar{g}, h]^\mu{}_\nu + O(\epsilon h^2) \ .$$

For a single metric truncation, to define the gauge fixing and ghost propagators, it is enough to keep:

$$\delta_\epsilon^{(Q)} h_{\mu\nu} = \bar{\nabla}_\mu \epsilon_\nu + \bar{\nabla}_\nu \epsilon_\mu + O(h)$$

York decomposition of the metric:

$$h_{\mu\nu} = h^{TT}{}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma - \frac{1}{d}\bar{g}_{\mu\nu}\bar{\nabla}^2\sigma + \frac{h}{d}\bar{g}_{\mu\nu}$$

and similarly for the diffeomorphism generator

$$\epsilon^\mu = \epsilon^{T\mu} + \bar{\nabla}^\mu \frac{1}{\sqrt{-\bar{\nabla}^2}}\psi \ ; \qquad \bar{\nabla}_\mu \epsilon^{T\mu} = 0$$

Transformations:

$$\delta_{\epsilon^T}\xi^\mu = \epsilon^{T\mu} \qquad \delta_\psi \sigma = \frac{2}{\sqrt{-\bar{\nabla}^2}}\psi \qquad \delta_\psi h = -2\sqrt{-\bar{\nabla}^2}\psi$$

Gauge invariant quantities

$$s = h - \bar{\nabla}^2\sigma \qquad h^{TT}_{\mu\nu}$$

To adsorbe some Jacobians one can redefine:

$$\xi'_\mu = \sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{d}}\xi_\mu \ ; \qquad \sigma' = \sqrt{-\bar{\nabla}^2}\sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{d-1}}\sigma$$

Gauge fixing and ghosts

We shall use here the so called physical gauge fixing: set to zero the gauge dependent fluctuations. Therefore in the path integral there remain two kind of gauge invariant fluctuations: s and $h_{\mu\nu}^{TT}$

We face two possible GF choices:

I: $\xi'_\mu = 0 \quad , \quad h = \text{const.}$

II: $\xi'_\mu = 0 \quad , \quad \sigma' = 0$

Remark: for compact spaces (e.g. a sphere) no diffeomorphism can change the constant mode of h.

Faddeev Popov determinants: varying the GF conditions.

$\delta(\xi'_\mu)$ $\quad$ $\det\left(\sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{d}}\right)$

$\delta(h-\text{const})$ $\quad$ $\det(\sqrt{-\bar{\nabla}^2})$

$\delta(\sigma')$ $\quad$ $\det\left(\sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{d-1}}\right)$

f(R) truncation: $\Gamma[g_{\mu\nu}] = \int \mathrm{d}^d x \sqrt{g} f(R)$

Using gauge invariant variables and the Lichnerowicz operators the Hessian looks very simple:

$$I^{(2)}_{exp} = -\frac{1}{4} f'(\bar{R}) h^{TT}_{\mu\nu} \Big(\Delta_2 - \frac{2}{d}\bar{R}\Big) h^{TT\,\mu\nu} + \frac{d-1}{4d} s \Big[\frac{2(d-1)}{d} f''(\bar{R}) \Big(\Delta_0 - \frac{\bar{R}}{d-1}\Big) + \frac{d-2}{d} f'(\bar{R})\Big] \Big(\Delta_0 - \frac{\bar{R}}{d-1}\Big) s + h\Big(\frac{1}{8} f(\bar{R}) - \frac{1}{4d} f'(\bar{R})\bar{R}\Big) h.$$

Zero on shell

Then on a maximally symmetric space (sphere)
and using the rescaled gauge invariant variable: $s' = \sqrt{-\bar{\nabla}^2 - \frac{R}{d-1}}\, s$

$$\frac{1}{2}\int dx \sqrt{\bar{g}} \Big[-\frac{1}{2} f'(\bar{R}) h^{TT}{}_{\mu\nu} \left(-\bar{\nabla}^2 + \frac{2\bar{R}}{d(d-1)}\right) h^{TT\mu\nu} + \frac{(d-1)^2}{d^2} s' \left(f''(\bar{R}) \left(-\bar{\nabla}^2 - \frac{\bar{R}}{d-1}\right) + \frac{(d-2)}{2(d-1)} f'(\bar{R}) \right) s' + h \left(\frac{1}{4} f(\bar{R}) - \frac{1}{2d} \bar{R} f'(\bar{R}) \right) h$$

To be compared with…
(Codello-Percacci_Rahmede, Demmel-Saueresseig-Zanusso, Eichhorn)

Imposing a physical gauge on ξ'_μ and h it is easy to see that
only the vector ghots contribution survives. In Einstein gravity the scalar does not propagate.
Same result if using De Donder gauge fixing.

Flow Equation

We consider the following family of coarse-graining schemes, depending on three endomorphisms. which affect the summation of the fluctuations of different spin.

(optimized cutoff)

$$P_k(\Box) = \Box + R_k(\Box)\,, \quad \Box = \Delta + E_{(s)}\,, \quad \Delta = -\nabla^2\,,$$

$$R_k(z) = (k^2 - z)\theta(k^2 - z)$$ (Litim)

The endos are defined by three real parameters: $E_{(2)} = -\alpha\bar{R}, \quad E_{(0)} = -\beta\bar{R}, \quad E_{(1)} = -\gamma\bar{R}$

gauge invariant (spin=2,0) — vector ghost (spin=1)

Flow of the effective average action (we consider d=4):

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}_{(2)}\left[\frac{\dot{f}'(\bar{R})R_k(\Box) + f'(\bar{R})\dot{R}_k(\Box)}{f'(\bar{R})\left(P_k(\Box) - E_{(2)} + \frac{2}{d(d-1)}\bar{R}\right)}\right]$$
$$+\frac{1}{2}\mathrm{Tr}_{(0)}\left[\frac{\dot{f}''(\bar{R})R_k(\Box) + f''(\bar{R})\dot{R}_k(\Box)}{f''(\bar{R})\left(P_k(\Box) - E_{(0)} - \frac{1}{d-1}\bar{R}\right) + \frac{d-2}{2(d-1)}f'(\bar{R})}\right]$$
$$-\frac{1}{2}\mathrm{Tr}_{(1)}\left[\frac{\dot{R}_k(\Box)}{P_k(\Box) - E_{(1)} - \frac{1}{d}\bar{R}}\right],$$

This scheme is spectrally adjusted: coarse-graining is affected by the shape of f(R)!

The traces can be evaluated in different ways:
- Heat kernel expansion
- spectral sums on the sphere

Both have some limitations in the IR and for large R.

Flow from Heat Kernel

The Heat Kernel expansion valid at small R leads to the following well known construction:

$$\mathrm{Tr}_{(s)}[W(\Box)] = \frac{1}{(4\pi)^{d/2}} \int_{S^d} d^d x \sqrt{\bar{g}} \sum_{n \geq 0} b_{2n}^{(s)} Q_{d/2-n}[W] \bar{R}^n \qquad Q_m[W] = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} W[z]$$

Then we get the following HK coefficients

	b_0	b_2	b_4	b_6
Spin 0	1	$\frac{1}{6}+\beta$	$\frac{-511+360\beta+1080\beta^2}{2160}$	$\frac{19085-64386\beta+22680\beta^2+45360\beta^3}{272160}$
Spin 1	3	$\frac{1}{4}+3\gamma$	$\frac{-607+360\gamma+2160\gamma^2}{1440}$	$\frac{37259-152964\gamma+45360\gamma^2+181440\gamma^3}{362880}$
Spin 2	5	$-\frac{5}{6}+5\alpha$	$\frac{-1-360\alpha+1080\alpha^2}{432}$	$\frac{311-126\alpha-22680\alpha^2+45360\alpha^3}{54432}$

Dimensionless variables: $r \equiv \bar{R}k^{-2}$ $\quad \varphi(r) = k^{-d} f(\bar{R})$ $\quad d = 4$

The flow equation, for an optimized cutoff reads

$$32\pi^2(\dot\varphi - 2r\varphi' + 4\varphi) = \frac{c_1(\dot\varphi' - 2r\varphi'') + c_2\varphi'}{\varphi'[6+(6\alpha+1)r]} + \frac{c_3(\dot\varphi'' - 2r\varphi''') + c_4\varphi''}{[3+(3\beta-1)r]\varphi'' + \varphi'} - \frac{c_5}{4+(4\gamma-1)r}$$

$$c_1 = 5 + 5\Big(3\alpha - \frac{1}{2}\Big)r + \Big(15\alpha^2 - 5\alpha - \frac{1}{72}\Big)r^2 + \Big(5\alpha^3 - \frac{5}{2}\alpha^2 - \frac{\alpha}{72} + \frac{311}{9072}\Big)r^3,$$
$$c_2 = 40 + 15(6\alpha - 1)r + \Big(60\alpha^2 - 20\alpha - \frac{1}{18}\Big)r^2 + \Big(10\alpha^3 - 5\alpha^2 - \frac{\alpha}{36} + \frac{311}{4536}\Big)r^3,$$
$$c_3 = \frac{1}{2}\Big[1 + \Big(3\beta + \frac{1}{2}\Big)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2 + \Big(\beta^3 + \frac{1}{2}\beta^2 - \frac{511}{360}\beta + \frac{3817}{9072}\Big)r^3\Big],$$
$$c_4 = 3 + (6\beta + 1)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2,$$
$$c_5 = 12 + 2(12\gamma + 1)r + \Big(12\gamma^2 + 2\gamma - \frac{607}{180}\Big)r^2.$$

Inclusion of the constant mode for h.

Add $\quad \dfrac{8}{3}\dfrac{r^2}{16 + 2\varphi - r\varphi'}$

Flow from Spectral Sum

The Heat Kernel expansion valid at small R leads to the following well known construction:

$$\mathrm{Tr}_{(s)}[W(\Delta+E_{(s)})] = \sum_l M_l(d,s)W(\lambda_l(d,s)+E_{(s)})$$

On the sphere the laplacian has eigenvalues $\lambda_l(d,s)$ with multiplicities $M_l(d,s)$

For an optimized cutoff then coarse-graining is regulated by $R_k(\lambda_l(d,s)+E_{(s)})$ with support $\lambda_l(d,s)+E_{(s)} \leq k^2$, that is $l \leq \bar{l}^{(s)}$ for

$$\bar{l}^{(2)} = -\frac{3}{2}+\frac{1}{2}\sqrt{48\frac{k^2}{R}+17+48\alpha}\,,\ \bar{l}^{(1)} = -\frac{3}{2}+\frac{1}{2}\sqrt{48\frac{k^2}{R}+13+48\gamma}\,,\ \bar{l}^{(0)} = -\frac{3}{2}+\frac{1}{2}\sqrt{48\frac{k^2}{R}+9+48\beta}$$

Averaging sums with $\bar{l}_{(s)}$ and $\bar{l}_{(s)}-1$ (Benedetti-Caravelli, Demmel-Saueresseig-Zanusso)

$$\begin{aligned} &32\pi^2(\dot{\varphi}-2r\varphi'+4\varphi) \\ &\quad = \frac{c_1(\dot{\varphi}'-2r\varphi'')+c_2\varphi'}{\varphi'[6+(6\alpha+1)r]} + \frac{c_3(\dot{\varphi}''-2r\varphi''')+c_4\varphi''}{[3+(3\beta-1)r]\varphi''+\varphi'} - \frac{c_5}{4+(4\gamma-1)r} \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{5}{108}[6+(6\alpha-1)r][6+(6\alpha+1)r][3+(3\alpha-2)r], \\ c_2 &= \frac{5}{108}[6+(6\alpha-1)r][144+9(20\alpha-3)r+2(6\alpha+1)(3\alpha-2)r^2], \\ c_3 &= \frac{1}{72}[2+(2\beta+3)r][3+(3\beta-1)r][6+(6\beta-5)r], \\ c_4 &= \frac{1}{8}[2+(2\beta-1)r][12+(12\beta+11)r], \\ c_5 &= 12+3(8\gamma+1)r+\left(12\gamma^2+3\gamma-\frac{19}{6}\right)r^2. \end{aligned}$$

Inclusion of the constant mode for h.

Add $$\frac{8}{3}\frac{r^2}{16+2\varphi-r\varphi'}$$

Quadratic FP "solutions"

With probability one in the space of the endomorphisms the asymptotic behavior goes like r^2

Writing the FP equation as N/D, N=0 is a quintic polynomial. The ansatz $\varphi(r) = g_0 + g_1 r + g_2 r^2$

solves N=0 for several sets of the unknowns $g_0,\ g_1,\ g_2,\ \alpha,\ \beta,\ \gamma$

Heat Kernel flow

$10^3\alpha$	$10^3\beta$	$10^3\gamma$	$10^3\tilde{g}_{0*}$	$10^3\tilde{g}_{1*}$	$10^3\tilde{g}_{2*}$	θ
-593	-73.5	-177	7.28	-8.42	1.71	3.78
-616	-70.7	-154	7.42	-8.64	1.74	3.75
-564	-80.3	-168	6.82	-8.77	1.83	3.70
-543	-87.4	-126	6.31	-9.47	2.06	3.43
-420	-100.5	-3.19	4.90	-10.2	2.83	2.93
-173	-2.98	244	4.53	-8.34	2.70	2.18
-146	-64973	250	2.90	-10.7	0.0006	2.58
-109	-22267	307	2.90	-10.4	0.0045	2.45
109	-3564	526	2.84	-7.83	0.094	C
377	-1305	794	2.57	-4.37	0.214	> 4

Spectral sum flow

$10^3\alpha$	$10^3\beta$	$10^3\gamma$	$10^3\tilde{g}_{0*}$	$10^3\tilde{g}_{1*}$	$10^3\tilde{g}_{2*}$	θ
-97.8	38.9	319	4.31	-7.46	2.85	2.03
-438	-122	-21.0	4.67	-10.4	3.14	3.2
134	-2765	551	2.82	-7.70	0.13	C
505	-715	922	2.16	-2.65	0.21	> 4
-564	-63.8	-147	7.83	-6.80	1.35	> 4

Various solutions have just two relavant directions (UV attractive) with real eigenvalues (- 4, - θ)

Note that for these values there exists at least one $r > 0$ for which D=0.
Thefore such solutions are defined everywhere apart from at least one isolated point!

The equation of motion $2f - Rf' = 0$ is satisfied by R>0 and there is no redundancy of the eigenperturbations in the domain of existence.

(Dietz-Morris)

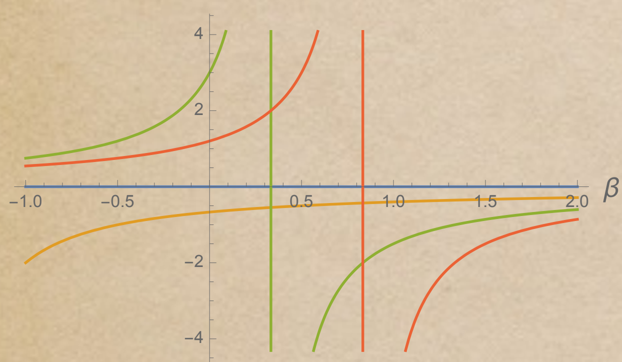
Global solutions: some parameter constraints

Are there non trivial and not simple global solutions? Analysis for the spectral sum flow.

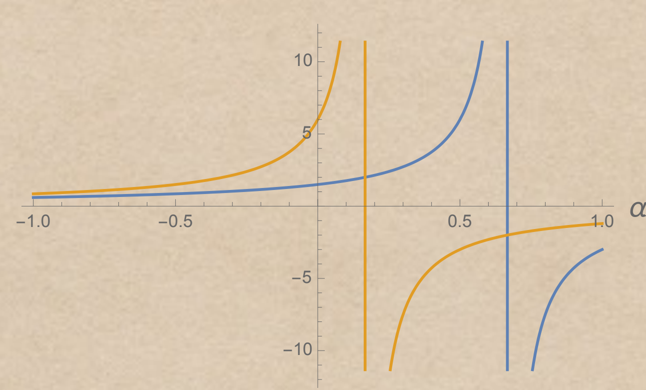
We assume at the beginning the absence of moving singularities.
Fixed singularities: look at coeff. of φ''', $1/\varphi'$ and at the ghost term.

φ'''

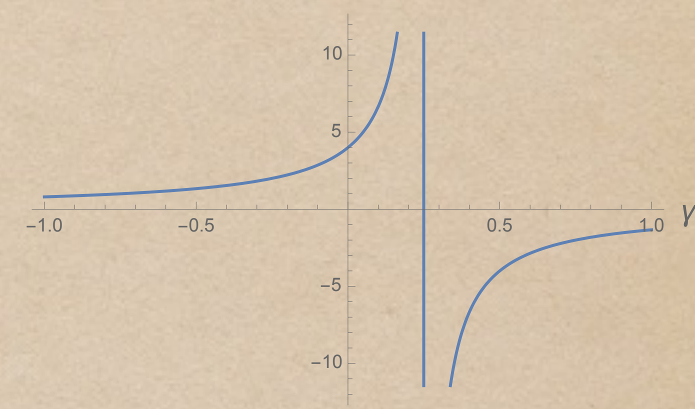
zeros of rc_3



$1/\varphi'$



Ghost term



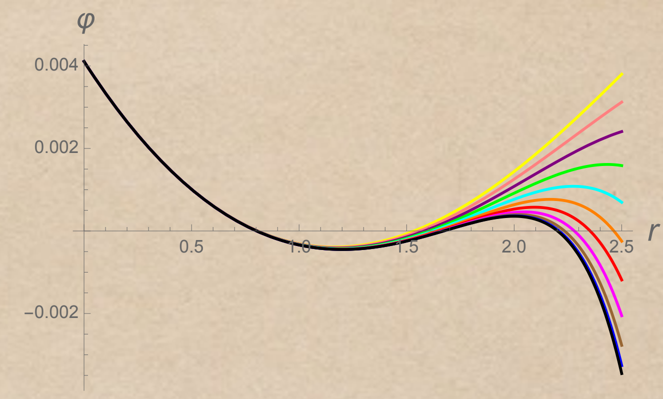
Example: $\alpha = -\frac{1}{6}\,, \beta = \frac{1}{3}\,, \gamma \geq \frac{1}{4}$

From β we get 2 fixed singularities at positive r=0 , 2.
From α we allow the existence of an extremum at r=6/5 , 3.
From γ we ensure that no other fixed singularities arise from the ghost term.

A global solution: numerical analysis

We consider the case: $\alpha = -\frac{1}{6}\,,\, \beta = \frac{1}{3}\,,\, \gamma = \frac{1}{2}$

Polynomial analysis around the origin up to order 16 (black): it becomes very stable and suggests indeed a global solution with a minimum in 6/5.



2 relevant direction. Real Critical exponents: 4, 1.83

Regularity conditions at r=0 and r=2 together with the condition of minimum at r=6/5 fix completely the three parameters for the Cauchy problem.

Strategy:

1) Provide analytical polynomial expansions in terms of two parameters at the points 0, 6/5, 2.

0	6/5	2
$\varphi'(0)\,,\,\varphi''(0)$	$\varphi(6/5)\,,\,\varphi''(6/5)$	$\varphi'(2)\,,\,\varphi''(2)$

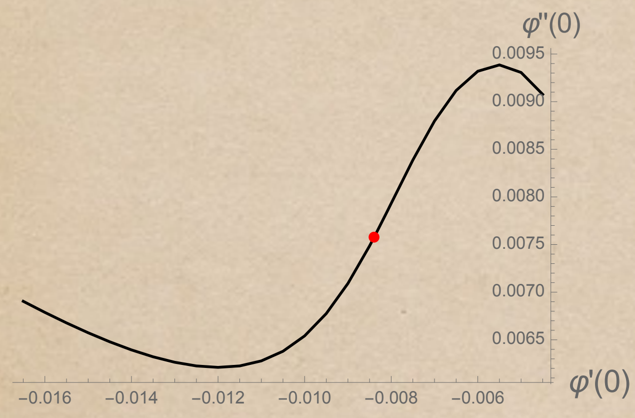
2) Evolve num. from 0+ to 6/5 imposing the condition of minimum ⟶ curve in $\varphi'(0)\,,\,\varphi''(0)$
Evolve num. from 2- to 6/5 imposing the condition of minimum ⟶ curve in $\varphi'(2)\,,\,\varphi''(2)$

3) Map the two curves into two curves in 6/5 in the plane $\varphi(6/5)\,,\,\varphi''(6/5)$

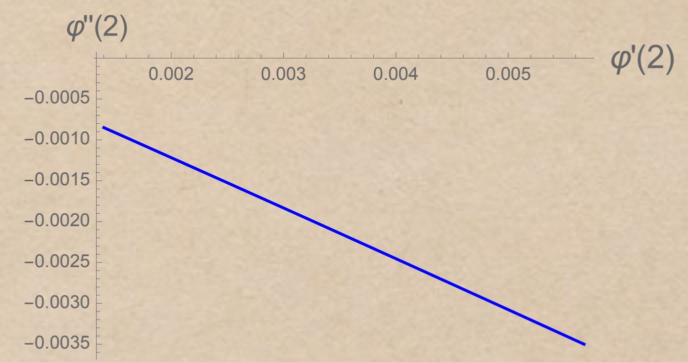
4) The intersection fixes completely the parameters of the solution.

Numerical analysis 2

Evolution from 0+ to 6/5
(red point at polynomial solution)

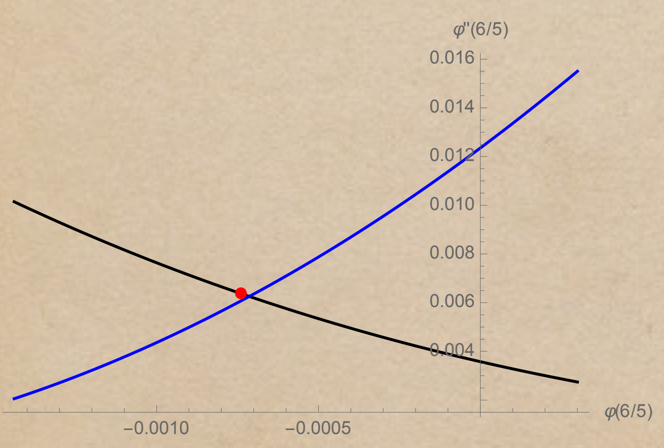


Evolution from 2- to 6/5



Remark: the evolution between betweek 2- and 6/5 encounters also <u>moving singularities</u> in <u>other</u> regions of the parameter plane!

Intersecting the curves mapped to r=6/5



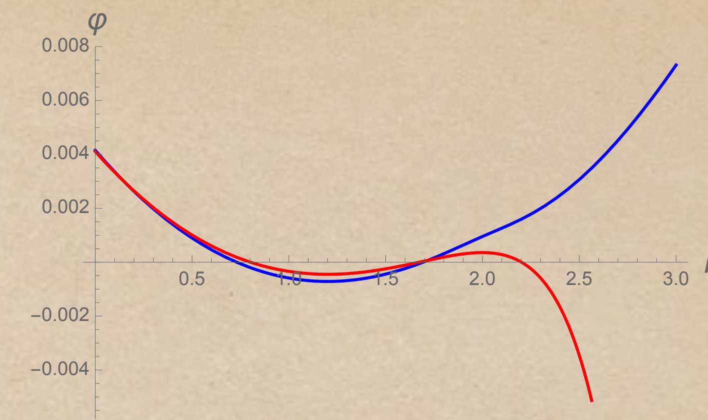
$$\varphi(6/5) = -0.0007136\cdots$$

$$\varphi''(6/5) = 0.006256\cdots$$

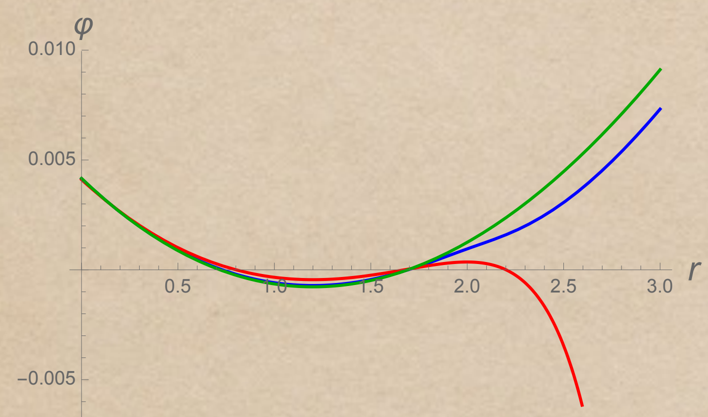
The intersecting solution is slightly deformed compared to the approximate polynomial solution.

Numerical analysis 3

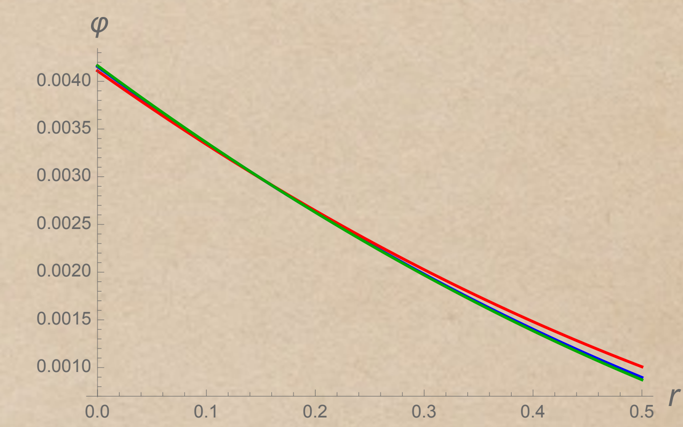
The solution (blue) is constructed from a numerical evolution in the intervals 6/5- to 0, 6/5+ to 2- and from 2+ to r>2 and using the analytic polynomial expansions around 6/5 and 2. Order 16 polynomial solution around the origin (red).



Possible to study at polynomial expansion around the minimum. It looks as a better approximation (green curve, order 16).

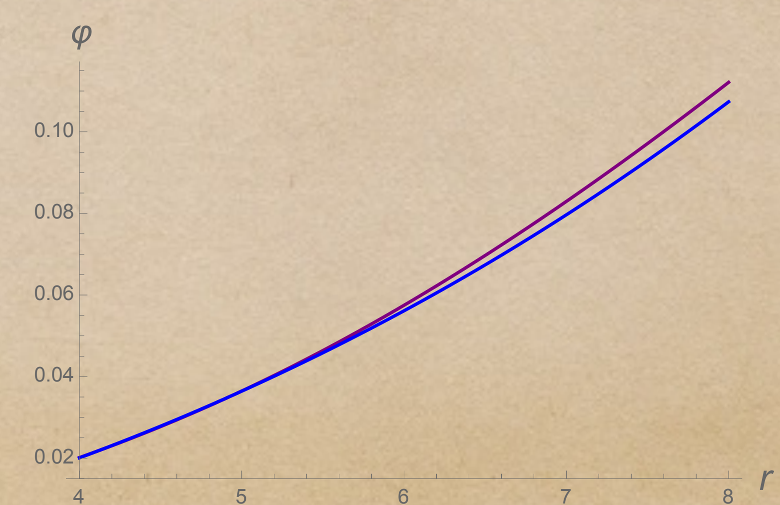


Zooming close to the origin:



Asymptotic expansion:

$$\varphi_{as}(r) = Ar^2 + \frac{1053Ar}{50 - 13824\pi^2 A} + \frac{1051066368\pi^4 A^2 + 107637120\pi^2 A - 1943075}{6144\pi^2\left(25 - 6912\pi^2 A\right)^2} + O\left(\frac{1}{r}\right)$$



Comparison of asymptotic expansion (purple) with numerical solution at large r (blue) after tuning the parameter A.

Some consideration on the domain of f(r).

The Heat Kernel expansion is reliable only for small $r = R/k^2$ so that there is no access at fixed R to the IR limit.

For fhe spectral sum approach we considered the average of the sums for all the spins of modes with two different upper bounds, both starting from 2 because of the Killing symmetries.

$$2 \leq l \leq \bar{l}^{(s)} \quad , \quad 2 \leq l \leq \bar{l}^{(s)} - 1$$

For large $r = R/k^2$ depending on the endomorphisms there might be no room to some modes.

On the other end it is also not clear what means a coarse-graining at length scales larger that the dimension of the compact manifold. Therefore one might question if at fixed R one shoud look for a global scaling solution defined on the full positive semiaxes. The situation looks better for noncompact background manifolds.

Conclusions

- We have revisited the f(R) truncation using a non linear parametrization of the exponential form in a single metric truncation.

- The choice of gauge invariant fluctuations and gauge fixing, based on a spin decomposition leads to simple structure of the Hessian.

- The flow equation are constructed with a spectrally adjusted coarse-graining scheme, using either and HK expansion or a spcetral sum on the sphere.

- Pure quadratic solutions valid everywhere apart from finite points exist. Other global solutions can be constructed numerically. The latter have typically 2 relevant directions.

- The sphere, which is a compact manifold poses some limitations in the cosntruction of a flow in the IR limit. As already noted an upper bound in the curvature may appears even if the equation defines f(R) everywhere.

Outlook

- Pure cutoff schemes lead to more complicated flow equations.
  Not yet analyzed.

- A similar analysis on non compact backgrounds has been started.

- An urgent issue is the one related to background independence.

- Inclusion of matter (e.g. scalar) at this level: $F_k(\rho, R)$

- Inclusion of the anomalous dimension.

- More general truncations.

Thank you!