From groups and knots to BH entropy

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0. Introduction

Loop Quantum gravity

Quantum theory of gravity.

Some of the **assumptions:**

- space-time 3+1 dimensional
- Einstein gravity
- quantization of geometry (not topology, not diff manifold,…)
- general covariance (—> use background structures as little as possible)

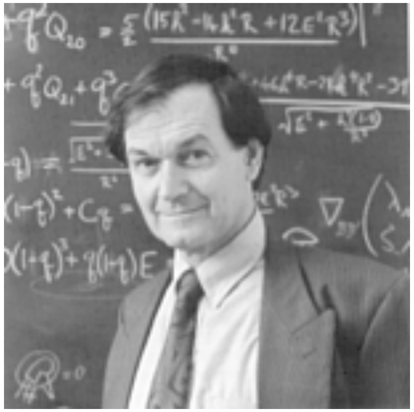
Some of the **consequences**:

- QFT of geometry
- discrete, combinatorial picture: "Atoms" of space(-time)
- not easy to interpret operationally

1. Quantized space

Spin networks

Penrose (1971):



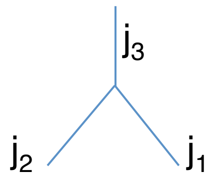
Spin network:

- directed graph (may be embedded in 3-dim manifold)
- SU(2) irrep (spin j_e) associated to each edge e
- invariant tensor at each vertex

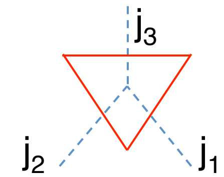
$$I_v \in \text{Inv}\left(\bigotimes_{e \text{ at } v} j_e\right)$$

Quantum (spatial) geometry <--> QM of spin:

Trivial example: existence of invariant tensor, quantum triangle



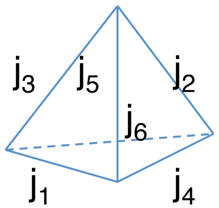
$$\longleftrightarrow \quad j_1 + j_2 + j_3 \geq 2\max(j_1, j_2, j_3) \quad \longleftrightarrow$$



Chromatic evaluation:

Penrose invents way to associate number to spin network via graphical calculus

Example:

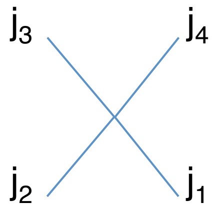


$$\longleftrightarrow \quad \left\{ \begin{matrix} j_1 & j_3 & j_5 \\ j_2 & j_4 & j_6 \end{matrix} \right\}$$

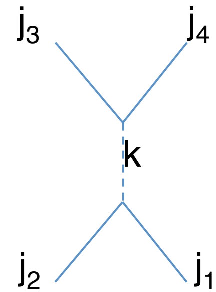
Uses it to give inner product on spin networks, beginning of quantum theory of geometry

Note for later:

Virtual edges can be used to always go to trivalent case:



$$\longleftrightarrow \qquad \sum_k c_k$$



c_k can be interpreted in terms of geometry.

Beautiful thing:

In loop quantum gravity:

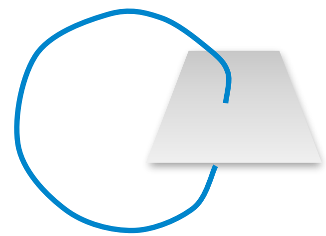
spin networks = states of non-commutative spatial geometry

is result of quantization of gravity.

Geometric operators: [Rovelli+Smolin, Ashtekar+Lewandowski]

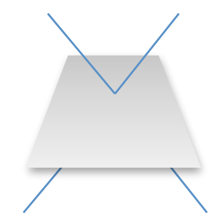
Area

$$\widehat{A}_{\triangle} \, |\underset{j}{\bigcirc}\rangle = 8\pi\beta l_p^2 \sqrt{j(j+1)} \, |\underset{j}{\bigcirc}\rangle$$



more generally:

$$\Delta_v = \left(\sum_{e \text{ above } S} \vec{J}_e - \sum_{e \text{ below } S} \vec{J}_e \right)^2$$



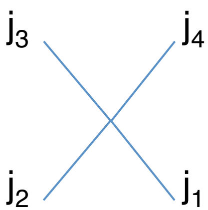
$$\widehat{A}|_v = 4\pi\beta\ell_P^2\sqrt{\Delta_v}$$

Spectrum is purely discrete

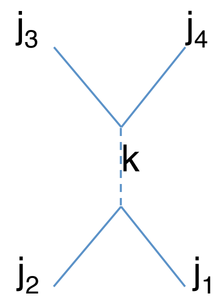
Area is non-commutative

$$[\widehat{A} \quad , \widehat{A} \quad] \neq 0$$

Volume: Action on intertwiner spaces



$$= \sum_k c_k$$



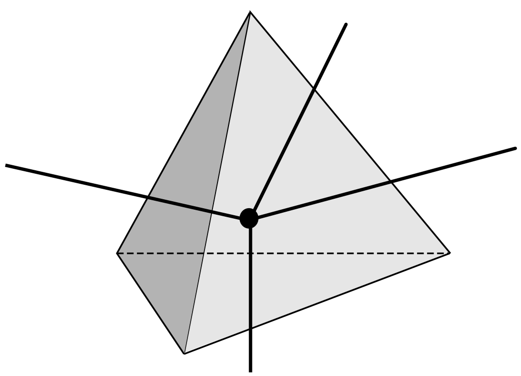
$$= \sum_k c_k \, |k\rangle$$

$$\widehat{V}_R \, |k\rangle = \sum_l V(v)_{kl} \, |l\rangle$$

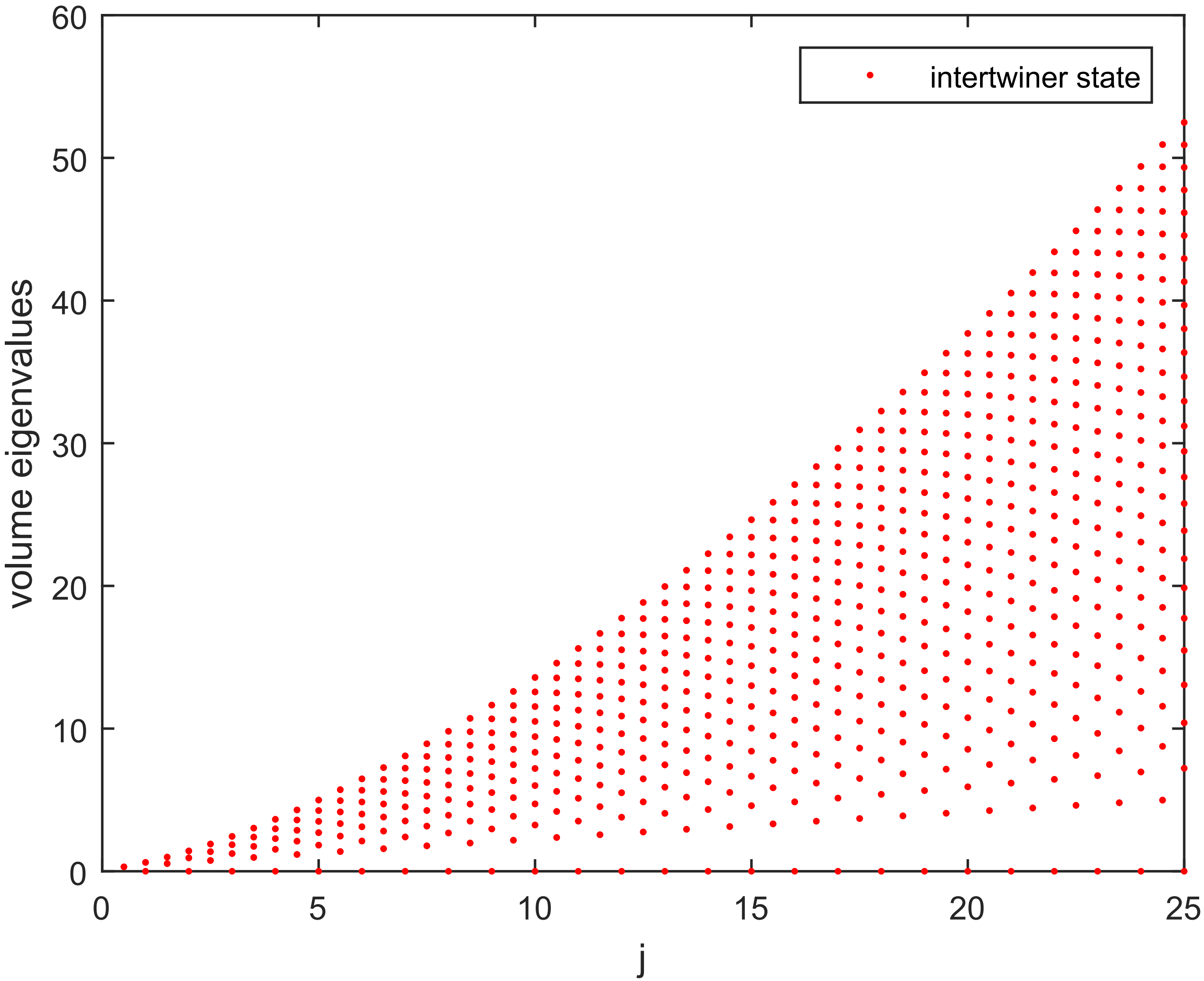
Picture: Vertices as atoms of space

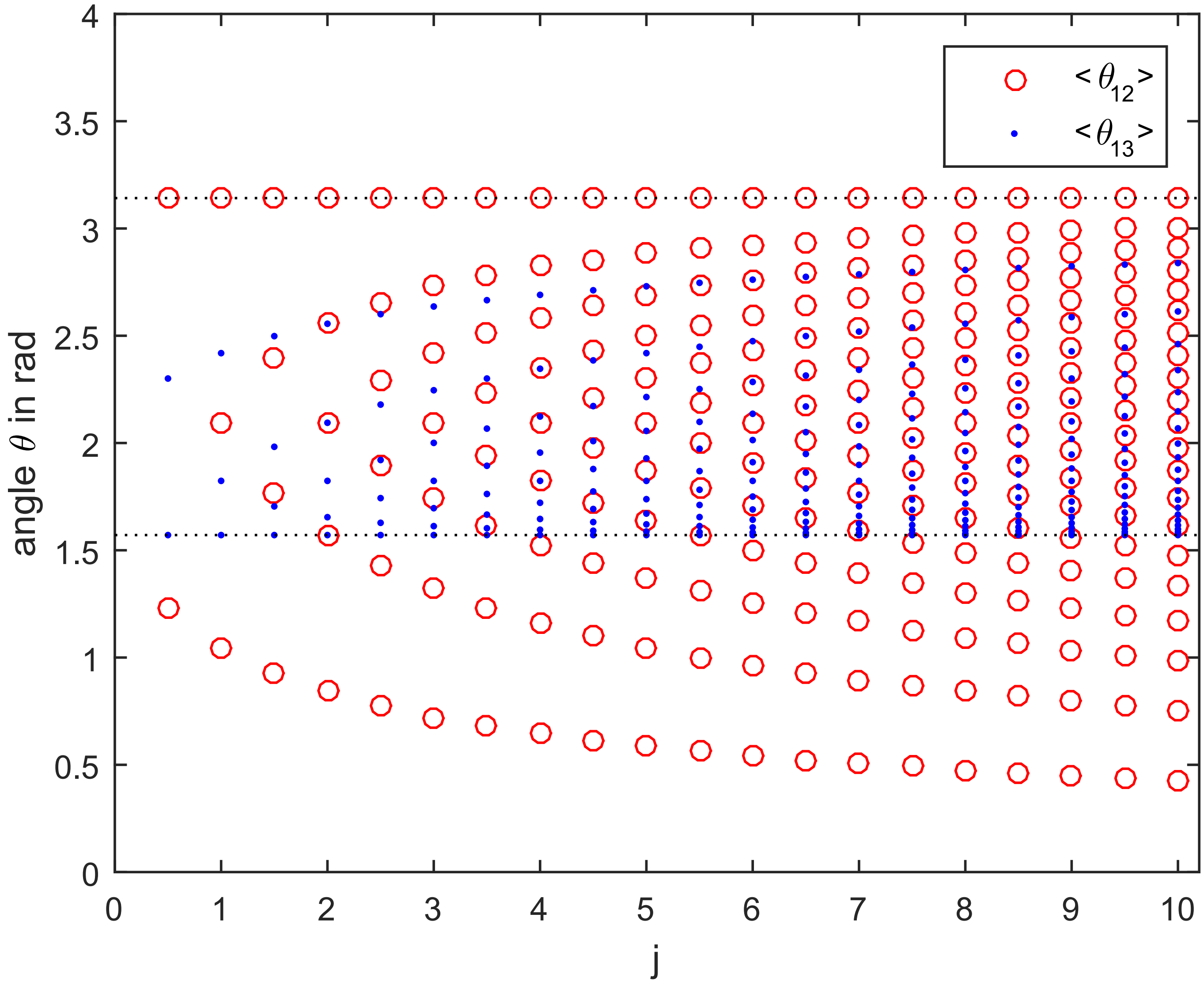
More quantitatively [Baez, … , Bianchi, Dona,Speziale]

n valent vertex <——> quantized flat polyhedron

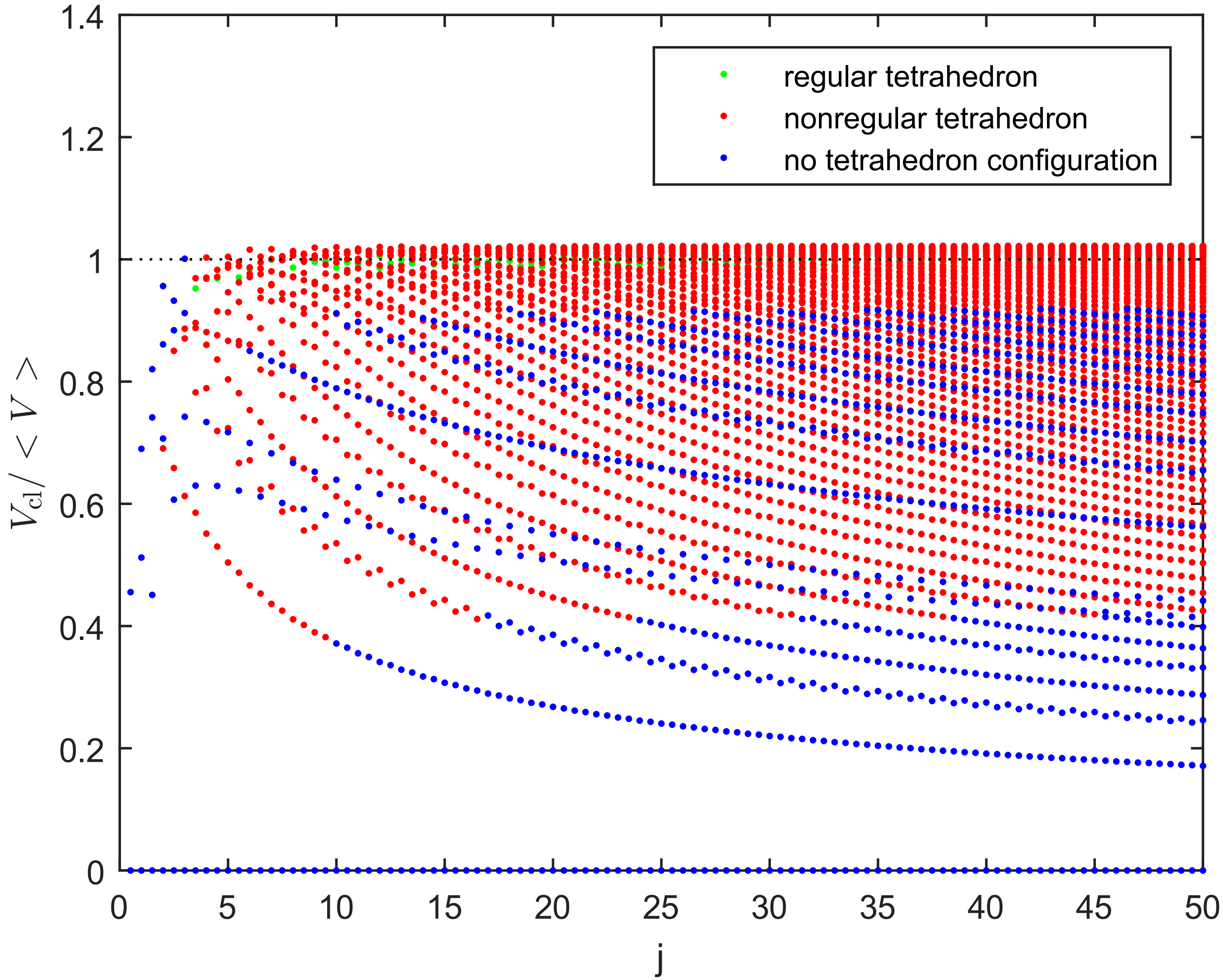


Some examples for monochromatic 4-vertex (from Bachelors thesis of K.Eder)


intertwiner state
volume eigenvalues
j
0
10
20
30
40
50
60
0
5
10
15
20
25



angle θ in rad
j
< θ12 >
< θ13 >


regular tetrahedron
nonregular tetrahedron
no tetrahedron configuration
$V_{\rm cl}/<V>$
j
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50

Derivations (sketch!)

Ashtekar's discovery:

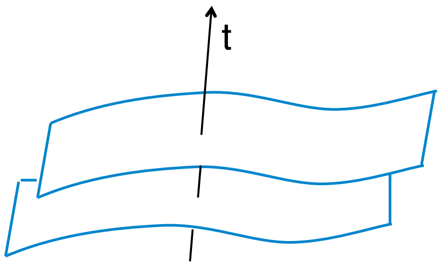
GR can be formulated such that (non-reduced) phase space is that of SU(2) Yang-Mills

$$S[e,\omega] \sim \int *(e\wedge e)\wedge F(\omega) + \frac{1}{\gamma} e\wedge e\wedge F(\omega)$$

- e^I_a : Tetrad
- ω_a^{IJ} : SO(3,1) connection
- $*$: "internal Hodge"

$$\sim \int dt \int_{\Sigma_t} E^a_i \dot{A}^i_a - \omega_0^{i0} G_i + e^i_0 C_a + e^0_0 C$$

- space-time split
- covering group SO(3,1) —> SL(2,C)
- partial gauge fixing (time gauge) SL(2,C) —> SU(2)



phase space formulation

- (A,E) phase space coordinates
- (first class) constraints G,C_i, C

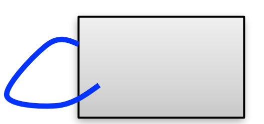
$$A \longmapsto h_e = \mathrm{Pexp} \int_e -A$$

$$E \longmapsto E_{S,f} = \int_S E^i f_i$$

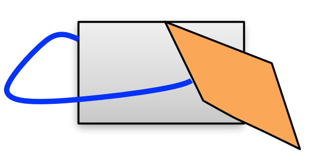
Quantum algebra

$$[h_\alpha, h_{\alpha'}] = 0$$

$$[E_{\ ,f}, \ ] = 8\pi\beta l_P^2 \, f(p)^I \ \ \text{I}$$

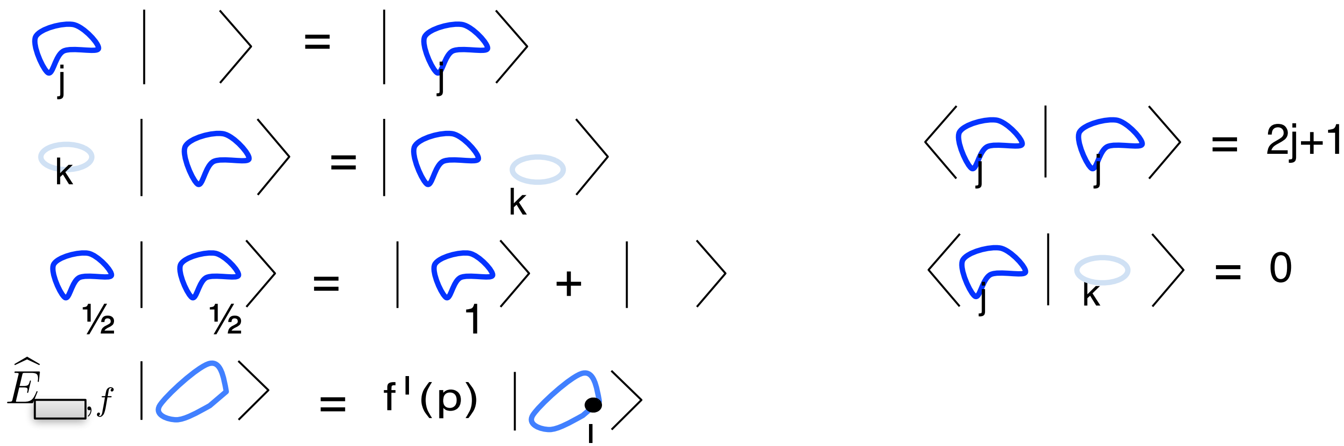


$$[[E_{\ ,f}, E_{\ ,g}], \ ] = (8\pi\beta l_P^2)^2 \, f(p)^I g(p)^J \ \ \text{IJ}$$



Representation: [Rovelli+Smolin, Ashtekar+Lewandowski]

Combinatorial description



No background geometry used anywhere. Spatial diffeomorphisms unitarily represented.

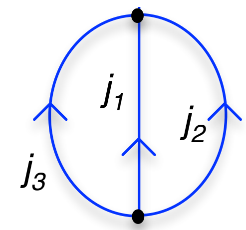
Uniqueness: [HS, LOST, Fleischhack]

This is the only cyclic rep with spatial diffeo invariant vacuum.

Analytic description:

Spin nets are associated to gauge invariant functionals.

$$\Psi[A] = \pi_{j_1}(h_{e_1}[A])^{m_1}{}_{n_1}\, \pi_{j_2}(h_{e_2}[A])^{m_2}{}_{n_2}\, \pi_{j_3}(h_{e_3}[A])^{m_3}{}_{n_3}\ \iota_{m_1 m_2 m_3}\iota^{n_1 n_2 n_3}$$



Inner product: [Ashtekar+Lewandowski]

$$\langle \Psi_\gamma | \Psi'_\gamma \rangle = \int_{\mathrm{SU}(2)^{|\gamma|}} \prod_i \mathrm{d}h_i\ \overline{\Psi(h_1, h_2, \ldots)}\Psi'(h_1, h_2, \ldots)$$

Spin nets are orthogonal (compare Peter-Weyl).

Hilbert space

$$\mathcal{H}_{\mathrm{AL}} = \lim_{\gamma \to \infty} \mathcal{H}_\gamma = L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, \mathrm{d}\mu_{\mathrm{AL}})$$

projective/inductive limits

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Note: Higher dimensions

Can embed gravity in D+1 in SO(D+1) -Yang Mills phase space.

[Bodendorfer, Thiemann, Thurn 2013]

Canonical pair:

- A: SO(D+1) connection
- π: tensor density corresponding momentum

with:

- simplicity constraints ⇒ π encodes d-bein
- Gauss, diffeo, Hamilton constraints

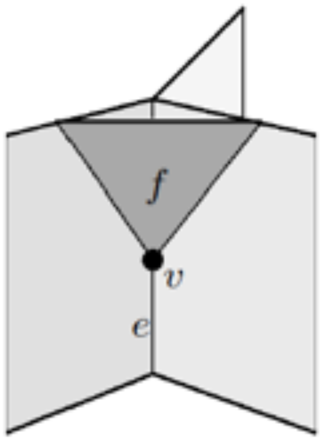
2. Quantized space-time

(see also D. Oriti's talk later this afternoon!)

Spin foam [Ponzano-Regge, Baez, Rovelli+Reisenberger,…]

For group G:

- oriented 2-complex (with boundary) C
- labeling of faces with irreps of G
- labeling of edges with interwiners between face-reps



Amplitude:

$$Z : \{\ \mathrm{C},\ \text{boundary data}\ \} \longrightarrow \mathbb{C}$$

Interpretation: Transition amplitude.

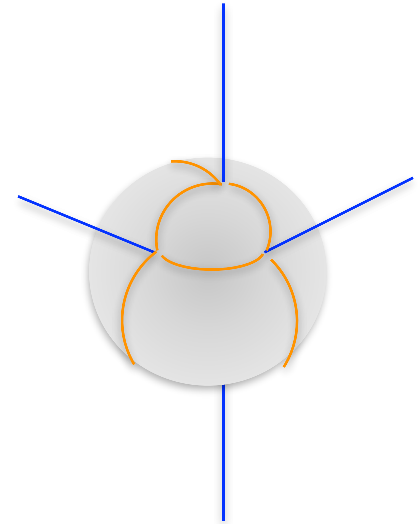
Typically:

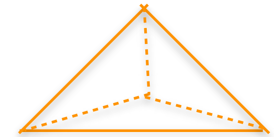
$$Z = \sum_{\{f\}\to \mathrm{Irrep}\, G} \sum_{\iota} \prod_f A_f(\pi) \prod_v A_v(\pi_e, \iota_n)$$

EPRL-FK vertex amplitude:

$$A_v(j_e, \iota_n) = (YT_{\{j_e,\iota_n\}})(\mathbb{I}_{\mathrm{SL}(2,\mathbb{C})})$$

- T: vertex spin network
- Y: map SU(2) spin nets —> SL(2,C) spin nets
  For choice of SU(2) subgroup





$$\mathcal{H}_{p,k} = \bigoplus_{j=k}^{\infty} \mathcal{H}^j_{p,k} \qquad Y: \quad \pi^j(\cdot)^m{}_{m'} \longmapsto \pi^j_{\beta j,j}(\cdot)^m{}_{m'}$$

Asymptotics

Vertex amplitude asymptotes to Regge-action when evaluated on boundary coherent state

$$\sum_{j_e}\sum_{\iota_n} A_v^{\mathrm{EPRL}}(j_e,\iota_n)\Psi^t(j_e,\iota_n) \;\simeq\; e^{iS_{\mathrm{Regge}}}+e^{-iS_{\mathrm{Regge}}}$$

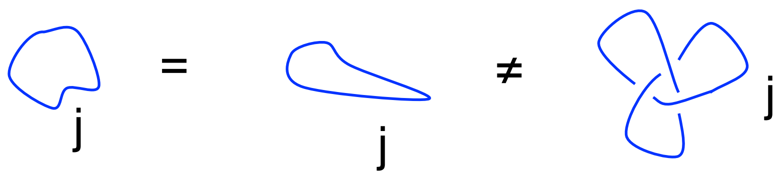
where Ψ is peaked on boundary geometry of 4-simplex, and S_{Regge} is Regge action.

Vertex: Atom of space-time?

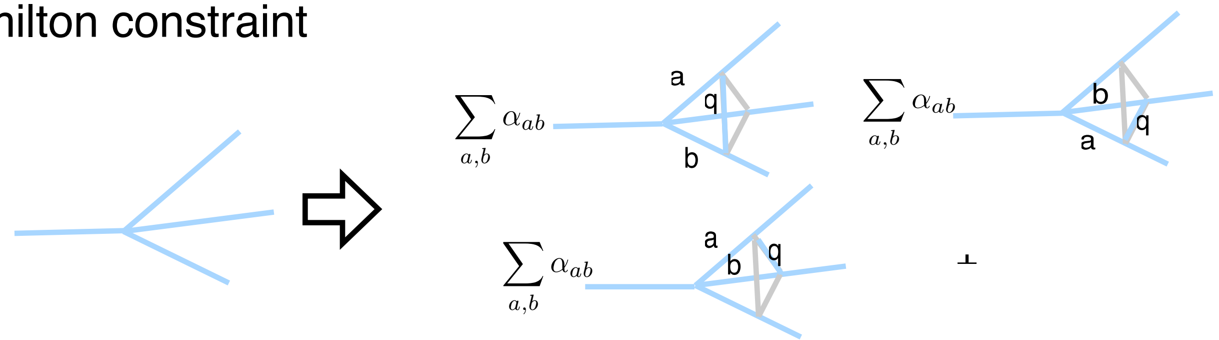
Derivations (sketch!)

1) Canonical

- Diff constraint: Roughly speaking



- Hamilton constraint



using formal expansion

$$\delta(\vec{x}) = \int \exp(i\vec{k}\cdot\vec{x})\mathrm{d}\vec{k} = \int (1 + i\vec{k}\cdot\vec{x} + \frac{1}{2}(i\vec{k}\cdot\vec{x})^2 + \ldots)\mathrm{d}\vec{k}$$

Projector on kernel

$$P_{\text{phys}} = \prod_{x} \delta(\widehat{C}(x)) = \int DN \exp(i \int dx N(x)\widehat{C}(x))$$
$$= \mathbf{1} + i \int DN \int dx N(x)\widehat{C}(x)$$
$$- \frac{1}{2}\int DN\, DN' \int dx\, dx' N(x)\widehat{C}(x)N'(x')\widehat{C}(x') + \ldots$$

Has expansion of matrix elements labeled by 2-complexes

2) covariant

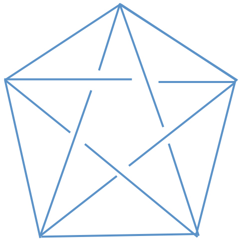
- $$S[B,\omega,\phi] = \int B \wedge F(\omega) + \phi\, B \wedge B$$

  with phi enforcing simplicity

$$B = *(e \wedge e) + \frac{1}{\beta} e \wedge e$$

- BF theory topological, Z can give exact path integral

$$Z = \sum_{\{\pi_f\}} \sum_{\{I_e\}} \prod_f A_f(\pi_f) \prod_v$$



$$(\mathbb{I})$$

- Y constructed in such a way that simplicity holds in a sense for the SL(2,C) representations

$$\mathcal{H}_{p,k} = \bigoplus_{j=k}^{\infty} \mathcal{H}^j_{p,k} \qquad Y: \quad \pi^j(\cdot)^m{}_{m'} \longmapsto \pi^j_{\beta j,j}(\cdot)^m{}_{m'}$$

But:

- divergencies in the Lorentzian case
- summing over spin-foams
- renormalization?

Ongoing work:

Course graining, renormalization [Dittrich, Oriti, Bahr, Ben Geloun, Steinhaus,…], Tensor models [Gurau, Benedetti,…]

3. Why could LQG be right?

Classical limit:

– covariant formulation ---> Regge gravity

– canonical formulation ---> coherent states describing approx. classical metrics

Cosmology:

– resolution of singularities

– consistent picture of inflationary phase of universe

3d gravity:

- canonical and covariant picture equivalent
- standard picture obtained

Black holes:

– Entropy for large class of black holes

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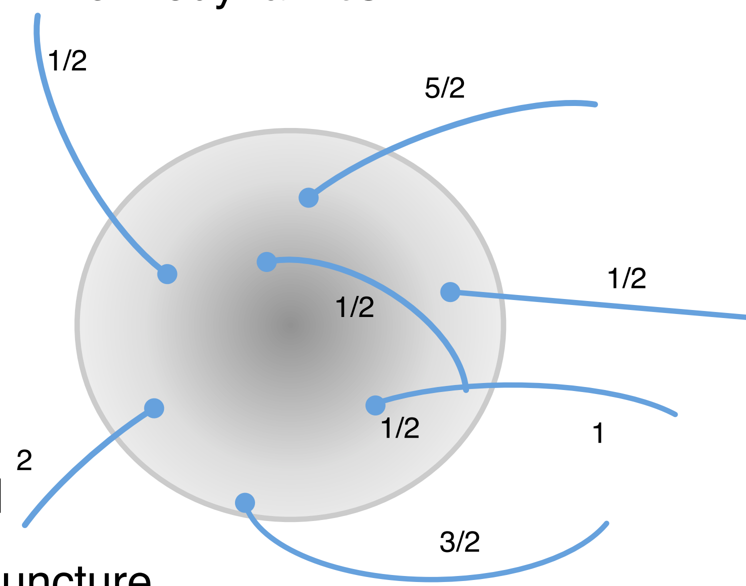
4. Black holes in LQG

Isolated horizons: [Ashtekar et al, Engle+Noui+Perez]

Quasilocal notion of BH horizon, strong enough for BH thermodynamics.

Boundary condition at horizon

$$\underset{\Leftarrow}{F}^I(A^\beta) = \frac{(1-\beta^2)\pi}{a_H}\underset{\Leftarrow}{\Sigma}^I$$



Symplectic structure aquires CS boundary term.

Quantum theory: [Ashtekar+Baez+Krasnov, many others]

LQG in bulk, SU(2) CS on boundary. Spin nets can puncture horizon and endow it with area.

Punctures = CS particles.

$$\mathcal{H}_{(j_1,j_2\ldots)} = \mathcal{H}_{CS}(j_1,j_2\ldots)\otimes\mathcal{H}_{\text{bulk}}$$

Entropy:

For given puncture structure

$$\mathcal{H}_{a_H} = \bigoplus_{4\pi\beta \sum_i \sqrt{j_i(j_i+1)} \approx a_H} \mathcal{H}_{(j_1, j_2 \ldots)}$$

Infinite dimensional. For entropy ignore bulk part.

Count: All sequences j_1, j_2, … such that

$$4\pi\beta \sum_i \sqrt{j_i(j_i+1)} \leq a_H$$

with multiplicity approx $\dim \mathrm{Inv}(j_1 \otimes j_2 \otimes \ldots)$

Result:

$$\ln N_{\leq}(a) = \frac{\widetilde{\beta}}{2\pi\beta} \frac{a_H}{4 l_P^2} - \frac{3}{2} \ln(a_H / l_P^2) + O(a_H^0)$$

reproduces Bekenstein-Hawking for correct choice of β.
Similar results for more general types of BH with same choice of β.

Intrinsic description?

Quantum boundary condition: Want

$$\underset{\Leftarrow}{\widehat{F}}^{I}(A^{\beta})\,\Psi = \frac{(1-\beta^2)\pi}{a_H}\underset{\Leftarrow}{\widehat{\Sigma}}^{I}\,\Psi$$

But F not well defined in LQG. Thus exponentiate, using non-abelian Stokes' theorem:

$$h_{\partial S}[A] = \mathcal{P}\exp\oiint_S \mathcal{F}[A]\,\mathrm{d}^2 s \qquad \boldsymbol{W}_S := \mathcal{P}\exp\oiint_S \mathcal{E}[A]\,\mathrm{d}^2 s$$

$$\mathcal{F} = hFh^{-1}[A] \qquad \mathcal{E} = c\,h(*E)h^{-1}$$

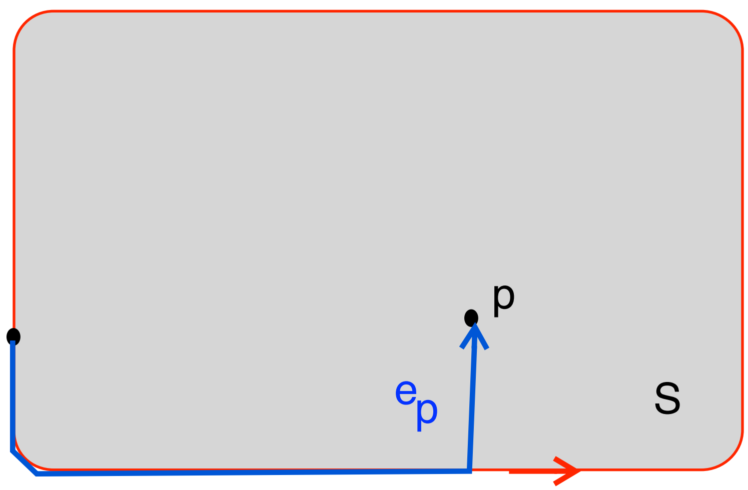
Then demand quantum boundary condition:

$$\mathrm{tr}(h_{\partial S})\Psi = \mathrm{tr}(W_S)\Psi$$

Can W_S be defined? Does the equation have solutions?

Non-Abelian Stokes theorem (I. Ya. Aref'eva 1980):

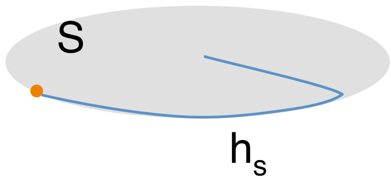
$$h_{\partial S}[A] = \mathcal{P}\exp \oiint_S \mathcal{F}[A]\mathrm{d}^2p$$



$$\mathcal{P}\exp \oiint_S \mathcal{F}[A]\,\mathrm{d}^2p := \mathcal{P}\exp \oiint_S h_{e_p}^{-1}F(p)h_{e_p}\,\mathrm{d}^2p$$
$$= \mathbb{I} + \int_S h_{e_p}^{-1}F(p)h_{e_p}\,\mathrm{d}^2p + \iint_{p,p'\in S^2:p\leq p'} h_{e_p}^{-1}F(p)h_{e_p}h_{e'_p}^{-1}F(p)h_{e'_p}\,\mathrm{d}^2p\mathrm{d}^2p' + \ldots$$

Key object: [HS+Thiemann]

$$\boldsymbol{W}_S := \mathcal{P}\exp \oiint_S \mathcal{E}[A,E](s)\,\mathrm{d}^2 s$$
$$= \mathbb{I}_2 + 8\pi i c \int_S \mathrm{Ad}_{h_s}(*\widehat{E}(s))$$
$$+ 8\pi i c \int_{S^2} K_{s,s'}\,\mathrm{Ad}_{h_s}(*\widehat{E}(s))\mathrm{Ad}_{h_{s'}}(*\widehat{E}(s'))$$
$$+ \dots$$



Can we make it well defined?

First step: LQG E is operator (matrix) valued distribution, factorizes:

$$\widehat{E}^a_I(s) = \widehat{E}^a(s)\widehat{E}_I(s): \qquad [\widehat{E}_I(s), \widehat{E}_J(s)] = \epsilon_{IJ}{}^K\,\widehat{E}_K(s)$$

$$\widehat{E}^a(s)h_e[A] = e^a(s)h_e[A], \quad e^a(s) = \int \mathrm{d}t\,\dot{e}^a(t)\delta^3(s, e(t)),$$

Two problems:

1) Delta functions at integration boundaries.

Solution: Standard procedure gives factor 1/n!

2) Ordering problem: How to order the E_I ?

Solution: Harish-Chandra/Duflo isomorphism

earlier suggested in somewhat different context [Alekseev et al, Freidel]

Harish-Chandra/Duflo map

Given semisimple Lie algebra $\mathfrak{g}$.

Quantization map

Kirillov-Kostant brackets

$$[T_I, T_J] = 0, \qquad \{T_I, T_J\} = f_{IJ}^K T_K$$

$$\Upsilon : \mathrm{Sym}(\mathfrak{g}) \longrightarrow \mathrm{U}(\mathfrak{g}) \qquad [T_I, T_J] = f_{IJ}^K T_K$$

$$\mathrm{Sym}^G(\mathfrak{g}) \longleftrightarrow \mathrm{Z}(\mathrm{U}(\mathfrak{g}))$$

which is an Isomorphism

It's a refinement of symmetric quantization (PBW) χ

$$\Upsilon = \chi \circ j^{\frac{1}{2}}(\partial)$$

where

$$j^{\frac{1}{2}}(x) = \det{}^{\frac{1}{2}} \left(\frac{\sinh \frac{1}{2}\mathrm{ad}x}{\frac{1}{2}\mathrm{ad}x} \right) = 1 + \frac{1}{48}\|x\|^2 + \dots \qquad \partial^I T_J = \delta^I_J$$

Example: For SU(2)

$$\Upsilon(\|E\|^2) = \Delta_{\mathrm{SU}(2)} + \frac{1}{8}\mathbb{I}$$

This makes W_S well defined (albeit hard to determine explicitly)

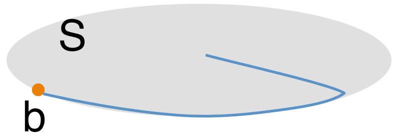
General properties:

For suitably chosen path systems

$$W_{S_1+S_2} = W_{S_1} W_{S_2}, \qquad W_S^\dagger = W_{-S}$$

Under gauge transformations

$$U_g W_S U_g^{-1} = g(b) W_S g(b)^{-1}$$

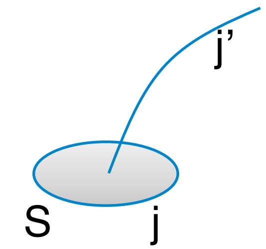


For SU(2):

$$\mathrm{tr}_j\,(W_S)\,|j', m\rangle = \underbrace{\frac{\sin\left[\pi c(2j+1)(2j'+1)\right]}{\sin\left[\pi c(2j'+1)\right]}}_{=:\lambda_{jj'}} |j', m\rangle \qquad j, j' \neq 0$$

$$W_S|0,0\rangle = \mathbb{I}\,|0,0\rangle$$

This is subject to quantization ambiguity



Note: Eigenvalues can be written in terms of quantum integers

$$\lambda_{j,j'} = \frac{[(2j+1)(2j'+1)]_q}{[2j'+1]_q} \qquad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}} \qquad q = e^{\pi i c}$$

They are related to

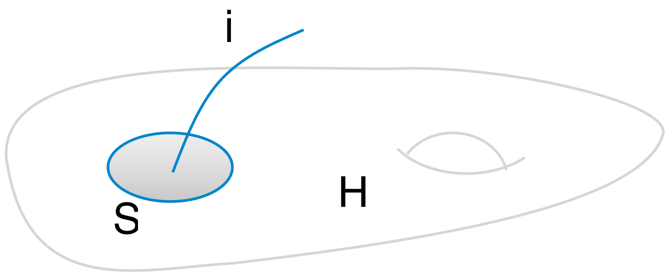
- Verlinde coefficients of $SU(2)_k$ rational CFT (k=1/c)
- Trace of the square of the R-matrix of $U_q(su(2))$ on j ⊗ j'

and are precisely what to expect for holonomy around particle in SU(2) CS [Witten]

Back to black holes

Quantum boundary condition: For any S in H and $c = 1/k = -\pi\beta(1-\beta^2)\ell_{\mathrm{P}}^2/2a_H$

$$\mathrm{tr}(h_{\partial S})\Psi = \mathrm{tr}(W_S)\Psi$$



If we had a solution Ψ

- formula for simple loops:

$$W_\alpha^{(j)}\,\Psi = \begin{cases} \lambda_{jj_i}\Psi & \alpha \text{ around } p_i \\ \Psi & \alpha \text{ trivial} \end{cases}$$

- inv under small diffeos fixing punctures:

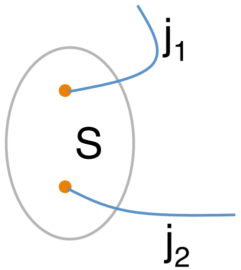
$$W_\gamma\Psi = W_{\phi(\gamma)}\Psi$$

- reps on H only different mod k
- nontrivial monodromy of punctures
- some fluxes transversal to surface well defined
- must be in new rep of HF algebra

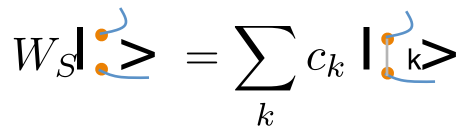
Question: Do we have a representation? (WIP)

Have a good representation for the bulk and simple loops in H.

Extension to non- simple loops? Difficult to answer due to



1) Action of W_S Complicated:



2)There are ∞ many Mandelstam identities to be satisfied. Checked some things, ex.

$$\mathrm{tr}_{\frac{1}{2}}\left(W_{S_1}\right)\mathrm{tr}_{\frac{1}{2}}\left(W_{S_2}\right)|j_1 j_2\rangle = \mathrm{tr}_{\frac{1}{2}}\left(W_{S_1+S_2}\right)\mathrm{tr}_{\frac{1}{2}}\left(W_{S_1}W_{-S_2}\right)|j_1 j_2\rangle$$

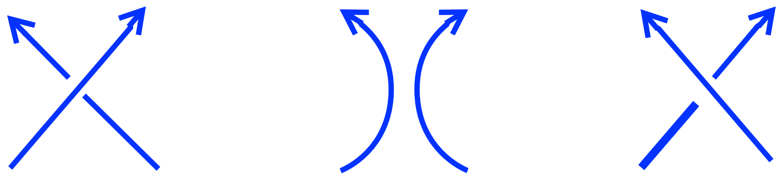
but not all.

Further question: Do we have SU(2) CS? (WIP)

- DOF remaining on horizon point to ISU(2) with particles
- would be nice: 3d Euclidean quantum gravity with particles on horizon
- Boundary conditions do not seem to fit constraints of ISU(2)-CS 100%

Jones polynomial

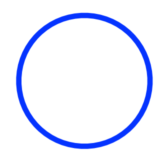
Invariant of oriented knots. Usually defined via skein relations:



L_+ L_0 L_-

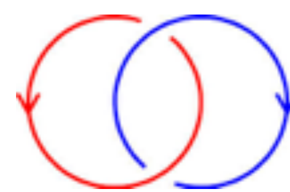
$$-qJ(L_+) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J(L_0) + q^{-1}J(L_-) = 0$$

Examples:



$$J(\bigcirc) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}} = q^{1/2} + q^{-1/2}$$

$$J(H_+) = 1 + q^{-1} + q^{-2} + q^{-3}$$



H_+

\

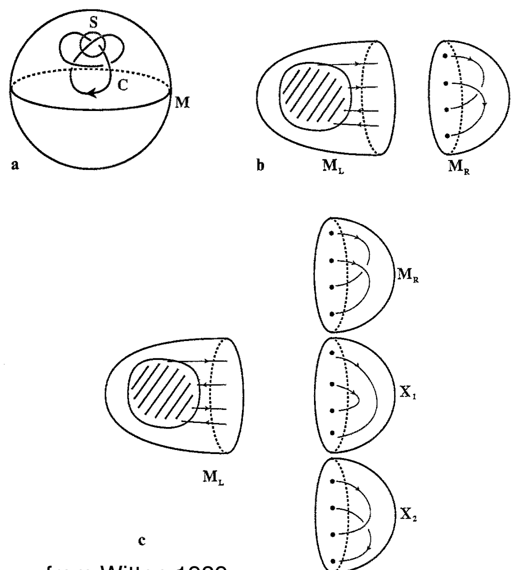
Jones polynomial and SU(2)

Witten: Jones polynomial

is expectation value of j=1/2 trace of holonomies in SU(2) CS

$$J(\alpha) = \langle \mathrm{tr}(h_\alpha) \rangle_{\mathrm{CS}} = \int_{\overline{\mathcal{A}}} \mathrm{tr}(h_{\partial_S})[A] e^{iS_{\mathrm{CS}}[A]}\, \mathrm{d}\mu[A]$$

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_M \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right)$$



from Witten 1989

To regularize the integral: Introduce framing.

Standard framing --> Jones polynomial.

Elegant proof via CQFT.

\

Jones polynomial from W_S

Using

$$-\frac{8\pi i}{k}\epsilon_{abc}\frac{\delta}{\delta A_c}e^{iS_{\mathrm{CS}}[A]} = F_{ab}e^{iS_{\mathrm{CS}}[A]}$$

can replace holonomies under the CS path integral by W_S, obtaining relations among expectation values:[HS+Thiemann]

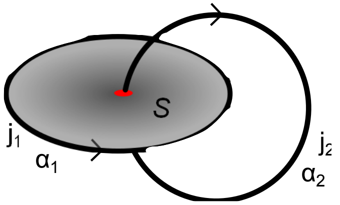
$$\begin{aligned}\langle L\,\mathrm{tr}_\rho(h_{\partial_S})\rangle &= \int_{\overline{\mathcal{A}}} L[A]\,\mathrm{tr}_\rho(h_{\partial_S})[A]e^{iS_{\mathrm{CS}}[A]}\,\mathrm{d}\mu[A] \\ &= \int_{\overline{\mathcal{A}}} L[A]\,\mathrm{tr}_\rho(W_S)e^{iS_{\mathrm{CS}}[A]}\,\mathrm{d}\mu[A] \\ &= \int_{\overline{\mathcal{A}}} (\mathrm{tr}_\rho(W_{-S})L)[A]e^{iS_{\mathrm{CS}}[A]}\,\mathrm{d}\mu[A] \\ &= \langle(\mathrm{tr}_\rho(W_{-S})L)\rangle.\end{aligned}$$

Enough to define functional in some cases. Note: Choice of S introduces framing.

\

Using Seifert surfaces for the W_S one can calculate some expectation values. For example for the Hopf link:

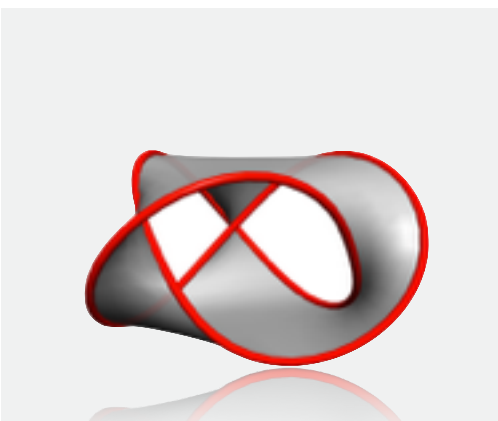
$$\langle H^+(j_1,j_2)\rangle_{\mathrm{CS}} = \lambda_{j_1 j_2}\lambda_{j_2 0} = \frac{\sin\left[\frac{\pi}{k}(2j_1+1)(2j_2+1)\right]}{\sin\left[\frac{\pi}{k}\right]}$$



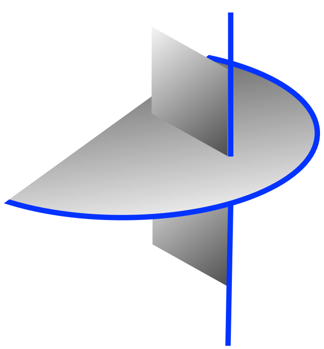
reproducing the known values for Kauffman bracket and Jones polynomial and their generalization.

As far as we know: First QFT calculation of such without using CFT

Nontrivial knots?

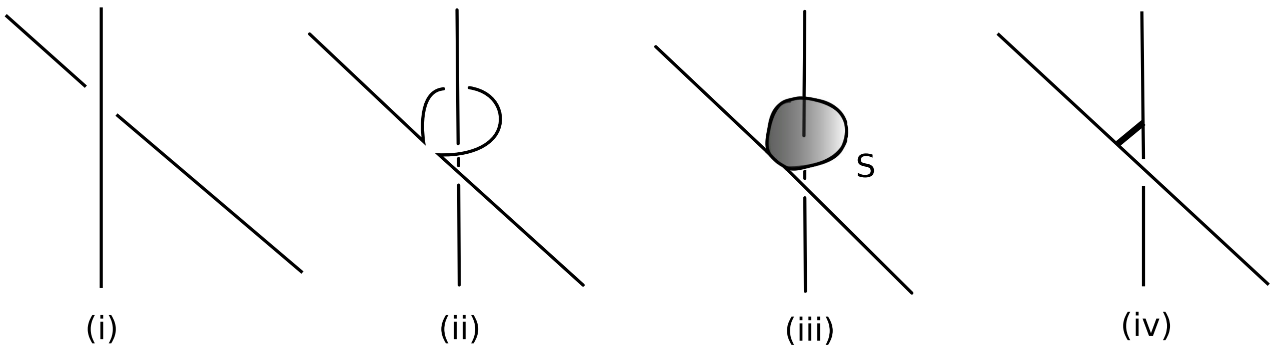


Seifert surfaces are not simply connected so Stokes does not work.



Contraction discs: Simply connected but have self intersections. Definition of W_S?

Skein relations: Idea



Need: Quantization of the exponential map

$$Q_D\left[\exp\left(-\frac{8\pi i}{k}\kappa^{ij}E_i\,(T_j)^A{}_D\right)\right]^C{}_B$$

Surprisingly hard to determine.
We have a result (Sahlmann+Zilker arXiv) but we do not like it.

To take home:

- Quantum field theory without background metric
- Canonical and covariant approach
- space-time geometry from combinatorics, representation theory

Understanding beginning to emerge about

- quantum geometry
- black holes, cosmology

Many open questions

- interpretation of transition amplitudes/solutions of constraints
- divergencies in the spinfoam sum
- …

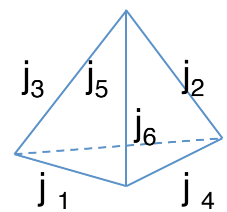
3d gravity

successfully approached with canonical and covariant LQG methods.

Euclidean works best but Lorentzian also possible. [Ponzano+Regge `68, many others later]

Spin foam picture:

$$Z = \sum_{\{f\}\to j} \prod_f (2j_f + 1) \prod_v$$



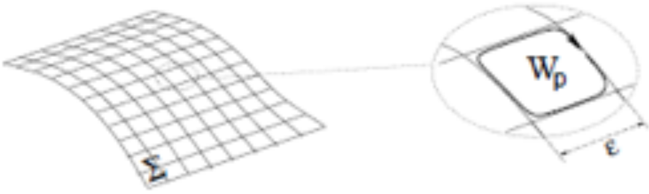
- Divergencies understood (twisted betty number) [Smerlak+Bonzom ‘10]
- SU(2) --> $SU_q(2)$ gives cosmological constant, [Turaev+Viro, Mizoguchi +Tada]
- Equivalence to CS treatment can be shown [Freidel+Louapre]
- Inclusion of particle “Feynman diagrams” gives effective NC field theory

$$S_{\text{eff}} = \int d^3x \left[(\partial\phi \star \partial\phi)(x) - \frac{1}{2}\frac{\sin^2 m\kappa}{\kappa^2}(\phi \star \phi)(x) + \frac{\lambda}{3!}(\phi \star \phi \star \phi)(x) \right]$$ [Freidel+Livine]

Canonical approach:

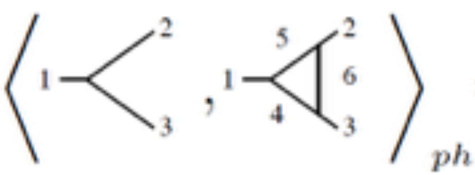
Hamilton constraint imposes flatness.

$$C = \prod_{\alpha} \delta(W_j(\alpha))$$

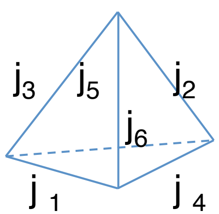


from Perez 06

Can recover spin foam picture. In particular, for physical inner product [Perez+Noui]



$$= \sqrt{\Delta_4\Delta_5\Delta_6}\begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{Bmatrix} \times \text{rest}$$



Connections to other canonical approaches can be made.

Loop quantum cosmology

[Bojowald,...]

Symmetric sector of gravity as testbed for LQG

- What happens at the big bang singularity?
- Can GR be reproduced far away from the singularity?

Some aspects *derivable* from full theory.

Example: FRW with massless scalar

[Ashtekar+Pawlowski+Singh]

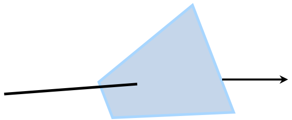
$$M = \mathbb{R} \times \mathbb{R}^3, \qquad \mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

Canonical variables

$$A^I_a = c\,\omega^I_a, \qquad E^a_I = p\,\sqrt{q_0}\,{}^0e^a_I$$

Quantization in terms of

- holonomies $\widehat{=} \exp(ilc)$
- Fluxes $\widehat{=} p$



Gravity Hilbert space spanned by $|l\rangle, \qquad l \in \mathbb{R}$:

$$\langle l|l'\rangle = \delta_{l,l'} = \begin{cases} 1 & \text{if } l = l' \\ 0 & \text{otherwise} \end{cases}$$

Representation

$$p|l\rangle = \frac{8\pi\beta l_{\mathrm{P}}^2}{3}, \qquad \exp(il'c)|l\rangle = |l+l'\rangle$$

Scalar field rep for ϕ, π_ϕ is standard.

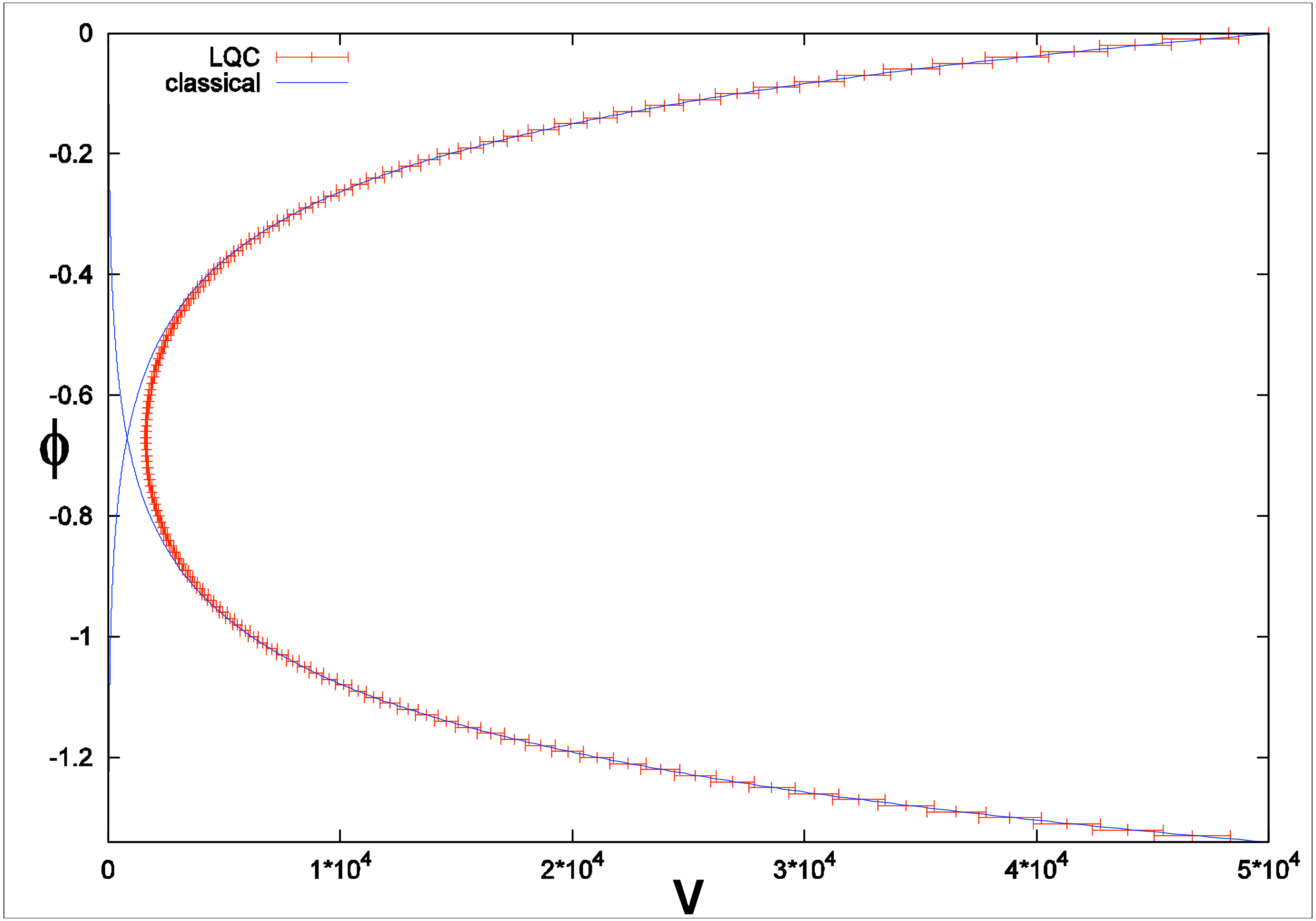
Gravity part of constraint with LQG methods:

$$\widehat{C}\,|l\rangle = f_0(l)|l\rangle + f_+(l)|l+\delta_+(l)\rangle + f_-(l)|l-\delta_-(l)\rangle$$

Constraint equation becomes

$$\partial_\phi^2 \Psi(\phi, l) = \widehat{C}\Psi(\phi, l)$$

scalar = time, physical Hilbert space = positive frequency space



Ashtekar, Pawlowski and Singh, Phys. Rev. Lett. 96 (2006)