Gravity, Fermions and Spin Base Invariance

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Phys. Rev. D **89**, no. 6, 064040 (2014)
Phys. Lett. B **743**, 415 (2015)
Phys. Rev. D **91**, no. 10, 104006 (2015)

November 6, 2015

Research Training Group
Quantum and Gravitational Fields

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Quantum and Gravitational Fields

Motivation

- aim at a theory of quantized matter and quantized forces
- matter is composed of fermions $\Rightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi}$ ✓
- SM forces mediated via gauge fields $\Rightarrow \mathcal{D}\mathcal{A}_{\mu}{}^{\mathrm{a}}$ ✓
- gravity encoded in spacetime curvature $\Rightarrow \mathcal{D}g_{\mu\nu}$, $\mathcal{D}e_{\mu}{}^{a}$, $\mathcal{D}?$ ✗

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$\Rightarrow$ need to describe fermions in curved spacetime

$\Rightarrow$ vielbein formalism is mandatory [Weyl '29, DeWitt '65]

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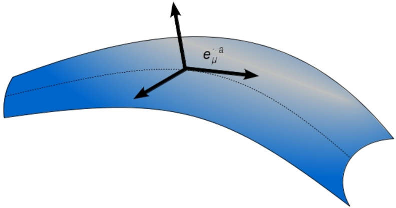
Hidden Spin Base Invariance:



- assume a vielbein $e_\mu{}^a$ with local $\mathrm{SO}(d-1,1)$ symmetry
- independence of the tangential spaces of every point of the manifold $\Rightarrow$ reason for locality of $\mathrm{SO}(d-1,1)$
- covariant derivative: $D_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0$

- fermions need Dirac structure
  $\Rightarrow$ Clifford algebra in tangential space:
  $$\{\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}\} = 2\eta_{ab}\mathrm{I}$$
- independence of the tangential spaces
  $\Rightarrow$ local changes of the Dirac matrix representation:
  $$\gamma_{(\mathrm{f})a} = \gamma_{(\mathrm{f})a}(x) \qquad \text{(hidden spin-base invariance)}$$
- behavior under spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$:
  [Schrödinger '32, Bargmann '32, Pauli '36]
  $$\gamma_{(\mathrm{f})a} \to \mathcal{S}\gamma_{(\mathrm{f})a}\mathcal{S}^{-1}$$

- standard spin connection for constant $\gamma_{(\mathrm{f})_a}$

$$D_\mu {e_\nu}^a + {\omega_\mu}^a{}_b {e_\nu}^b = 0 \Leftrightarrow D_\mu {e_\nu}^a \gamma_{(\mathrm{f})_a} + [\hat{\Gamma}_\mu, {e_\mu}^a \gamma_{(\mathrm{f})_a}] = 0$$

where $\quad \hat{\Gamma}_\mu = \frac{1}{8} {\omega_\mu}^{ab} [\gamma_{(\mathrm{f})_a}, \gamma_{(\mathrm{f})_b}]$

- behavior of spin connection $\hat{\Gamma}_\mu$ under spin-base transformations

$$\hat{\Gamma}_\mu \to \mathcal{S} \hat{\Gamma}_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S}) \mathcal{S}^{-1}$$

- it seems we first need ${e_\mu}^a$ and then $\gamma_{(\mathrm{f})_a}$ together with $\mathcal{S}$ to calculate the spin connection $\hat{\Gamma}_\mu$ for an arbitrary spin-base

- BUT: $\hat{\Gamma}_\mu$ can be derived from

$$D_\mu \gamma_\nu + [\hat{\Gamma}_\mu, \gamma_\nu] = 0, \quad \mathrm{tr}\, \hat{\Gamma}_\mu = 0$$

for arbitrary spin-bases $\Rightarrow$ relevant objects are the γ_μ

- introduction of vielbein is artificial split of γ_μ into ${e_\mu}^a$ and ${\gamma_{(\mathrm{f})}}_a$

- Dirac matrices are more general
  Kofink '49, Brill and Wheeler '57, Unruh '74, Finster et al. '99, Casals et al. '13, Gies and Lippoldt '13, Christiansen et al. '15

$\Rightarrow$ formalize construction without introduction of vielbein

Spin Base Invariance

Dirac Structure:

- Clifford algebra (irreducible representation):

  $$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \quad \gamma_\mu \in \mathbb{C}^{d_\gamma \times d_\gamma}, \quad d_\gamma = 2^{\lfloor d/2 \rfloor}$$

- Dirac conjugation with spin metric h:

  $$\bar{\psi} = \psi^\dagger h$$

- covariant derivative:

  $$\nabla_\mu \psi = \partial_\mu \psi + \hat{\Gamma}_\mu \psi$$

- perform coordinate transformation $\gamma_\mu \to \gamma'_\mu$

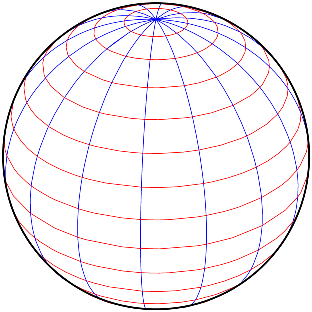
  $\{\gamma'_\mu, \gamma'_\nu\} = 2g'_{\mu\nu}\mathrm{I} = 2\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}g_{\rho\lambda}\mathrm{I} = \{\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \frac{\partial x^\lambda}{\partial x'^\nu}\gamma_\lambda\}$

  $\Rightarrow$ most general solution $\gamma'_\mu = \pm\frac{\partial x^\rho}{\partial x'^\mu}\mathcal{S}\gamma_\rho\mathcal{S}^{-1}$, $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$

- have two independent coordinate transformations
  - spacetime coordinate transformations:
    $\gamma_\mu \to \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \quad \psi \to \psi' = \psi, \quad h \to h' = h$
  - spin base transformations:
    $\gamma_\mu \to \gamma'_\mu = \pm\mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad h \to \pm(\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}$

Global Spin Base on the 2-sphere

- spherical coordinates (ϑ, φ)
- metric $(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \vartheta \end{pmatrix}$
- zweibein $({e_\mu}^a) = \begin{pmatrix} 1 & 0 \\ 0 & \sin \vartheta \end{pmatrix}$



- zweibein spin-connection: $\omega_1{}^{12} = 0, \quad \omega_2{}^{12} = -\cos\vartheta$
- flat Dirac matrices: $(\gamma_{(\mathrm{f})a}) = \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}$
- eigenfunctions $\psi^{(s)}_{\pm,n,l}$ of the Dirac operator read
  Camporesi and Higuchi '96

$$\psi^{(-)}_{\pm,n,l}(\vartheta,\varphi) = \frac{c_2(n,l)}{\sqrt{4\pi}} \mathrm{e}^{-\mathrm{i}(l+\frac{1}{2})\varphi} \begin{pmatrix} \Phi_{n,l}(\vartheta) \\ \pm\mathrm{i}(-1)^{n-l}\Phi_{n,l}(\pi-\vartheta) \end{pmatrix}$$

$$\psi^{(+)}_{\pm,n,l}(\vartheta,\varphi) = \frac{c_2(n,l)}{\sqrt{4\pi}} \mathrm{e}^{\mathrm{i}(l+\frac{1}{2})\varphi} \begin{pmatrix} \mathrm{i}(-1)^{n-l}\Phi_{n,l}(\pi-\vartheta) \\ \pm\Phi_{n,l}(\vartheta) \end{pmatrix}$$

the eigenfunctions have *strange* properties

$$\psi^{(s)}_{\pm,n,l}(\vartheta, \varphi + 2\pi) = -\psi^{(s)}_{\pm,n,l}(\vartheta, \varphi)$$

$$\psi^{(s)}_{\pm,n,l=0}(\vartheta = 0, \varphi) \sim \mathrm{e}^{s\mathrm{i}\frac{\varphi}{2}} \begin{pmatrix} 1 - s \\ \pm(1 + s) \end{pmatrix}$$

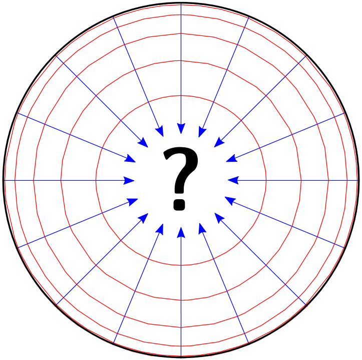
$$\psi^{(s)}_{\pm,n,l=0}(\vartheta = \pi, \varphi) \sim \mathrm{e}^{s\mathrm{i}\frac{\varphi}{2}} \begin{pmatrix} 1 + s \\ \pm(1 - s) \end{pmatrix}$$

$\Rightarrow$ **Why is that?**

- spherical coordinates are ill defined at the poles!
- need at least two patches in position space to cover the whole 2-sphere
- perform coordinate transformation

  $\Rightarrow$ zweibein is ill defined at the poles (hairy ball theorem)

  $\Rightarrow$ zweibein spin connection is singular at the poles



- cure problems by introducing globally well defined spin base
  $$\gamma_\mu = \mathcal{S} e_\mu{}^a \gamma_{(\mathsf{f})_a} \mathcal{S}^{-1}, \text{ with } \mathcal{S} = \mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}\sigma_3} \mathrm{e}^{-\mathrm{i}\frac{\vartheta-\pi}{2}\sigma_1}$$
- circumvent hairy ball theorem by suitable distribution of zeros of components $(\gamma_\mu)^I{}_J$
- spin connection $\hat{\Gamma}_\mu = \frac{\mathrm{i}}{2}\gamma_\mu$ is well defined

- globally well defined eigenfunctions of the Dirac operator

$$\hat{\psi}^{(s)}_{\pm,n,l} = \mathcal{S}\psi^{(s)}_{\pm,n,l}$$

$$\hat{\psi}^{(s)}_{\pm,n,l}(\vartheta, \varphi + 2\pi) = \hat{\psi}^{(s)}_{\pm,n,l}(\vartheta, \varphi)$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\vartheta = 0, \varphi) \sim \begin{pmatrix} \pm(1+s) \\ 1-s \end{pmatrix}$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\vartheta = \pi, \varphi) \sim \begin{pmatrix} 1+s \\ \pm(1-s) \end{pmatrix}$$

Path Integral

- can construct action S if γ^μ are known
- naive way: $\int\mathcal{D}\gamma\, \mathrm{e}^{\mathrm{i}S}$ (and fermions, gauge fields, ...)
- but the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$ prohibits arbitrary variations of γ_μ
  $\Rightarrow$ determine degrees of freedom from Clifford algebra

Weldon theorem [Weldon '01] ($d = 4$), [Lippoldt '15] ($d \geq 2$)

$$\delta\gamma_\mu = \tfrac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \quad \mathrm{tr}\,\delta\mathcal{S}_\gamma = 0$$

$\delta g_{\mu\nu}$: metric fluctuations

$\delta\mathcal{S}_\gamma$: spin-base fluctuations, $\mathrm{SL}(d_\gamma, \mathbb{C})$

want to show that the $\mathcal{S}_\gamma$ part factorizes

ingredients for proof:

- action as a functional of the fermions ψ, $\bar{\psi}$, the metric $g_{\mu\nu}$ and the spin-base $\mathcal{S}_\gamma$: $S = S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$
- spin-base invariance of S:
  $S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma] \to S[\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \mathcal{S}'_\gamma] \equiv S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]$
- spin-base invariance of the measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$:
  $\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}(\mathcal{S}\psi)\mathcal{D}(\bar{\psi}\mathcal{S}^{-1}) = \mathcal{D}\psi\mathcal{D}\bar{\psi}$

- study expectation value of an operator $\hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma)$, which is a scalar under spin-base transformations

- consider integration over fermions and metric

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle = \int \mathcal{D}g \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \mathrm{e}^{\mathrm{i}S[\psi, \bar{\psi}, g; \mathcal{S}_\gamma]}$$

- with spin-base invariance we find

$$\left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}_\gamma) \right\rangle \equiv \left\langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{S}'_\gamma) \right\rangle$$

$\Rightarrow$ integration over $\mathrm{SL}(d_\gamma, \mathbb{C})$ is trivial

summary

- impose full nontrivial symmetry of Clifford algebra
  $\Rightarrow$ spinbase transformations: $\mathrm{SL}(d_\gamma, \mathbb{C})$
- spin metric h and canonical part of the spin connection $\hat{\Gamma}_\mu$ are determined by γ_μ
- existence of a global spin base on the 2-sphere
- purely metric based path integral in presence of fermions

Thank you for your attention!