Operator Product Expansion

S. Hollands

UNIVERSITÄT LEIPZIG

based on joint work with J. Holland and Ch. Kopper

SIFT Jena
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Introduction

Operator Product Expansion [Wilson '69]

Products of composite fields can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \underbrace{\ldots}_{\text{Spectators}} \rangle \sim \sum_B \underbrace{\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \ldots \rangle$$

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  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$

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- Asymptotic short distance expansion:
  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$
- Practical application e.g. in deep-inelastic scattering
- Plays fundamental role in conformal field theory
  (Conformal bootstrap, "Vertex operator algebras", ...)
- Plays fundamental role in QFTCST
  (State-independent definition of QFT!)

Topics of today's talk:

1. In what sense does the OPE converge? N-point functions $\leftrightarrow$ 1-point functions & OPE coefficients
2. What are algebraic relations between OPE coefficients? Vertex algebras in d-dims.
3. A novel recursion scheme for OPE coefficients New self-consistent construction method

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Model: Perturbative, Euclidean φ^4_4-theory

The model: Euclidean φ^4-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1)\ldots\mathcal{O}_{A_N}(x_N)\rangle := \mathcal{N}\int \mathcal{D}\varphi \, \exp\left[-S\right] \, \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,,$$

where the action is given by

$$S(\varphi) := \int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu\varphi)^2(x) + \frac{m^2}{2}\varphi^2(x) + g\varphi(x)^4 - \text{counterterms}\right)$$

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- We use a "renormalization group flow equation" approach [Wilson, Polchinski, Kopper-Keller-Salmhofer, Wetterich]

Outline

1. OPE factorisation

2. OPE convergence

3. Recursive construction of OPE

Outline

The OPE factorises

Theorem (Holland-SH)

At any arbitrary but fixed loop order:

$$\mathcal{C}_{A_1\dots A_N}^{B}(x_1,\dots,x_N)=\sum_{C}\mathcal{C}_{A_1\dots A_M}^{C}(x_1,\dots,x_M)\mathcal{C}_{C A_{M+1}\dots A_N}^{B}(x_M,\dots,x_N)$$

holds on the domain $\frac{\max_{1\leq i\leq M}|x_i-x_M|}{\min_{M<j\leq N}|x_j-x_M|}<1$. *(Sum over* C *absolutely convergent !)*

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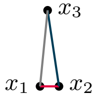
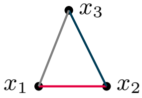
For $N = 3$: $\xi = \frac{|x_1 - x_2|}{|x_2 - x_3|} < 1$



for $\xi \ll 1$



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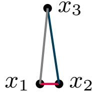

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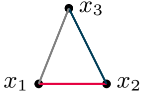

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This shows associativity really holds!

- Vertex Algebras (Borcherds property) also in $4d$.
- $\mathcal{C}^{B}_{A_1\ldots A_N}$ uniquely determined in terms of $\mathcal{C}^{B}_{A_1 A_2}$
- "Bootstrap construction" of OPE coefficients possible

Quantitative bound

Theorem

Up to any perturbation order $r \in \mathbb{N}$ the bound

$$\left|\textit{Remainder in associativity}\right| \leq K \left(\frac{D}{\varepsilon}\right)^{8^{r+1}(\sum_{i=1}^{N}[A_i]+[B])} \left[(1+\varepsilon)^{8^{r+1}}\xi\right]^D \frac{\max\limits_{1\leq i\leq N}(\frac{1}{m}, |x_i - x_N|)^{[B]+1}}{\min\limits_{1\leq i<j\leq N}|x_i - x_j|^{\sum_j [A_j]+1}}$$

holds for any sufficiently small ε, where

$$\xi := \frac{\max_{1\leq i\leq M}|x_i - x_M|}{\min_{M<j\leq N}|x_j - x_M|}$$

and where K is a constant which does not depend on D.

Outline

Bound on OPE remainder I

Theorem (Holland-Kopper-SH)

At any perturbation order r and for any $D \in \mathbb{N}$,

$$\Bigg|\overbrace{\Big\langle \Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^{B}_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big) \underbrace{\hat{\varphi}(p_1)\cdots\hat{\varphi}(p_n)}_{\text{Spectator fields}} \Big\rangle}^{\text{OPE-Remainder}}\Bigg|$$

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$$\leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N}|x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

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- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu,\varepsilon)$, where $\varepsilon = \min_{I\subset\{1,\dots,n\}} |\sum_I p_i|$
  ε: distance of $(p_1,\dots,p_n)$ to "exceptional" configurations

Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \; \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

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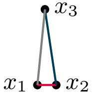
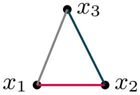
   - ratio of max. and min. distances is large, e.g. for $N = 3$



Slow convergence



Fast convergence

Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

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M: mass for $m > 0$ or renormalization scale μ for massless fields

$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2+M^2)^s\hat{f}(p)|$ (Schwartz norm)

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1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$:

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$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2+M^2)^s\hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$:

Bound on OPE remainder II

Consider now smeared spectator fields $\varphi(f_i) = \int f_i(x)\varphi(x)\,\mathrm{d}^4x$.

Theorem (Holland-Kopper-SH)

At any perturbation order r *and for any* $D \in \mathbb{N}$*, there exists a* $K > 0$ *such that*

$$\Big|\Big\langle\Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\leq D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big)\varphi(f_1)\cdots\varphi(f_n)\Big\rangle\Big|$$

$$\leq \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM\max_{1\leq i\leq N}|x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\ \sup\left(1, \frac{|P|}{M}\right)^{(D+2)(r+5)}$$

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   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$: Bound vanishes as $D \to \infty$
   $\Rightarrow$ OPE converges at any finite distances!

Outline

1. OPE factorisation
2. OPE convergence
3. Recursive construction of OPE

Motivation for a new construction method

Textbook method (roughly):

- Write down correlation function with operator insertions
- Perform short distance/large momentum expansion (in some clever way)
- Argue that the coefficients obtained this way are state independent

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- Argue that the coefficients obtained this way are state independent

Not entirely satisfying:

- Relies on correlation functions $\Rightarrow$ OPE not 'fundamental'
- State independence not obvious
- Hard to study general properties of OPE

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

Coupling constant derivatives of OPE coefficients in $g\varphi^4$-theory can be expressed as

$$\partial_g \mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N) = -\int \mathrm{d}^4 y \Big[\mathcal{C}^{B}_{\varphi^4 A_1 \dots A_N}(y, x_1, \dots, x_N) - \sum_{i=1}^{N} \sum_{[C] \le [A_i]} \mathcal{C}^C_{\varphi^4 A_i}(y, x_i)\, \mathcal{C}^B_{A_1 \dots \widehat{A_i}\, C \dots A_N}(x_1, \dots, x_N) - \sum_{[C] < [B]} \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N)\, \mathcal{C}^B_{\varphi^4 C}(y, x_N) \Big] .$$

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- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

OPE coefficients at perturbation order $(r+1)$ can be expressed as

$$(\mathcal{C}_{r+1})^{B}_{A_1\dots A_N}(x_1,\dots,x_N) = -\int \mathrm{d}^4 y \Bigg[(\mathcal{C}_r)^{B}_{\varphi^4 A_1\dots A_N}(y,x_1,\dots,x_N) - \sum_{s=0}^{r}\sum_{i=1}^{N}\sum_{[C]\leq[A_i]} (\mathcal{C}_s)^{C}_{\varphi^4 A_i}(y,x_i)\,(\mathcal{C}_{r-s})^{B}_{A_1\dots\widehat{A_i}\,C\dots A_N}(x_1,\dots,x_N) - \sum_{s=0}^{r}\sum_{[C]<[B]} (\mathcal{C}_s)^{C}_{A_1\dots A_N}(x_1,\dots,x_N)\,(\mathcal{C}_{r-s})^{B}_{\varphi^4 C}(y,x_N)\Bigg].$$

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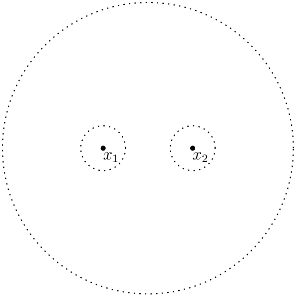
- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.
- State independence obvious.
  No other objects enter the construction.
- The formula depends on the renormalisation conditions.
  (Here BPHZ)

Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$



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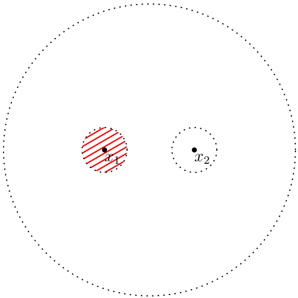
UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]=0}^{\infty} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

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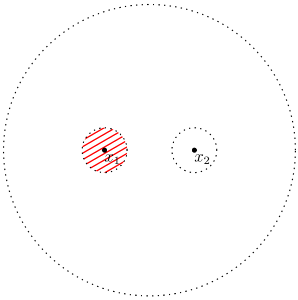

Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4y \Big[\sum_{[C]>[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y,x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1,x_2)$$

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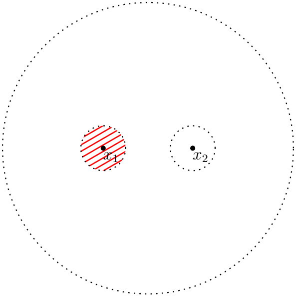

Built-in renormalisation (Example: $N = 2$)

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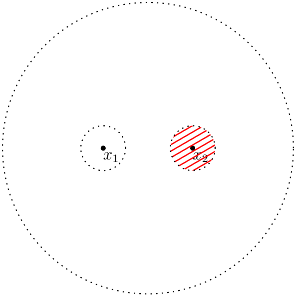
UV-region II ($y \approx x_2$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]>[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

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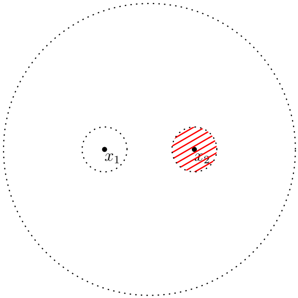
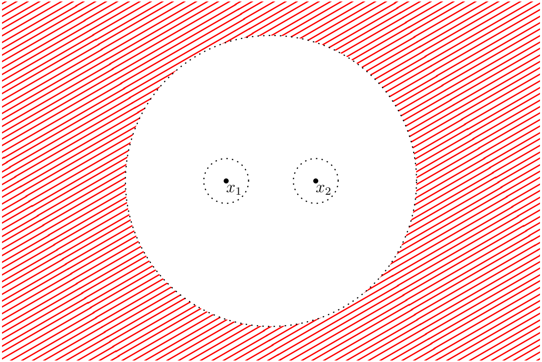


Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]\geq[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) \Big]$$

IR-region ($|y - x_2| \gg |x_1 - x_2|$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N=2$)

$$\int \mathrm{d}^4y \Big[\sum_{[C]\geq[B]} \mathcal{C}_{A_1A_2}^{C}(x_1,x_2)\,\mathcal{C}_{\varphi^4 C}^{B}(y,x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y,x_1)\,\mathcal{C}_{CA_2}^{B}(x_1,x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y,x_2)\,\mathcal{C}_{A_1 C}^{B}(x_1,x_2)\Big]$$

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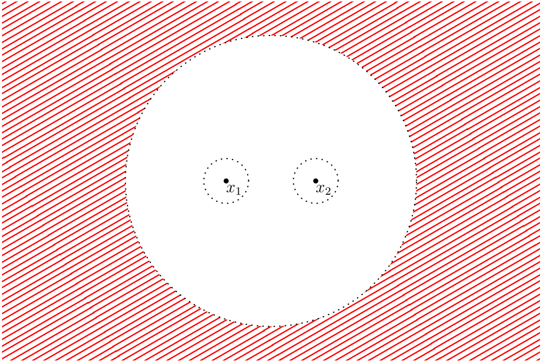
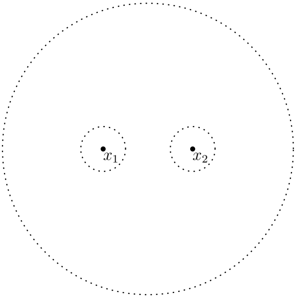


Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\mathcal{C}^{B}_{\varphi^4 A_1 A_2}(y, x_1, x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}^{C}_{\varphi^4 A_1}(y, x_1)\, \mathcal{C}^{B}_{C A_2}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}^{C}_{\varphi^4 A_2}(y, x_2)\, \mathcal{C}^{B}_{A_1 C}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}^{C}_{A_1 A_2}(x_1, x_2)\, \mathcal{C}^{B}_{\varphi^4 C}(y, x_2) \Big]$$

The integral is absolutely convergent due to the factorisation property.



Conclusions & Outlook

In Euclidean perturbation theory, we found that:

1. The OPE converges at finite distances.
2. The OPE factorises (associativity).
3. The OPE satisfies a recursion formula.

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Possible Generalisations

- Gauge theories (in progress)
- Curved manifolds
- Minkowski space
- ...

Applications of the Recursion Formula

- Does the algorithm facilitate computations?
- Does the perturbation series for OPE coefficients converge?