Regge - Wheeler Lattice Theory of Gravity

“Strongly-Interacting Field Theories”

General reference: “Quantum Gravitation” (Springer 2009), ch. 4 & 6

FSU Jena, Nov. 5 2015

[with R.M. Williams and R. Toriumi]

Why the Lattice?

- *Discretization/regularization of the Feynman P.I.*
- *Starts from a manifestly covariant formulation*
- *No need for gauge fixing (as in Lattice QCD)*
- *Dominant paths are nowhere differentiable*
- *Allows for non-perturbative calculations*
- *Extensively tested in QCD & Spin Systems*
- *30 years experience / high accuracy possible*

Dominant Paths are Nowhere Differentiable

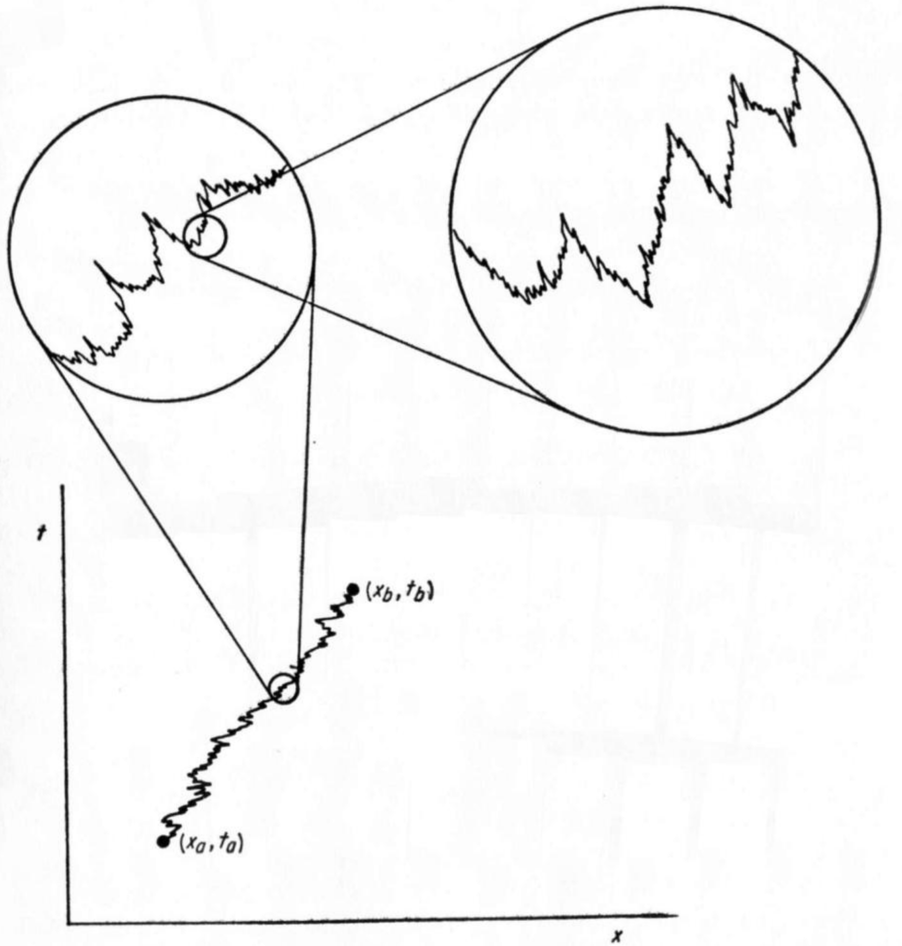


Fig. 7-1 Typical paths of a quantum-mechanical particle are highly irregular on a fine scale, as shown in the sketch. Thus, although a mean velocity can be defined, no mean-square velocity exists at any point. In other words, the paths are nondifferentiable.

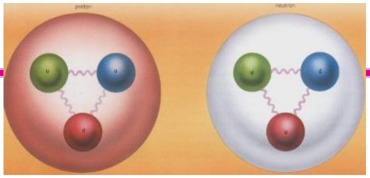
R. P. Feynman
Tolman Professor of Physics
California Institute of Technology

A. R. Hibbs
Jet Propulsion Laboratory
California Institute of Technology

McGraw-Hill Book Company

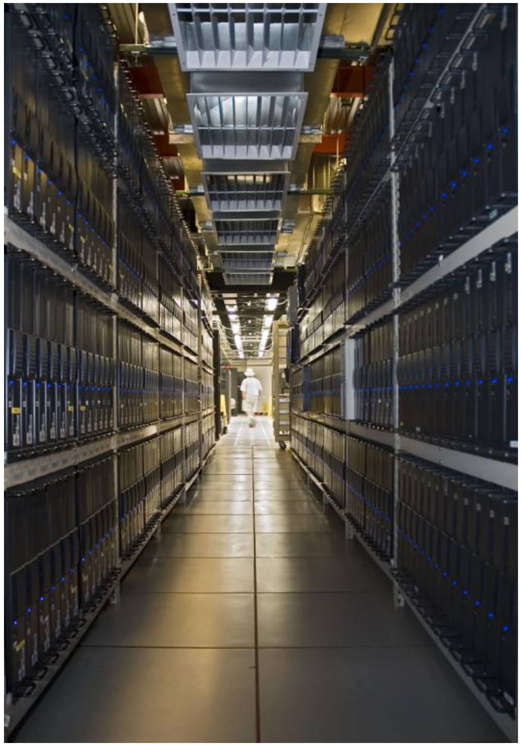
Prototype: Wilson's Lattice QCD

- *In QCD Pert. Th. is next to useless at low energies*
  - $\Rightarrow$ *Non-perturbative regularization*
- *Clear correspondence betw. Lattice and Cont. ops.*
- *Nontrivial measure (Haar)*
- *Confinement is almost immediate (Area law)*
- *Physical Vacuum bears little resemblance to pert. Vacuum*
- *Nontrivial Spectrum (glueballs) / Vacuum chromo-electric condensate / Quark condensates*



QCD is Hard. Very Hard.

Big Supercomputers.



Fermilab LQCD Cluster

Lattice Gauge Theory *Works*

Running of α strong :

$$\alpha_S(\mu) = \frac{4\pi}{\beta_0 \ln \mu^2/\Lambda^2_{\overline{MS}}} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln\left[\ln \mu^2/\Lambda^2_{\overline{MS}}\right]}{\ln \mu^2/\Lambda^2_{\overline{MS}}} + \ldots \right]$$

Wilson's lattice gauge theory provides to this day the only convincing theoretical evidence for : confinement and chiral symmetry breaking in QCD.

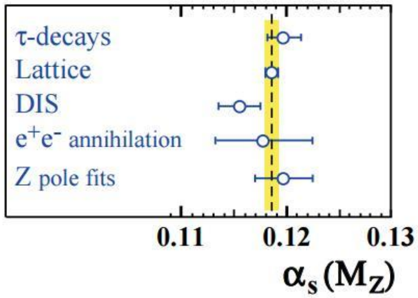
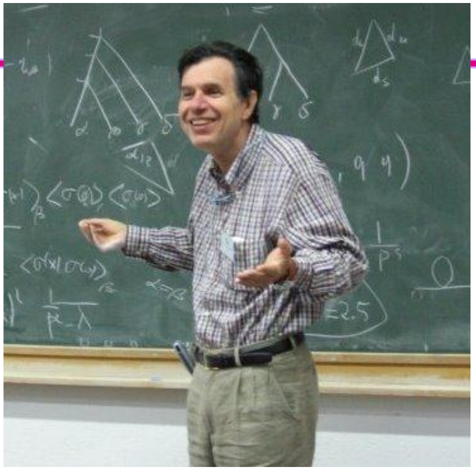


Figure 9.3: Summary of values of $\alpha_s(M_Z^2)$ obtained for various sub-classes of measurements (see Fig. 9.2 (a) to (d)). The new world average value of $\alpha_s(M_Z^2) = 0.1185 \pm 0.0006$ is indicated by the dashed line and the shaded band.

[Particle Data Group LBL, 2015]

"Non-Renormalizability"



Nuclear Physics B100 (1975) 368--388

THE THEORY OF NON-RENORMALIZABLE INTERACTIONS

The large N expansion

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Received 23 June 1975

A particular class of non-renormalizable interactions is studied in the infinite cut-off limit. In this paper we consider the quadrilinear interaction of an N-component field; the Lagrangian is invariant under the action of the $O(N)$ group. The Green functions are expanded in powers of $1/N$; we prove that this expansion is finite and renormalizable at all orders in not too high dimensions, the outputs are not C^∞ in the coupling constant around the origin: this property explains why divergences are present in the standard perturbative expansion. The interactions of both spin-zero and spin-$\frac{1}{2}$ fields have been studied: peculiar problems arise in the case of a current-current interaction.

1. Introduction

In quantum field theory the interactions are traditionally classified as superrenormalizable, renormalizable and non-renormalizable. In the first two cases the perturbation expansion in the coupling constant has been constructed: all divergences disappear after renormalization [1]. Existence theorems have been proved for particular supernormalizable interactions, e.g. $\lambda\phi^{2N}$ in 2 dimensions and $\lambda\phi^4$ in 3 di-

Gravity in 2+ε Dimensions

Wilson 1973
Weinberg 1977 …
Kawai, Ninomiya 1995
Kitazawa, Aida 1998

Wilson's double expansion … Formulate theory in 2+ε dimensions.

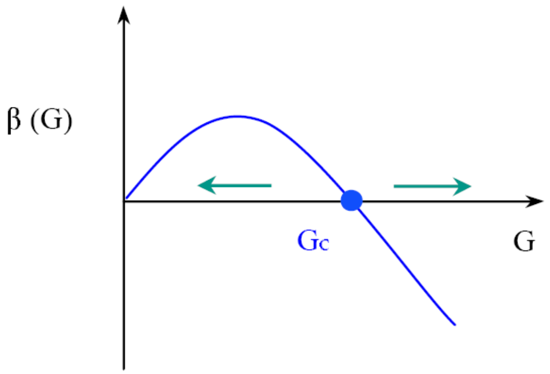
G is dim-less, so theory is now *perturbatively renormalizable*

$$\beta(G) = (d-2)\,G - \frac{2}{3}\,(25-n_s)\,G^2 - \frac{20}{3}\,(25-n_s)\,G^3 + \dots$$

(pure gravity : $n_s = 0$)

with a non-trivial UV fixed point :

$$\begin{cases} G_c = \dfrac{3}{2\,(25-n_s)}\,(d-2) - \dfrac{45}{2(25-n_s)^2}\,(d-2)^2 + \dots \\ \nu^{-1} = -\beta'(G_c) = (d-2) + \dfrac{15}{25-n_s}\,(d-2)^2 + \dots \end{cases}$$

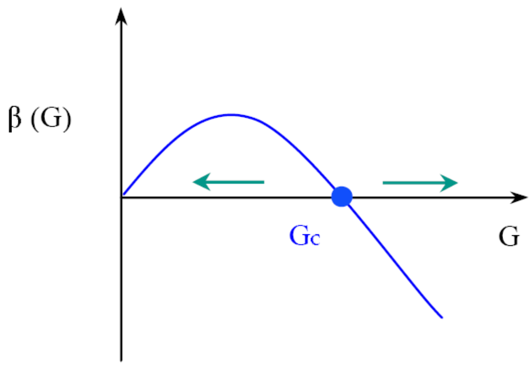


… suggests the existence of *two* phases

2+ε Cont'd

Running of Newton's G(k) in 2+ε is of the form:

$$G(k^2) \simeq G_0 \left[1 \pm c_0 \left(\frac{1}{\xi^2 \, k^2} \right)^{1/2\nu} + \ldots \right]$$



$$\nu^{-1} = -\beta'(G_c) = (d-2) + \frac{15}{25-n_s}(d-2)^2 + \ldots$$

Two key quantities : i) the universal exponent ν

ii) the new nonperturbative scale ξ

What is left of the above QFT scenario in 4 dimensions ?

Path Integral for Quantum Gravitation

$$\|\delta g\|^2 = \int d^d x \, \delta g_{\mu\nu}(x) \, G^{\mu\nu,\alpha\beta}(g(x)) \, \delta g_{\alpha\beta}(x)$$

DeWitt approach to measure : introduce a *Super-Metric G*

$$G^{\mu\nu,\alpha\beta}(g(x)) = \tfrac{1}{2}\sqrt{g(x)}\left[g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda\, g^{\mu\nu}(x)g^{\alpha\beta}(x)\right]$$

In d=4 this gives a “volume element” :

$$\int [d\, g_{\mu\nu}] = \boxed{\int \prod_x \prod_{\mu \geq \nu} d g_{\mu\nu}(x)} .$$

$$Z_{cont} = \int [d\, g_{\mu\nu}] \exp\left\{ -\lambda_0 \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g} R \right\}$$

Proper definition of F. Path Integral requires a *Lattice* (Feynman & Hibbs, 1964).

Perturbation theory in 4D about a flat background is *useless* … badly divergent

Conformal Instability

Euclidean Quantum Gravity - in the Path Integral approach - is affected by a fundamental instability, which cannot be removed.

The latter is apparently only overcome in the lattice theory (for G>Gc), because of the entropy (functional measure) contribution.

$$I_E = \lambda_0 \int dx \sqrt{g} - \frac{1}{16\pi G} \int dx \sqrt{g} R$$

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

Ω^2 (x) = conformal factor

$$I_E(\tilde{g}) = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \left(\Omega^2 R + 6\, g^{\mu\nu} \partial_\mu \Omega\, \partial_\nu \Omega \right) ,$$

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$$

$$\sqrt{g} (R - 2\lambda) = 6 g^{\mu\nu} \partial_\mu \Omega\, \partial_\nu \Omega - 2\lambda \Omega^4$$

Gibbons and Hawking PRD 15 1977;
Hawking, PRD 18 1978;
Gibbons , Hawking and Perry, NPB 1978.

Only One Coupling

Pure gravity path integral:

$$Z = \int [d\, g_{\mu\nu}]\, e^{-I_E[g]}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g}\, R$$

In the absence of matter, only one dim.-less coupling:

Rescale metric (edge lengths):

$$\tilde{G} \equiv G_0\, \lambda_0^{(d-2)/d}$$

$$g'_{\mu\nu} = \lambda_0^{2/d}\, g_{\mu\nu} \qquad g'^{\mu\nu} = \lambda_0^{-2/d}\, g^{\mu\nu}$$

… similar to g of Y.M.

$$I_E[g] = \Lambda^d \int dx \sqrt{g'} - \frac{1}{16\pi G_0\, \lambda_0^{\frac{d-2}{d}}} \Lambda^{d-2} \int dx \sqrt{g'}\, R'$$

Lattice Theory of Gravity

T. Regge 1961, J.A. Wheeler 1964

[MTW, ch. 42]

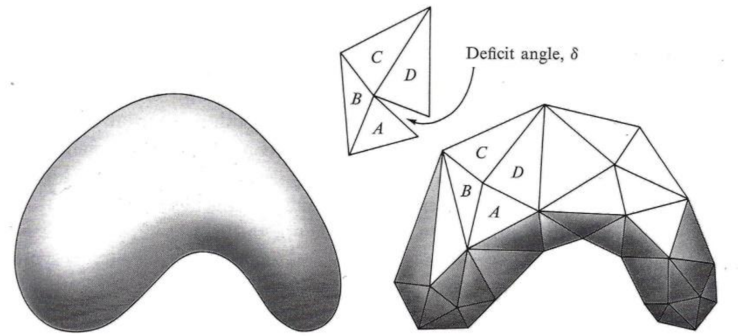
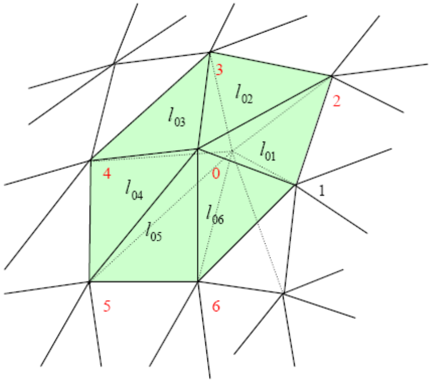


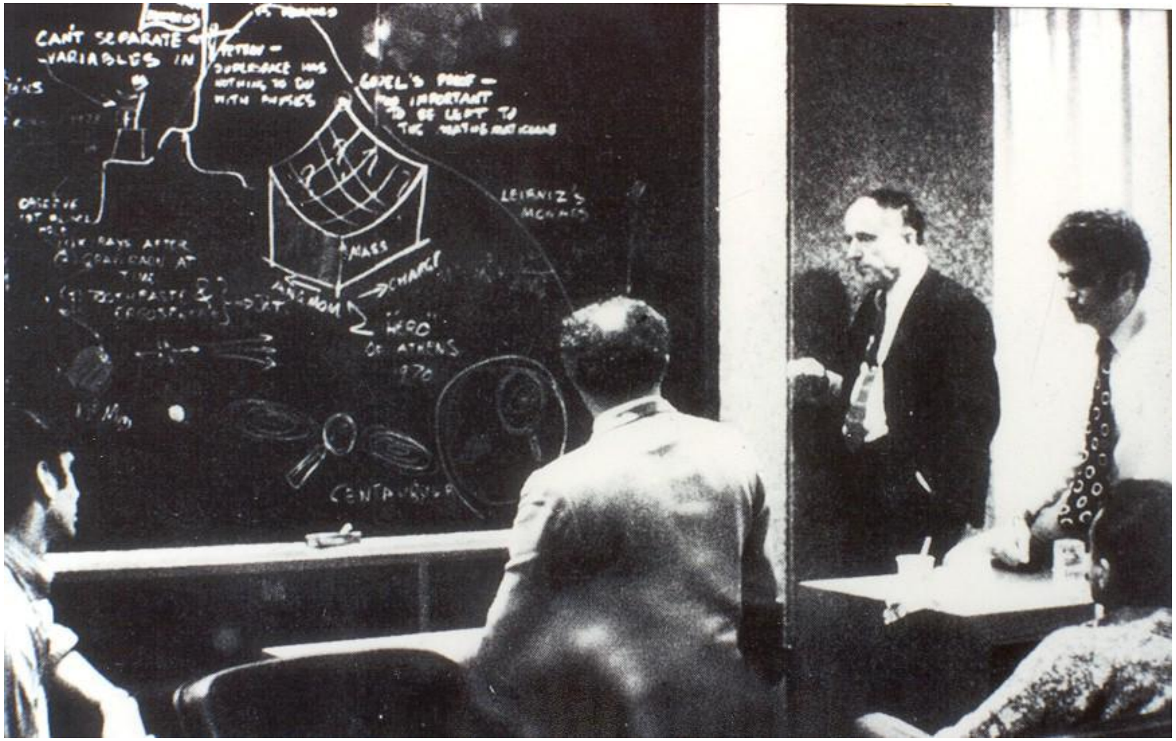
Figure 42.1.
A 2-geometry with continuously varying curvature can be approximated arbitrarily closely by a polyhedron built of triangles, provided only that the number of triangles is made sufficiently great and the size of each sufficiently small. The geometry in each triangle is Euclidean. The curvature of the surface shows up in the amount of deficit angle at each vertex (portion *ABCD* of polyhedron laid out above on a flat surface).

- Based on a *dynamical lattice*
- Incorporates *continuous local invariance*
- Puts within the *reach of computation* problems which in practical terms are beyond the power of analytical methods
- Affords in principle *any desired level of accuracy* by a sufficiently fine subdivision of space-time



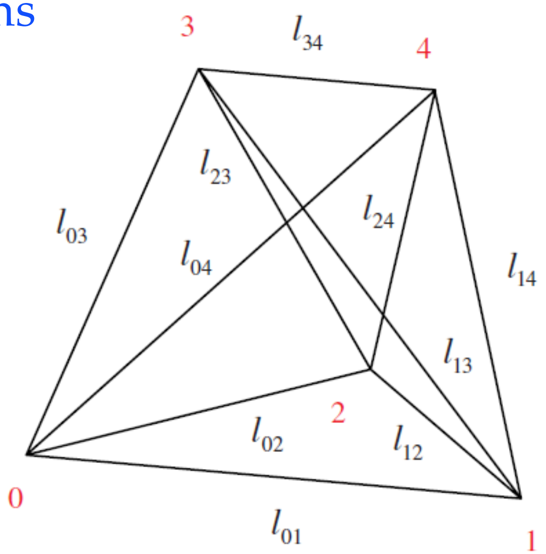
"Simplicial Quantum Gravity"

T. Regge, J.A. Wheeler and R. Ruffini, ca 1971



Elementary Building Block = *4-Simplex*

The *metric* (a key ingredient in GR) is defined in terms of the *edge lengths :*



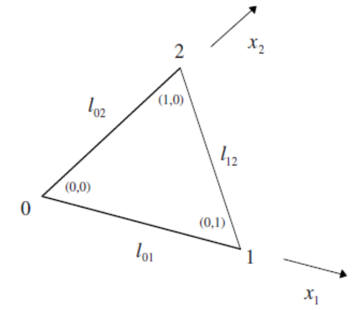
$$g_{ij}(s) = \frac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$$

The local *volume element* is obtained from a determinant :

$$V_n(s) = \frac{1}{n!}\sqrt{\det g_{ij}(s)}$$

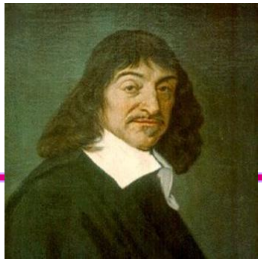
… or more directly in terms of the edge lengths :

$$V_n(s) = \frac{(-1)^{\frac{n+1}{2}}}{n!\,2^{n/2}} \begin{vmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & l_{01}^2 & \dots \\ 1 & l_{10}^2 & 0 & \dots \\ 1 & l_{20}^2 & l_{21}^2 & \dots \\ \dots & \dots & \dots & \dots \\ 1 & l_{n,0}^2 & l_{n,1}^2 & \dots \end{vmatrix}^{1/2}$$

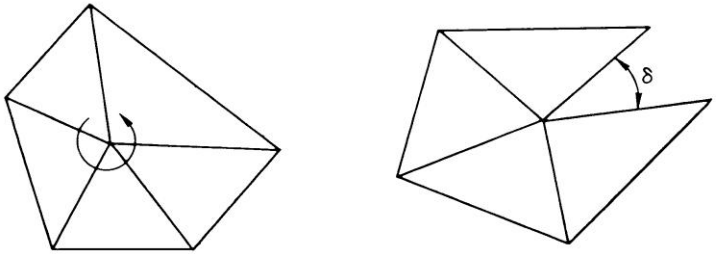


For more details see eg. Ch. 6 in “Quantum Gravitation” (Springer 2009), and refs therein.

Curvature - Described by Angles

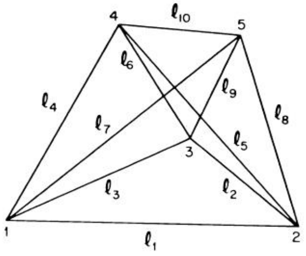


Edge lengths replace the Metric



$d = 2$

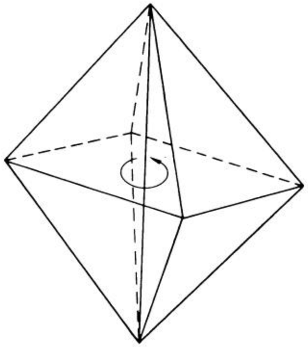
$$g_{ij}(s) = \tfrac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$$

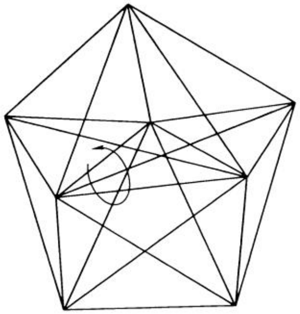


$$V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$\sin\theta_d = \frac{d}{d-1}\,\frac{V_d\, V_{d-2}}{V_{d-1}\, V'_{d-1}}$$

$$\delta_h = 2\pi - \sum_{\substack{\text{d-simplices} \\ \text{meeting on h}}} \theta_d$$





$d = 3$ $d = 4$

Curvature determined by edge lengths

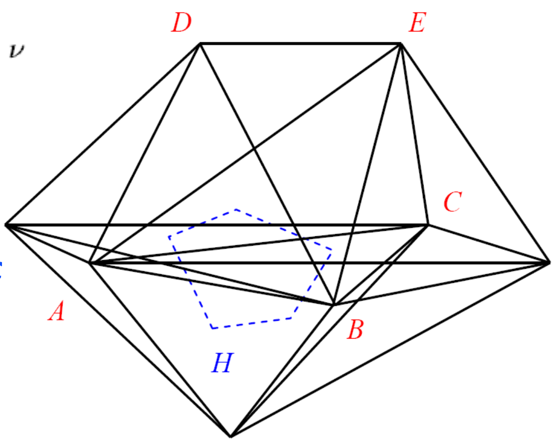
T. Regge 1961

J.A. Wheeler 1964

Rotations & Riemann tensor

$$\phi^{\mu}(s_{n+1}) = R^{\mu}{}_{\nu}(P)\,\phi^{\nu}(s_1) \qquad R^{\mu}{}_{\nu} = \left[P\, e^{\int_{\substack{\text{path}\\ \text{between simplices}}} \Gamma_\lambda dx^\lambda} \right]^{\mu}{}_{\nu}$$

$$\mathbf{R}(C) = \mathbf{R}(s_1, s_n) \cdots \mathbf{R}(s_2, s_1)$$



Elementary polygonal path around a hinge (triangle) in four dimensions.

Due to the hinge's intrinsic orientation, only components of the vector in the plane *perpendicular to the hinge* are rotated:

$$U_{\mu\nu}(h) = \mathcal{N}\, \epsilon_{\mu\nu\alpha_1 \alpha_{d-2}}\, l^{\alpha_1}_{(1)} \cdots l^{\alpha_{d-2}}_{(d-2)}$$

$$R^{\mu}{}_{\nu}(C) = \left(e^{\delta U} \right)^{\mu}{}_{\nu}$$

$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)}\, U_{\mu\nu}(h)\, U_{\lambda\sigma}(h)$$

$$R(h) = 2\,\frac{\delta(h)}{A_C(h)}$$ $\longrightarrow$ Regge Action

Exact lattice Bianchi identity (Regge)

$$\prod_{\substack{\text{hinges h}\\ \text{meeting on edge p}}} \left[e^{\delta(h) U(h)} \right]^{\mu}{}_{\nu} = 1$$

Curvature Squared Terms

- Riemann squared

$$U^{(h)}_{\mu\nu} = \frac{1}{2A_h}\,\epsilon_{\mu\nu\rho\sigma}\,l^{\rho}_{(a)} l^{\sigma}_{(b)}$$

$$\begin{aligned} R^{(i)}{}_{\mu\nu\rho\sigma} R^{(j)\mu\nu\rho\sigma} &\equiv \left[\frac{A\delta}{V} U_{\mu\nu} U_{\rho\sigma}\right]_{(i)} \left[\frac{A\delta}{V} U^{\mu\nu} U^{\rho\sigma}\right]_{(j)} \\ &= \frac{\delta_i A_i \delta_j A_j}{V_i V_j} \frac{1}{4{A_i}^2 {A_j}^2} \Big[(a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)\Big]^2 \end{aligned}$$

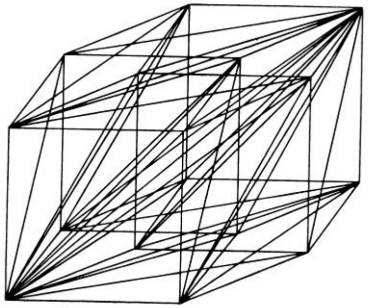
- Ricci squared, R^2, Weyl squared, Euler density …

$$\begin{aligned} R^{(i)}{}_{\mu\nu} R^{(j)\mu\nu} &\equiv \left[\frac{A\delta}{V} U_{\mu}{}^{\rho} U_{\rho\nu}\right]_{(i)} \left[\frac{A\delta}{V} U^{\sigma\mu} U_{\sigma}{}^{\nu}\right]_{(j)} \\ &= \frac{\delta_i A_i \delta_j A_j}{V_i V_j} \frac{1}{16 A_i^2 A_j^2} \\ &\times \Big[a^2c^2(b\cdot d)^2 + a^2d^2(b\cdot c)^2 + b^2c^2(a\cdot d)^2 + b^2d^2(a\cdot c)^2 \\ &- 2\Big[a^2(b\cdot c)(c\cdot d)(d\cdot b) + b^2(a\cdot d)(c\cdot d)(d\cdot a) + c^2(a\cdot b)(b\cdot d)(d\cdot a) \\ &+ d^2(a\cdot b)(b\cdot c)(c\cdot a)\Big] + 2\Big[(a\cdot b)(c\cdot d)\Big[(a\cdot c)(b\cdot d) + ((a\cdot d)(b\cdot c)\Big]\Big]\Big] \end{aligned}$$

Lattice Weak Field Expansion

Only propagating mode is : *One transverse traceless (TT) mode*

… start from the Regge action



$$-k\sum_h \delta_h(l^2)A_h(l^2)$$

… call small edge fluctuations "*e*" :

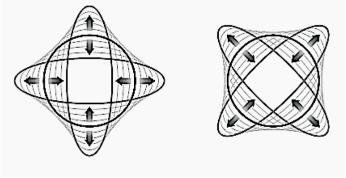
$$\delta^2 I_R \propto \sum_{mn} \varepsilon_m^T M_{mn}\,\varepsilon_n$$

… then Fourier transform, and express result in terms of metric deformations :

$$\delta g_{ij}(l^2) = \tfrac{1}{2}\left(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2\right)$$

… obtaining in the vacuum gauge precisely the familiar *TT form* in k → 0 limit:

$$\frac{1}{4}\mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k})\; h_{ij}^{TT}(\mathbf{k})$$



R.M.Williams, M. Rocek, 1981

WFE - Part 2

- More in detail …

$$M_\omega = \begin{pmatrix} A_{10} & B & 0 \\ B^\dagger & 18 I_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathscr{L}_{sym} = -\tfrac{1}{2}\partial_\lambda h_{\alpha\beta} V_{\alpha\beta\mu\nu} \partial_\lambda h_{\mu\nu} + \tfrac{1}{2} C^2$$

$$V_{\alpha\beta\mu\nu} = \tfrac{1}{2}\eta_{\alpha\mu}\eta_{\beta\nu} - \tfrac{1}{4}\eta_{\alpha\beta}\eta_{\mu\nu}$$

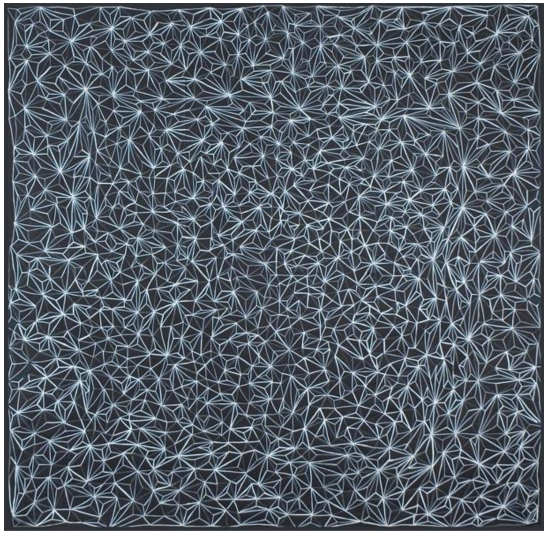
$$C_\mu = \partial_\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h_{\nu\nu}$$

$$\begin{aligned} \varepsilon_1 &= \tfrac{1}{2} h_{11} + O(h^2) \\ \varepsilon_3 &= \tfrac{1}{2} h_{12} + \tfrac{1}{4}(h_{11} + h_{22}) + O(h^2) \\ \varepsilon_7 &= \tfrac{1}{6}(h_{12} + h_{13} + h_{23}) + \tfrac{1}{6}(h_{23} + h_{13} + h_{12}) \\ &\quad + \tfrac{1}{6}(h_{11} + h_{22} + h_{33}) + O(h^2) \quad , \end{aligned}$$

Coincides with the expected continuum action (in the WFE)

$$\mathscr{L}_0 = \lambda_0 (1 + \kappa \tfrac{1}{2} h^\alpha{}_\alpha) + \tfrac{1}{2} h_{\alpha\beta} V^{\alpha\beta\mu\nu} (\partial^2 + \lambda_0 \kappa^2) h_{\mu\nu}$$

Choice of Lattice Structure

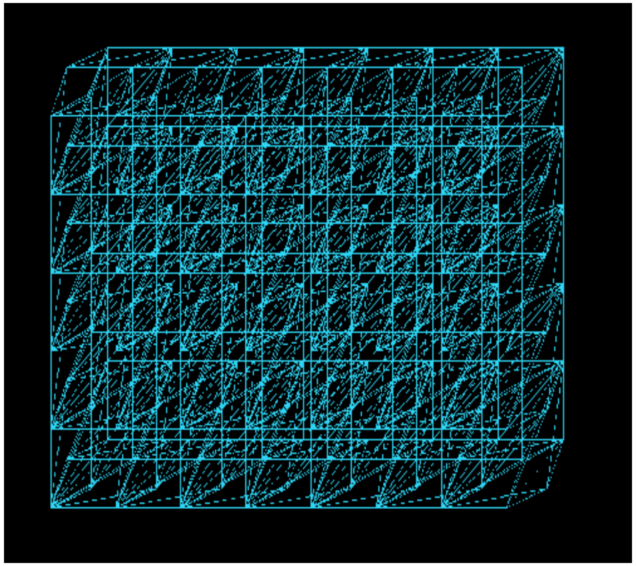


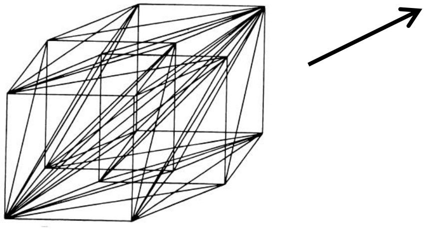
Timothy Nolan,

Carl Berg Gallery, Los Angeles

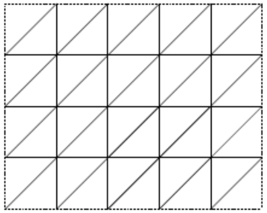
A not so regular lattice …

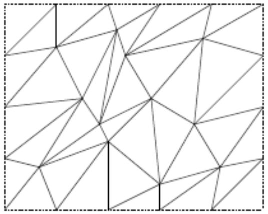
… and a more regular one:



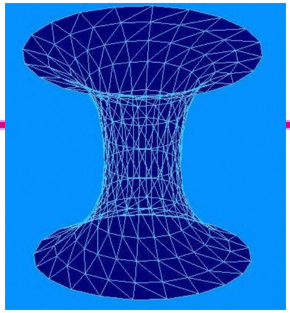


Regular geometric objects can be *stacked.*





Lattice Path Integral



Lattice path integral follows from edge assignments,

$$I_E[g] \;=\; \lambda_0\,\Lambda^d \int dx\,\sqrt{g} \;-\; \frac{1}{16\pi\,G_0}\,\Lambda^{d-2}\int dx\,\sqrt{g}\,R \;\longrightarrow\; I_L \;=\; \lambda_0 \sum_h V_h(l^2) - 2\,\kappa_0 \sum_h \delta_h(l^2)\,A_h(l^2)$$

$$\int [d\,g_{\mu\nu}] \;=\; \int \prod_x \left[g(x)\right]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \;\longrightarrow\; \int [d\,l^2] \;\equiv\; \int_0^\infty \prod_{ij} dl_{ij}^2 \prod_s \left[V_d(s)\right]^\sigma \;\Theta(l_{ij}^2)$$

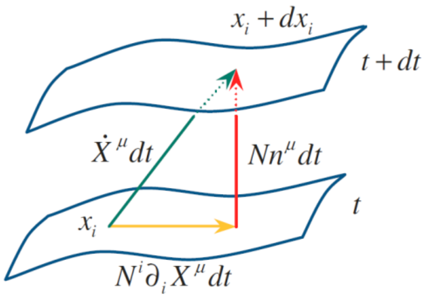
Schrader / Hartle / T.D. Lee measure ;
Lattice analog of the DeWitt measure

$$Z \;=\; \int [d\,g_{\mu\nu}]\; e^{-\lambda_0 \int d^dx\,\sqrt{g} + \frac{1}{16\pi G}\int d^dx \sqrt{g}\,R} \;\longrightarrow\; Z_L \;=\; \int [d\,l^2]\, e^{-I_L[l^2]}$$

Without loss of generality, one can set bare $\lambda_0 = 1$;

Besides the cutoff Λ, the only relevant coupling is κ (or G).

Lattice Hamiltonian



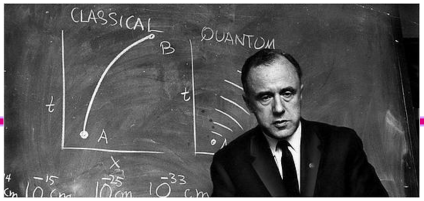
- ADM split space-time into **space** and **time** (3+1)

- Evolve spatial geometry forward in time according to Einstein's field equations … Introduce Momenta :

$$g_{ij}(t, \mathbf{x}) \quad \text{and} \quad \Pi^{ij}(t, \mathbf{x}) = \frac{\delta \mathcal{S}_{\text{Einstein}}}{\delta \dot{g}_{ij}(t, \mathbf{x})}$$

- Dynamics determined by the constraints (via lapse and shift functions)

Wheeler-DeWitt Equation

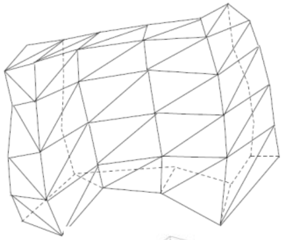


- Position rep. → Functional Schrödinger equation :

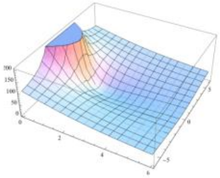
$$\hat{g}_{ij}(\mathbf{x}) \;\rightarrow\; g_{ij}(\mathbf{x}) \qquad \hat{\pi}^{ij}(\mathbf{x}) \;\rightarrow\; -i\hbar \cdot 16\pi G \cdot \frac{\delta}{\delta g_{ij}(\mathbf{x})}$$

$$\left\{ -(16\pi G)^2 \, G_{ij,kl}(\mathbf{x}) \, \frac{\delta^2}{\delta g_{ij}(\mathbf{x}) \, \delta g_{kl}(\mathbf{x})} \;-\; \sqrt{g(\mathbf{x})} \; \left({}^3R(\mathbf{x}) \;-\; 2\lambda \right) \right\} \Psi[g_{ij}(\mathbf{x})] \;=\; 0$$

- Discretized form (use Hartle latt. supermetric G):



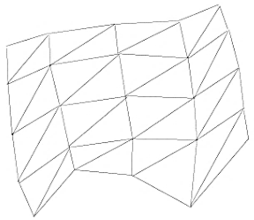
$$\left\{ -(16\pi G)^2 \sum_{i,j \subset \sigma} G_{ij}\,(\sigma)\, \frac{\partial^2}{\partial l_i^2 \, \partial l_j^2} \;-\; 2\, n_{\sigma h} \sum_{h \subset \sigma} l_h \, \delta_h \;+\; 2\lambda \, V_\sigma \right\} \Psi[\, l^2 \,] \;=\; 0$$



…a bit like Hamiltonian lattice gauge theory (K-S)

[RMW & HWH, Phys Rev D 2011]

Wheeler-DeWitt in 2+1



In 2+1 dimensions an *exact wavefunction* can be obtained:

$$\Psi \sim e^{-i x} \, {}_1F_1 \left(a, \, b, \, 2 \, i \, x \right)$$

$$\begin{aligned} a &\equiv \tfrac{1}{4} N_\Delta - \frac{\sqrt{2} \pi \, i}{\sqrt{\lambda} \, G} \chi \; = \; \tfrac{1}{4} N_\Delta - \frac{i}{2 \sqrt{2 \, \lambda} \, G} \int d^2 y \, \sqrt{g} \, R \\ b &\equiv \tfrac{1}{2} N_\Delta \\ x &\equiv \frac{\sqrt{2 \lambda}}{G} A_{tot} \; = \; \frac{\sqrt{2 \lambda}}{G} \int d^2 y \, \sqrt{g} \; . \end{aligned}$$

$$N_\Delta \; = \; \frac{1}{< A_\Delta >} A_{tot} \; = \; \frac{1}{< A_\Delta >} \int d^2 y \, \sqrt{g} \; .$$

From it one can compute the Total Area Fluctuation:

$$\chi_A \; = \; \frac{1}{N_\Delta} \left\{ < (A_{tot})^2 > - < A_{tot} >^2 \right\} \; .$$

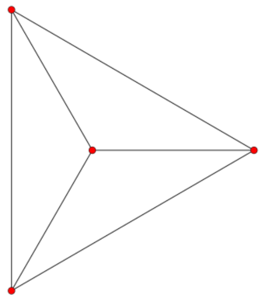
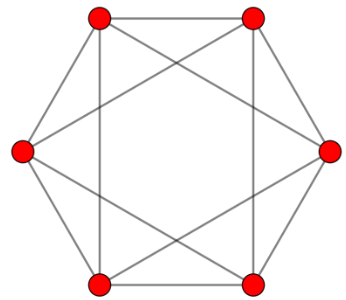
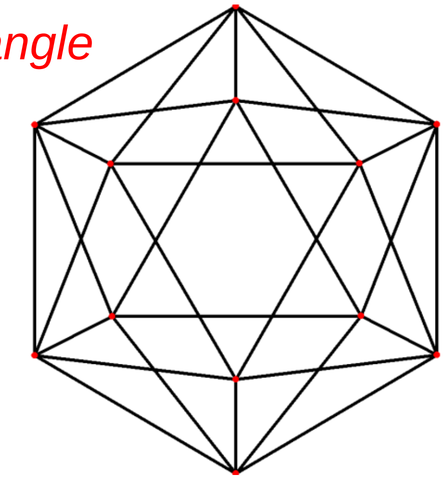
$$\boxed{\nu \; = \; \frac{6}{11} \; = \; 0.5454... \; .}$$

Exact value for ν in 3D

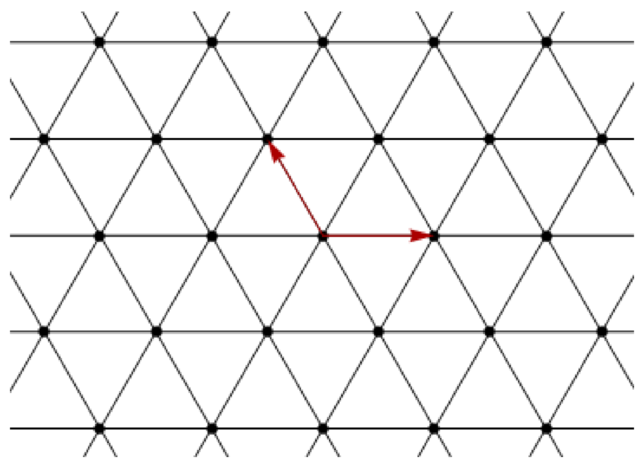
R.M. Williams, R. Toriumi and H.H., PRD 2013

Regular Triangulations in 2D

In 2+1 dimensions *one WdW equation for each lattice triangle*

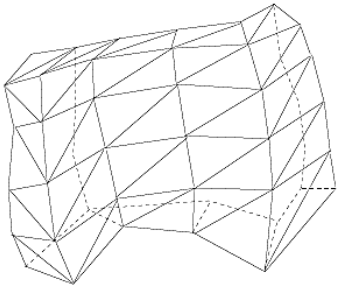





- Single triangle,
- tetrahedron,
- octahedron,
- icosahedron,
- triangular lattice



There are of course many more irregular ones.

Wheeler-DeWitt in 3+1



In 3+1 dimensions *one WdW equation for each lattice tetrahedron*

Can study single Tetrahedron, Five-cell, 16-cell and 600-cell complex.

Set for the asymptotic wavefunction (see eg. Schiff QM) :

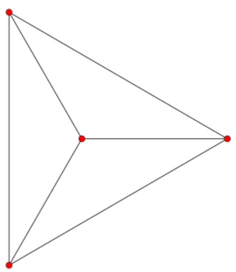
$$\psi \sim \exp\left\{ \pm i \left(\alpha \int d^3x \sqrt{g} + \beta \int d^3x \sqrt{g}\, R + \gamma \int d^3x \sqrt{g}\, R^2 + \delta \int d^3x \sqrt{g}\, R_{\mu\nu} R^{\mu\nu} + \cdots \right) \right\}$$

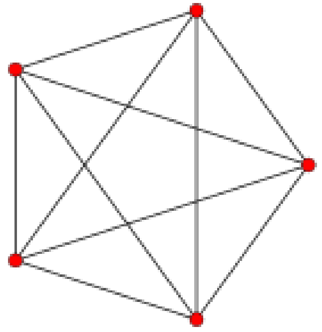
From it one can compute the Volume and Curvature fluctuation;

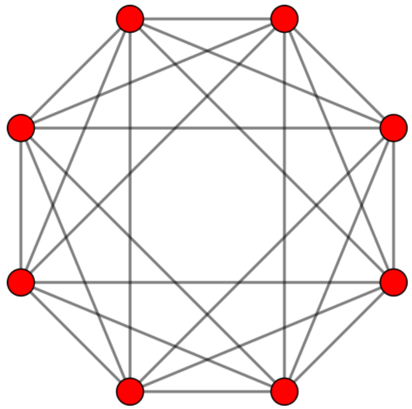
No value for universal critical exponent ν in 3+1 yet.

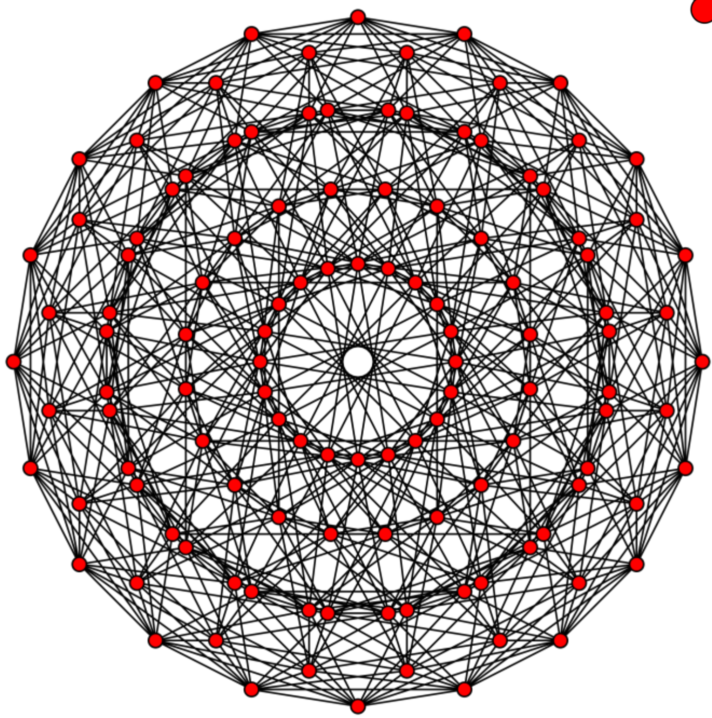
R.M. Williams, R. Toriumi and H.H., PRD 2014

Regular Triangulations in 3D









- Single tetrahedron,
- 4-simplex (5 tetrahedra),
- 16-cell (16 tetrahedra),
- 600 cell (600 tetrahedra).

There are of course many more irregular ones

3+1 … WF in Small Curvature Limit

What remains of the W-DW Eq. in 3+1 dimensions is the “Master Equation” :

$$\frac{\partial^2\psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_R \frac{\partial \psi}{\partial R} + c_{VR} \frac{\partial^2 \psi}{\partial V \partial R} + c_{RR} \frac{\partial^2 \psi}{\partial R^2} + c_\lambda \psi + c_{curv} \psi = 0 \, .$$

$$\begin{aligned}
c_V &= \frac{11+9\,q}{2\,q^2} \cdot \frac{N_3}{V} = \frac{11+9\,q_0}{2\,q_0^2} \cdot \frac{N_3}{V} + \frac{22+9\,q_0}{48\sqrt{2}\,3^{1/3}\,\pi\,q_0} \cdot \frac{N_3^{1/3}\,R}{V^{4/3}} + \mathcal{O}(R^2) \\
c_R &= -\frac{2}{9}\frac{R}{V^2} + \frac{11+9\,q_0}{6\,q_0^2} \cdot \frac{N_3\,R}{V^2} + \mathcal{O}(R^2) \\
c_{VR} &= \frac{2}{3}\frac{R}{V} + \mathcal{O}(R^2) \\
c_{RR} &= \frac{2}{9}\frac{R^2}{V^2} \\
c_\lambda &= \frac{32\,\lambda}{q^2\,G^2} = \frac{32}{G^2\,q_0^2} + \frac{4\sqrt{2}\lambda}{3\,3^{1/3}\,\pi\,q_0\,G} \cdot \frac{R}{N_3^{2/3}\,V^{1/3}} + \mathcal{O}(R^2) \\
c_{curv} &= -\frac{16}{G^2\,q^2} \cdot \frac{R}{V} = -\frac{16}{G^2\,q_0^2} \cdot \frac{R}{V} + \mathcal{O}(R^2) \, .
\end{aligned}$$

$$q_0 \equiv \frac{2\,\pi}{\cos^{-1}(\frac{1}{3})} = 5.1043$$

$$\psi(V,\,R) \simeq e^{-\frac{4\,i\sqrt{2\lambda}\,V}{q_0\,G}} \cdot \frac{\Gamma\left(\frac{(11+9\,q_0)\,N_3}{4\,q_0^2} + \frac{i\sqrt{2}\,R}{q_0\,G\sqrt{\lambda}}\right)}{\Gamma\left(1 - \frac{(11+9\,q_0)\,N_3}{4\,q_0^2} + \frac{i\sqrt{2}\,R}{q_0\,G\sqrt{\lambda}}\right)}$$

$$\times\, {}_1F_1\left(\frac{(11+9\,q_0)N_3}{4\,q_0^2} - \frac{i\sqrt{2}\,R}{q_0\sqrt{\lambda}\,G},\, \frac{(11+9\,q_0)N_3}{2\,q_0^2},\, \frac{8\,i\sqrt{2\lambda}\,V}{q_0\,G}\right)$$

Confluent Hypergeometric & Gamma functions with complex arguments

PHYSICAL REVIEW D **88**, 084012 (2013)

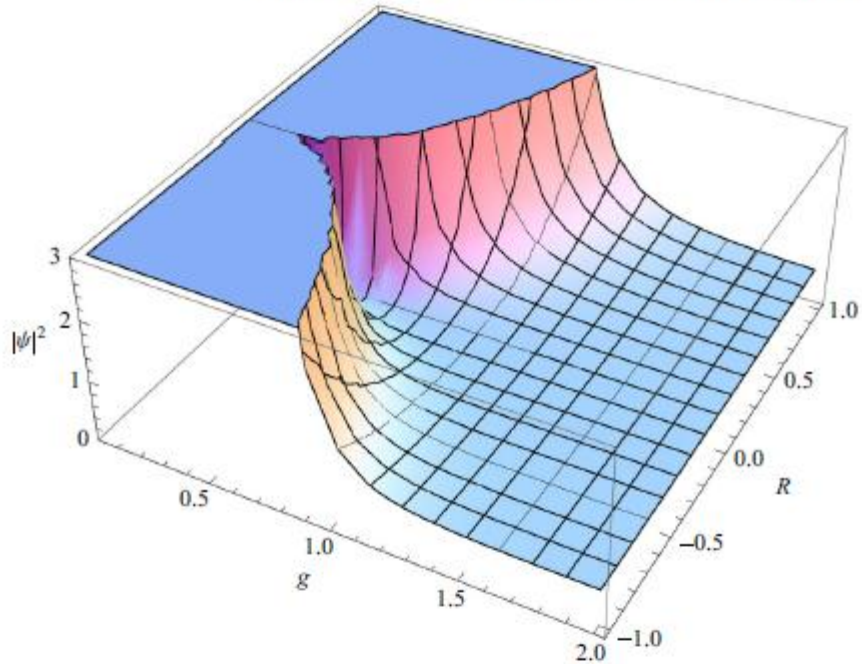


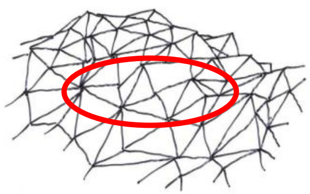
FIG. 5 (color online). Curvature distribution in R as a function of the coupling $g = \sqrt{G}$.

$G_c \approx 0.5672$

(Euclidean 4D : $G_c \approx 0.6231$)

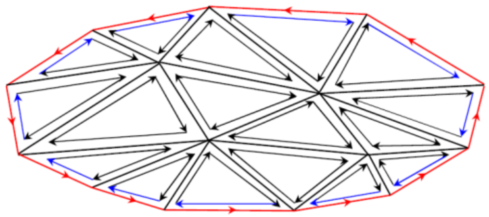
In the 3+1 dimensional lattice theory the weak coupling phase looks non-perturbatively unstable (no continuum limit).

Gravitational Wilson Loop



- Parallel transport of a vector done via lattice rotation matrix

$$R^{\alpha}{}_{\beta}(C) \;=\; \Big[\, \mathcal{P} \, \exp\Big\{ \oint_{\textbf{path C}} \Gamma^{\cdot}_{\lambda \cdot} dx^{\lambda} \Big\} \Big]^{\alpha}_{\;\beta}$$



For a *large* closed circuit obtain *gravitational Wilson loop;*
compute at *strong coupling (G large) …*

$$W(\Gamma) \sim \mathrm{Tr}\, \mathcal{P} \exp\Big[\int_C \Gamma^{\lambda}_{\cdot\cdot}\, dx_{\lambda} \Big] \;\underset{A\to\infty}{\sim}\; \exp\left(- A_C/\xi^2\right)$$

"Minimal area law" follows from loop tiling.

… then compare to *semi-classical result* (from Stokes' theorem)

$$R^{\alpha}{}_{\beta}(C) \sim \Big[\exp\Big\{ \tfrac{1}{2} \int_{S(C)} R^{\cdot}{}_{\cdot\,\mu\nu} A_C^{\mu\nu} \Big\} \Big]^{\alpha}_{\;\beta} \qquad A_C^{\mu\nu} = \tfrac{1}{2} \oint dx^{\mu}\, x^{\nu}$$

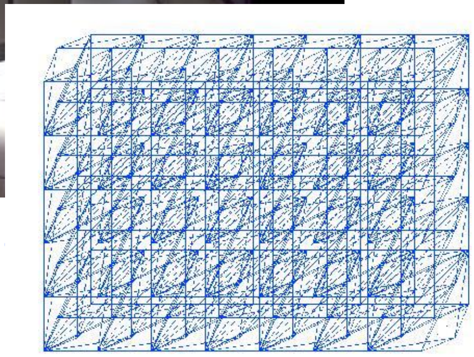
- suggests ξ related to curvature.
- argument can only give a positive cosmological constant.

$$\lambda_{obs} \simeq +\frac{1}{\xi^2}$$

R.M. Williams and H.H.,
Phys Rev D 76 (2007) ; D 81 (2010)

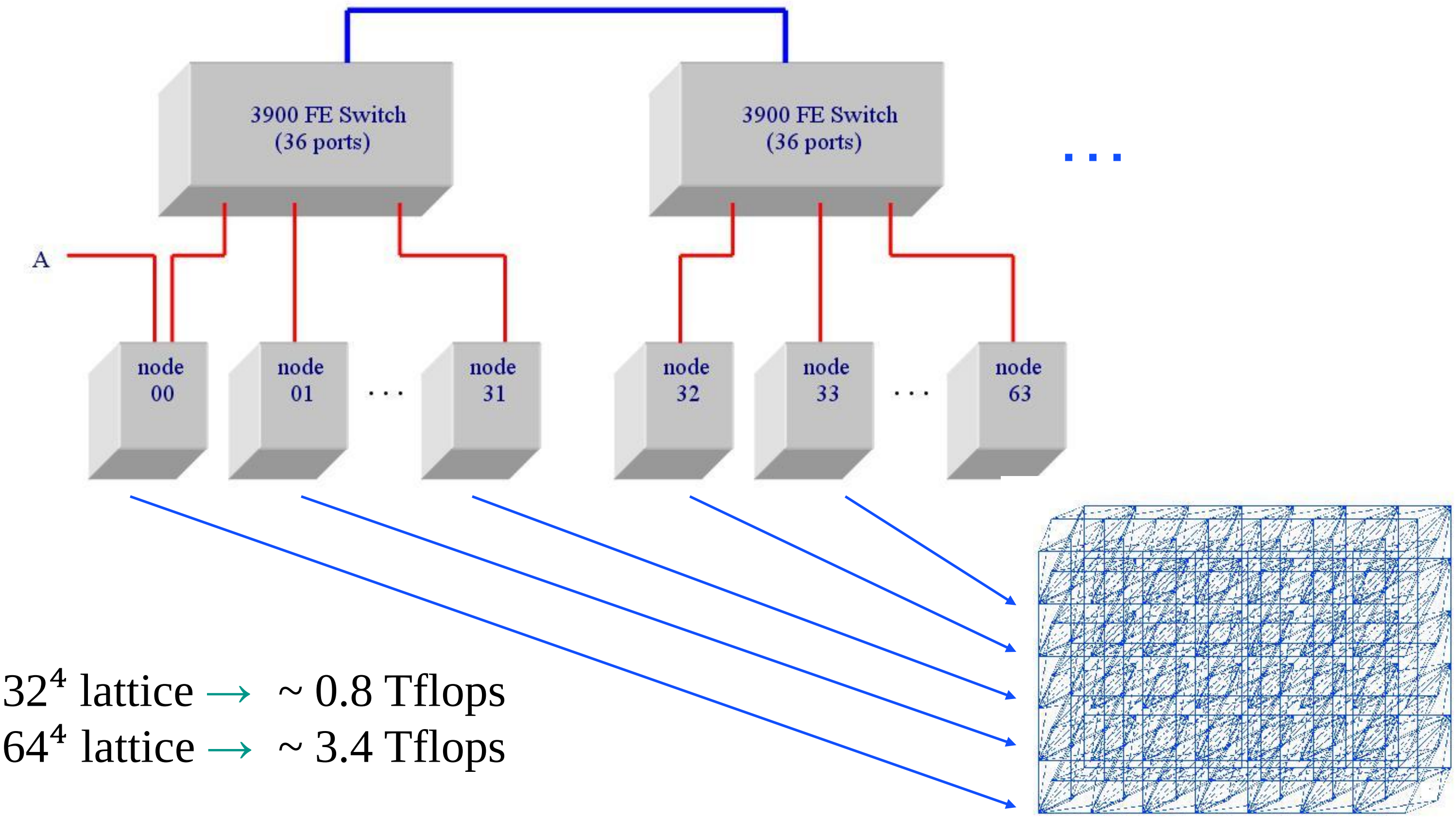
[Peskin and Schroeder, page 783]

Numerical Evaluation of *Z*



Lattice Sites are Processed in Parallel

Distribute Lattice Sites on, say, 1024 Processor Cores

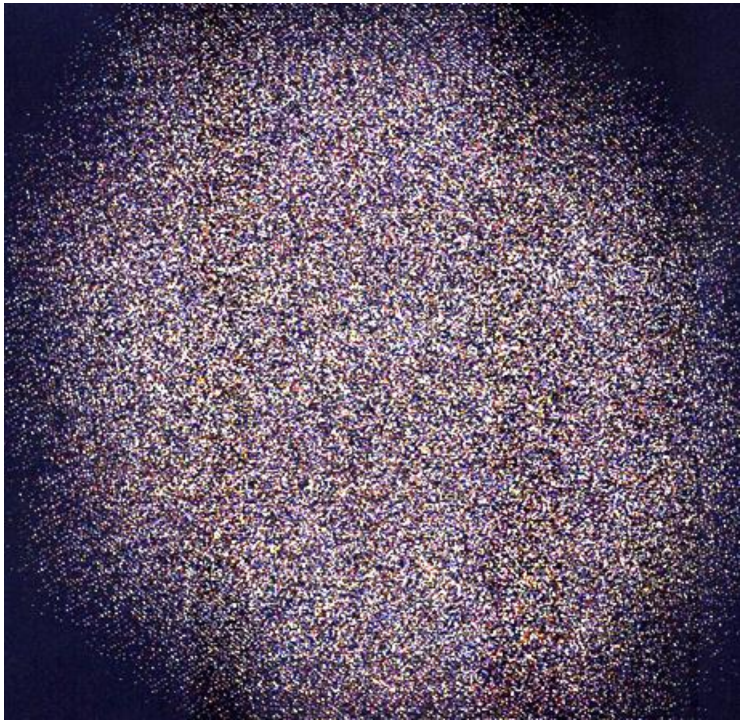


MPI

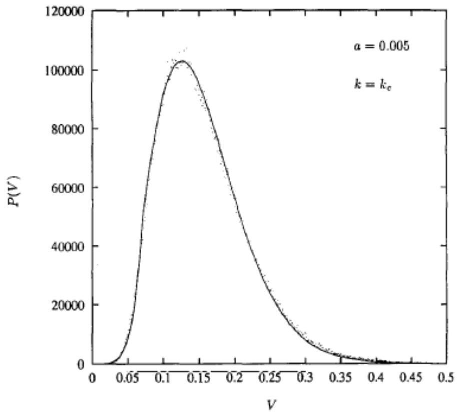
256 cores on 32^4 lattice $\longrightarrow$ ~ 0.8 Tflops

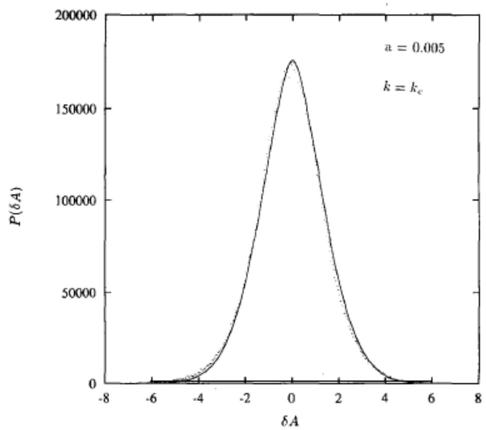
1024 cores on 64^4 lattice $\longrightarrow$ ~ 3.4 Tflops

Edge length / metric distributions



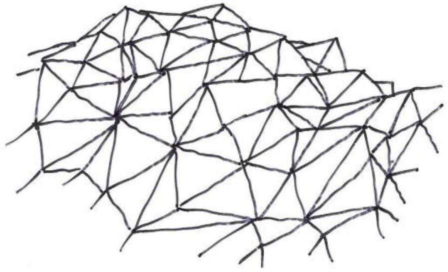
- 4^4 sites → 6,144 simplices
- 8^4 sites → 98,304 simplices
- 16^4 sites → 1.5 M simplices
- 32^4 sites → 25 M simplices
- 64^4 sites → 402 M simplices





Phases of (E.) Lattice Quantum Gravity

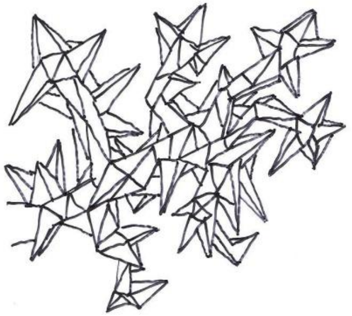
L. Quantum Gravity has two phases …



$G > G_c$ Smooth phase: R ≈ 0

$$\langle g_{\mu\nu} \rangle \approx c\, \eta_{\mu\nu}$$

Physical



$G < G_c$ Unphysical (branched polymer-like, d ≈ 2)

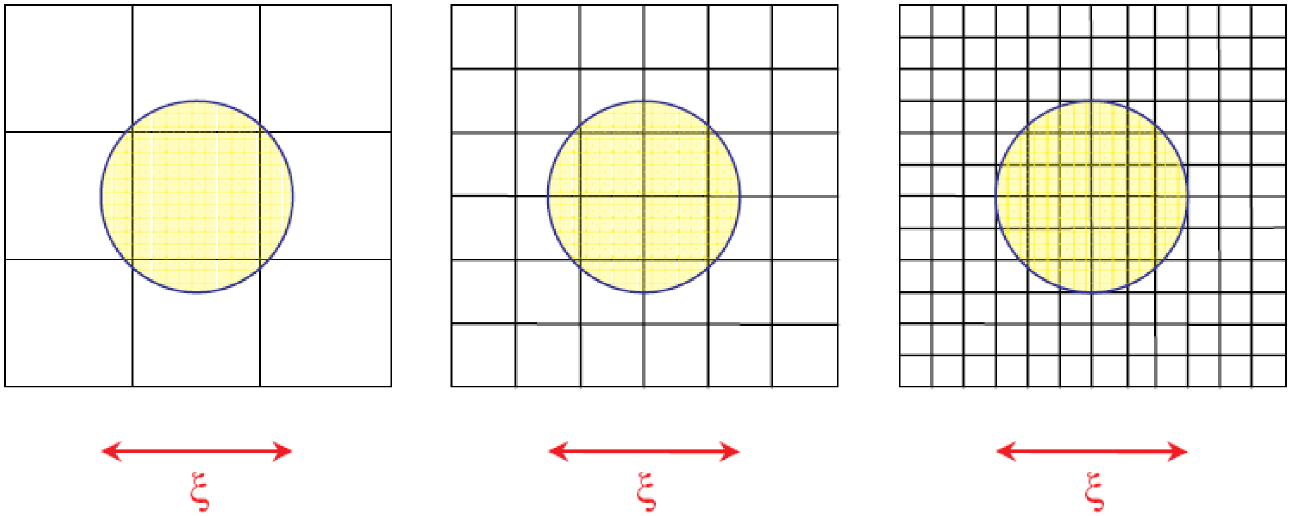
$$\langle g_{\mu\nu} \rangle = 0$$

Unphysical

R. Williams and HWH , NPB, PLB 1984 ;
B. Berg 1985 , …

(Lattice manifestation of conformal instability)

Lattice Continuum Limit



The lattice quantum continuum limit is gradually approached by considering sequences of lattices with increasingly larger correlation lengths ξ in lattice units. Such a limit requires the existence of an ultraviolet fixed point, where quantum field correlations extend over many lattice spacing.

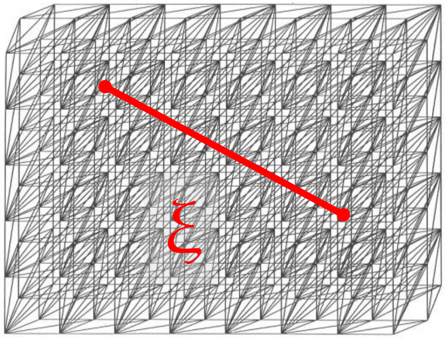
Continuum limit <u>requires</u> the existence of an UV fixed point.

(Lattice) Continuum Limit $\Lambda \to \infty$

Use Standard Wilson procedure in cutoff field theory

$$\frac{\partial G}{\partial \log \Lambda} = -\frac{1}{\nu} (G - G_c) + \dots$$

integrated to give :



$$\xi = 1/m \qquad m \underset{G(\Lambda) \to G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0 \, G_c} \right]^{\nu}$$

RG *invariant* correlation length ξ is kept fixed

UV cutoff $\Lambda \to \infty$ (average lattice spacing → 0)

Bare G *must* approach UV fixed point at G_c .

The *very same* relation gives the RG running of G(μ) close to the FP.

Determination of Scaling Exponents

$$\mathscr{R}(k) \sim \frac{< \int dx \sqrt{g} R(x) >}{< \int dx \sqrt{g} >} \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z \underset{k \to k_c}{\sim} - A_{\mathcal{R}} \, (k_c - k)^{\delta} \qquad \nu = \frac{1 + \delta}{d}$$

$$\chi_{\mathcal{R}}(k) \sim \frac{< (\int d^4x \sqrt{g} \, R)^2 > - < \int d^4x \sqrt{g} \, R >^2}{< \int d^4x \sqrt{g} >} \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z \underset{k \to k_c}{\sim} \delta \, A_{\mathcal{R}} \, (k_c - k)^{-(1-\delta)}$$

Use standard Universal Scaling assumption:

$$F_{sing}(G) \sim \xi^{-d}$$

$$\xi(k) \underset{k \to k_c}{\sim} A_{\xi} \, (k_c - k)^{-\nu}$$

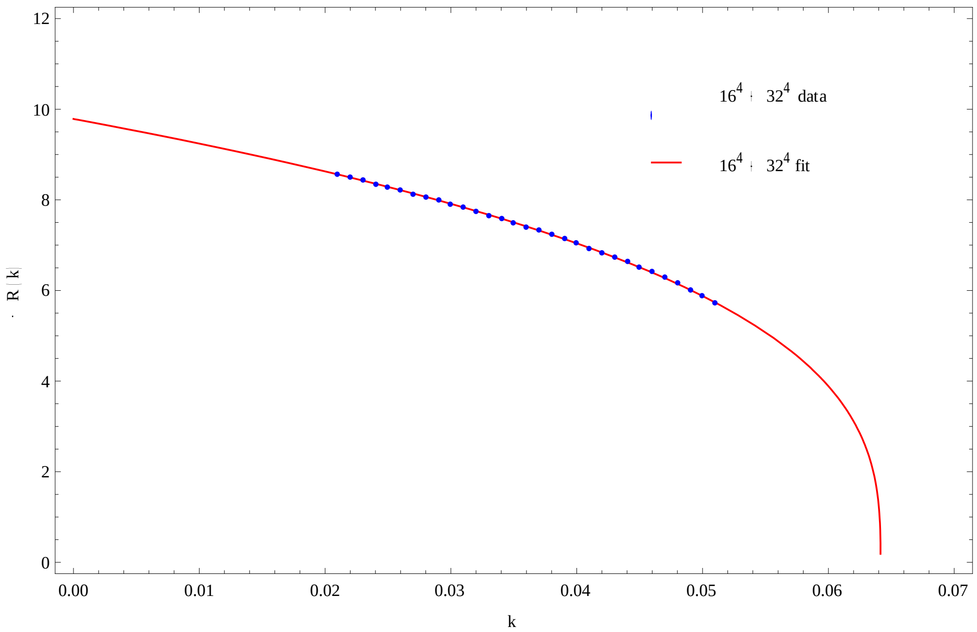
$$k_c = 0.063862(18) \qquad \nu = 0.334(4)$$

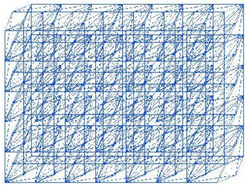
Find value for ν close to $1/3$:

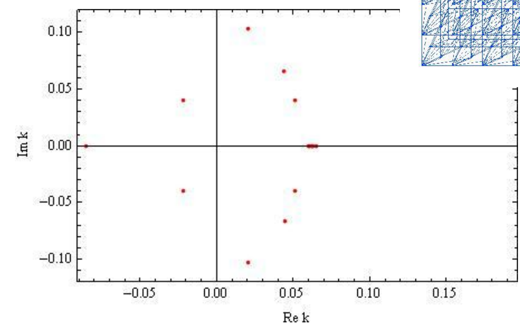
$$\nu \approx 1/3$$

(Phys Rev D Sept. 2015)

Recent runs on 2400 node cluster



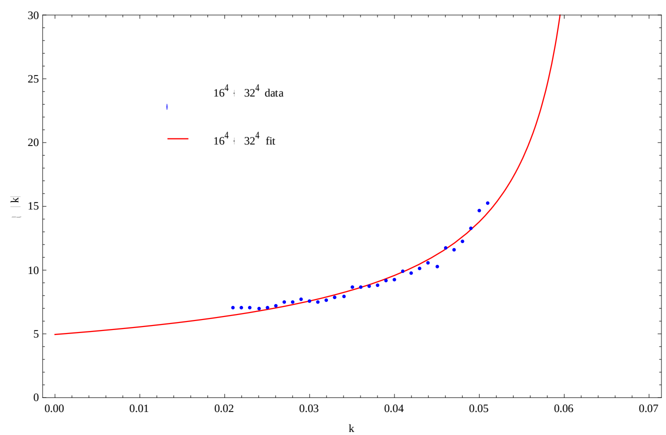




Distribution of zeros in complex k space

$$\chi_{\mathscr{R}}(k) \sim \frac{< (\int dx \sqrt{g} R)^2 > - < \int dx \sqrt{g} R >^2}{< \int dx \sqrt{g} >}$$

$$\mathscr{R}(k) \sim \frac{< \int dx \sqrt{g} R(x) >}{< \int dx \sqrt{g} >} \qquad G_c = \frac{1}{8\pi k_c} = 0.623042(25)$$



$$\nu = 0.334(4)$$

More calculations in progress …

Finite Size Scaling (FSS) Analysis

$$\mathcal{R}(k,L) \;=\; L^{-(4-1/\nu)} \left[\tilde{\mathcal{R}}\left((k_c - k)\, L^{1/\nu} \right) \,+\, \mathcal{O}(L^{-\omega}) \right]$$

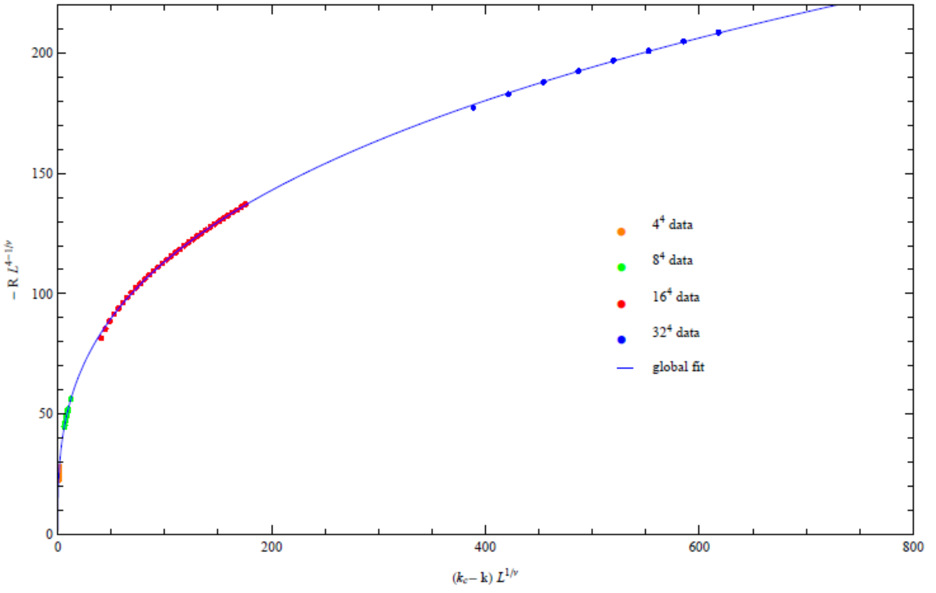


Figure 16: Finite size scaling behavior of the scaled curvature $\mathcal{R}(k,L) \cdot L^{4-1/\nu}$ versus the scaled coupling $(k_c - k) \cdot L^{1/\nu}$. Here $L = 4, 8, 16, 32$ for the lattice with L^4 sites. Statistical errors are comparable to the size of the dots. The continuous line represents a best fit to a scaling function of the form $a + b\,x^c$, and finite size scaling predicts that all points should lie on the same universal curve. The continuous line corresponds to a critical point $k_c = 0.06388(32)$ and exponent $\nu = 0.3334(4)$.

FSS for Curvature Fluctuation

$$\chi_{\mathcal{R}}(k,L) \;=\; L^{2/\nu-4}\,\left[\,\tilde{\chi_{\mathcal{R}}}\left((k_c-k)\,L^{1/\nu}\right)\;+\;\mathcal{O}(L^{-\omega})\,\right]$$

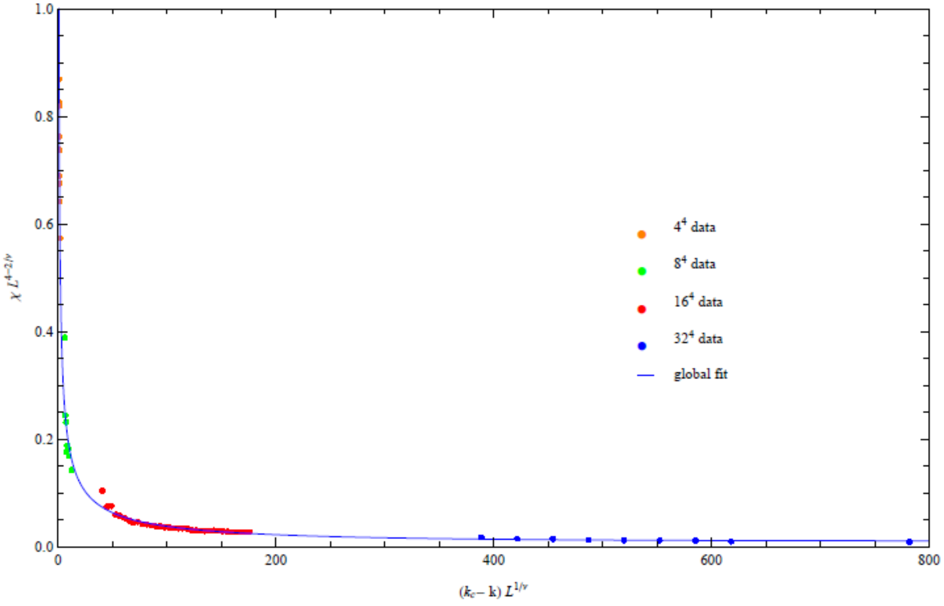


Figure 17: Finite size scaling behavior of the scaled curvature fluctuation $\chi_{\mathcal{R}}(k,L)\cdot L^{4-2/\nu}$ versus the scaled coupling $(k_c-k)\cdot L^{1/\nu}$. Here $L=4,8,16,32$ for a lattice with L^4 sites. The continuous line represents a best fit to a scaling function of the form $1/(a+b\,x^c)$, and finite size scaling predicts that all points should lie on the same universal curve. The continuous line corresponds to a critical point $k_c=0.06384(40)$ and an exponent $\nu=0.3389(56)$.

Gravitational Correlation Length ξ

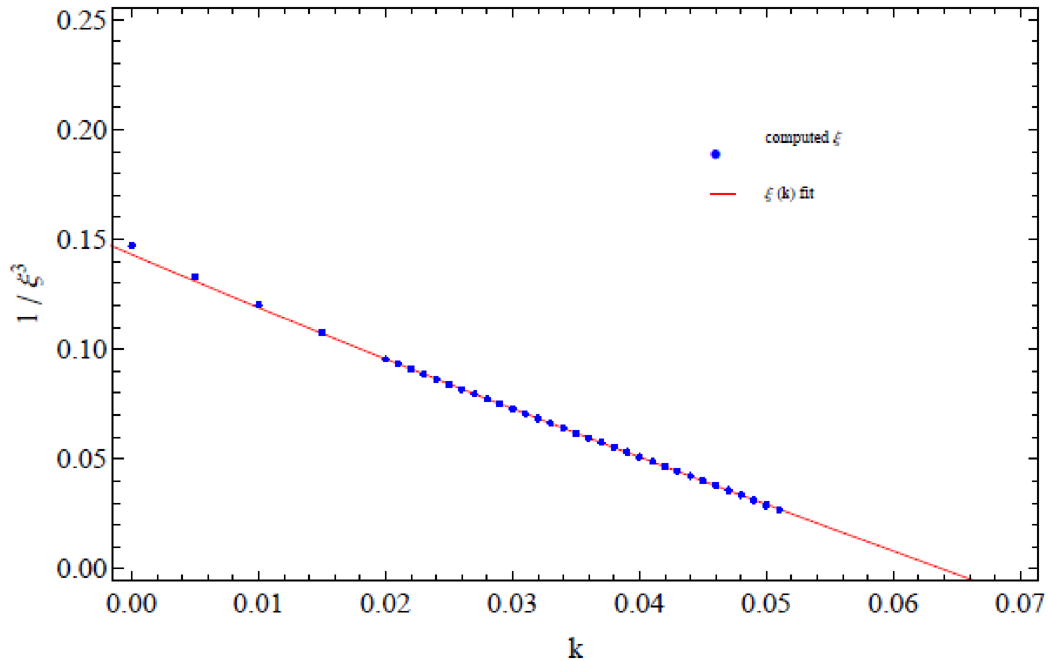


Figure 19: Estimate for the gravitational correlation length $\xi(k)$ versus bare coupling k. For a correlation length exponent $\nu = 1/3$ [see Eq. (42)], $1/\xi(k)^3$ is expected to be linear in k close to the critical point k_c.

$$A_\xi \;=\; 0.80(3)$$

Summary of Numerical Results

Observables used to compute k_c and ν	Critical Point k_c	Universal Exponent ν
Average Curvature $\mathcal{R}$ vs. k	0.06336(28)	0.331(4)
Average Curvature $\mathcal{R}^3$ vs. k	0.06367(29)	0.332(2)
Average Curvature $\mathcal{R}^3$ vs. k	0.06407(24)	-
Curvature Fluctuation $\chi_{\mathcal{R}}$ vs. k	0.05383(102)	0.350(56)
Curvature Fluctutation $\chi_{\mathcal{R}}$ vs. k	-	0.321(12)
Curvature Fluctuation $\chi_{\mathcal{R}}^{-3/2}$ vs. k	0.06369(84)	-
Logarithmic Derivative $2\langle l^2\rangle\chi_{\mathcal{R}}/\mathcal{R}$ vs. k	0.06338(56)	0.336(8)
Curvature Fluctuation $\chi_{\mathcal{R}}$ vs. $\mathcal{R}$	-	0.332(7)
$\mathcal{R}(k, L)$ Finite Size Scaling	0.06388(11)	0.333(2)
$\chi_{\mathcal{R}}(k, L)$ Finite Size Scaling	0.06384(18)	0.339(6)
Size Dependence of the Critical Point $k_c(L)$	0.063862(30)	-

TABLE I. Summary of results for the critical point k_c and the universal gravitational critical exponent ν, as obtained from the largest lattices studies so far.

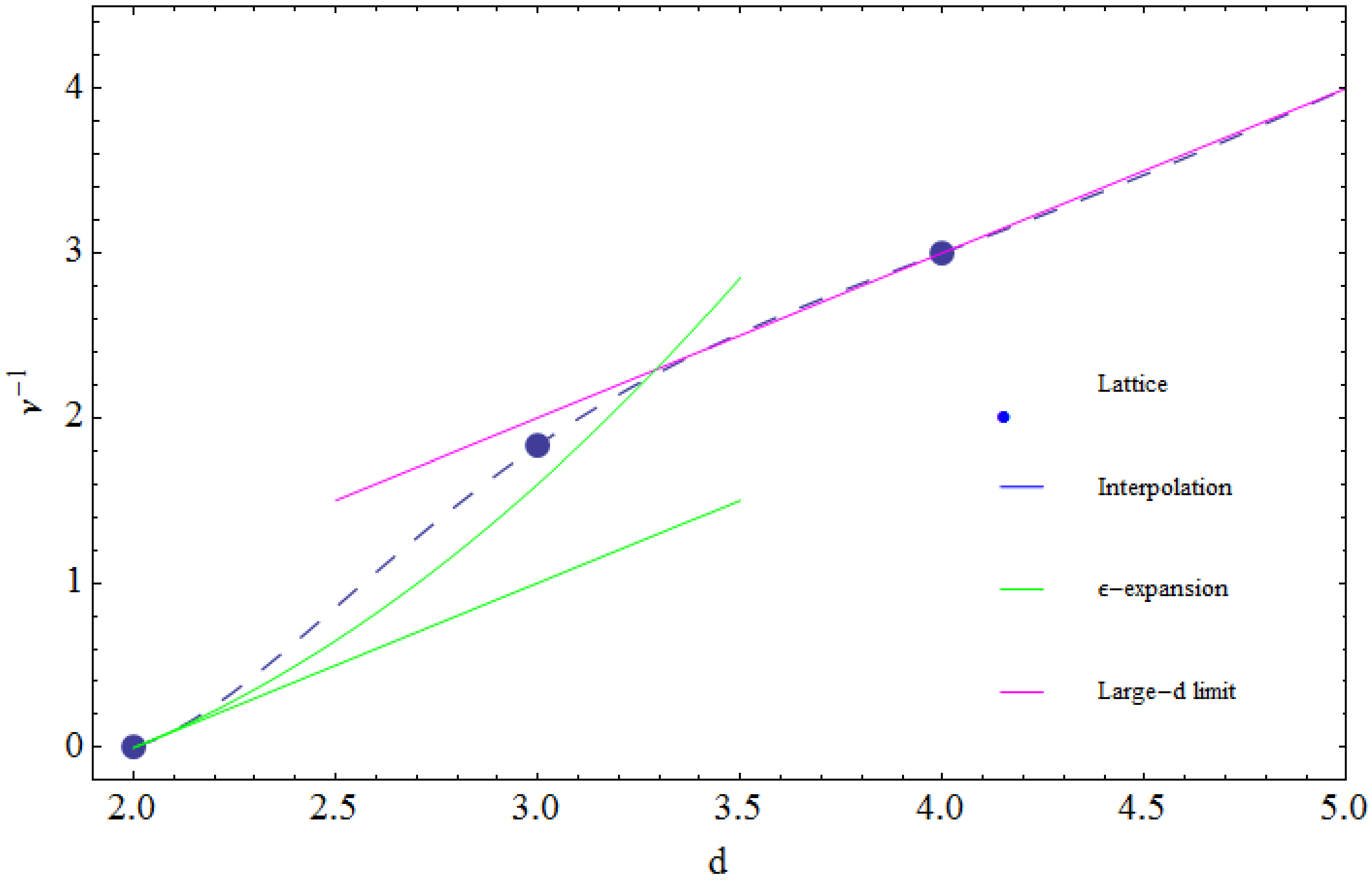
Exponent Comparison (D=4)

Method used to compute ν in d=4	Universal Exponent ν
Euclidean Lattice Quantum Gravity (this work)	$\nu^{-1} = 2.997(9)$
Perturbative $2+\epsilon$ expansion to one loop [22]	$\nu^{-1} = 2$
Perturbative $2+\epsilon$ expansion to two loops [23]	$\nu^{-1} = 22/5 = 4.40$
Einstein-Hilbert RG truncation [56]	$\nu^{-1} \approx 2.80$
Recent improved Einstein-Hilbert RG truncation [57]	$\nu^{-1} \approx 3.0$
Geometric argument [33] $\rho_{vac\ pol}(r) \sim r^{d-1}$	$\nu^{-1} = d - 1 = 3$
Lowest order strong coupling (large G) expansion [29]	$\nu^{-1} = 2$
Nonlocal field equations with $G(\Box)$ for the static metric [46]	$\nu^{-1} = d - 1$ for $d \geq 4$

Exponent Comparison (D=3)

Method used to compute ν in $d = 3$	Universal Exponent ν
Euclidean Lattice Quantum Gravity [58]	$\nu^{-1} = 1.72(5)$
Exact solution of Lorentzian Gravity (Wheeler-DeWitt Eq.) in 2+1 dim. [47]	$\nu^{-1} = 11/6 = 1.8333$
Perturbative $2 + \epsilon$ expansion to one loop [22]	$\nu^{-1} = 1$
Perturbative $2 + \epsilon$ expansion to two loops [23]	$\nu^{-1} = 8/5 = 1.6$
Einstein-Hilbert RG truncation [56]	$\nu^{-1} \approx 1.33$
Large d geometric argument [33] $\rho_{vac\ pol}(r) \sim r^{d-1}$	$\nu^{-1} = d - 1 = 2$

Comparison for Exponent $1/\nu$



Running Newton's Constant G

In gravity there is a new RG invariant scale ξ : $m \equiv \xi^{-1} = \Lambda F(G)$

Running of G determined largely by scale ξ and exponent ν :

$$G(k^2) \;=\; G_0 \left[1 \,+\, c_0 \left(\frac{1}{\xi^2 \, k^2} \right)^{1/2\nu} \,+\, \ldots \right] \qquad c_0 \approx 8.02.$$

Almost identical to 2 + ε expansion result, but with a 4-d exponent $\nu = 1/3$ and a calculable coefficient c_0 … "Covariantize" : $k^2 \to -\Box$

$$G(\Box) \;=\; G_c \left[1 \,+\, c_0 \left(\frac{1}{-\xi^2 \, \Box^2} \right)^{3/2} \,+\, \ldots \right]$$

Running of Newton's G(□)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu}$$

$\nu = 1/3$

$$G(\Box) = G_0 \left[1 + c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} + \dots \right]$$

New RG invariant scale of gravity $\xi \sim 1/\sqrt{\lambda}$ (infrared cutoff)

$\Rightarrow$ Expect small deviations from GR on largest scales

Eg.
- Matter density perturbations in comoving gauge
- Gravitational "slip" function in Newtonian gauge

Infrared RG Running of G

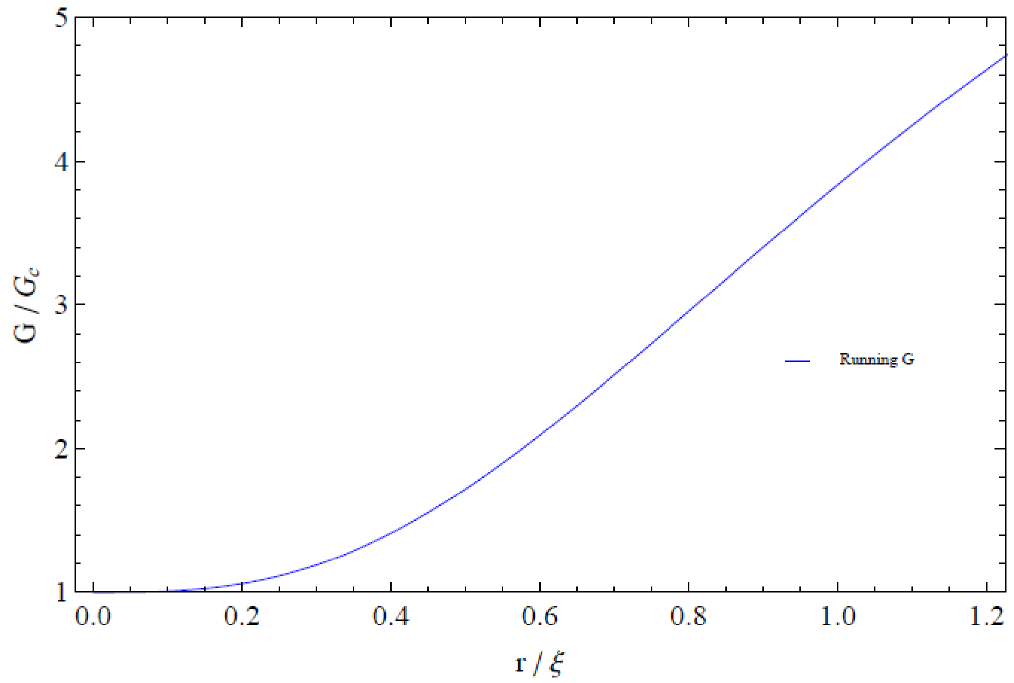


Figure 21: Running gravitational coupling $G(r)$ versus r, obtained from the $G(k)$ in Eq. (59) by setting $q \sim 1/r$, with the exponent $\nu = 1/3$ and amplitude $a_0 \simeq 8.02(55)$. The lattice quantum gravity calculations done so far suggest roughly a 5% effect on scales of $0.187 \times 4890 Mpc \approx 910 Mpc$, and a 10 % effect on scales of $0.238 \times 4890 Mpc \approx 1160 Mpc$.

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Vacuum Condensate Picture of QG

- Lattice Quantum Gravity: Curvature condensate

See also J.D.Bjorken, PRD '05

$$\langle R \rangle \simeq \frac{1}{\xi^2} \qquad \tfrac{1}{3}\,\lambda_{obs} \simeq +\frac{1}{\xi^2}\,. \qquad \xi \simeq \sqrt{3/\lambda} \approx 4890 Mpc$$

- Quantum Chromodynamics: Gluon and Fermion condensate

$$\langle \tfrac{\alpha_S}{\pi} F_\mu^2 \simeq \frac{1}{\xi^4} \simeq (440\, MeV)^4$$

$$\langle \bar{\psi}\psi \rangle \simeq \frac{1}{\xi^3} \simeq (290\, MeV)^3$$

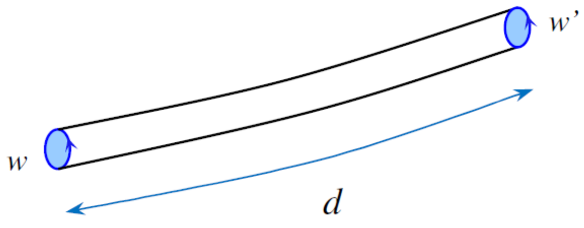
$$\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}} \simeq 210 MeV$$

- Electroweak Theory: Higgs condensate

Curvature Correlation Functions

Need the geodesic distance between any two points :

$$d(x, y \,|\, g) \;=\; \min_{\xi} \int_{\tau(x)}^{\tau(y)} d\tau \sqrt{g_{\mu\nu}(\xi) \frac{d\xi^{\mu}}{d\tau} \frac{d\xi^{\nu}}{d\tau}}$$



Curvature correlation function :

$$G_R(d) \;\sim\; < \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y| - d) >_c \underset{|x-y|\to\infty}{\sim} \frac{1}{|x-y|^{2n}} \quad .$$

But for $\nu = 1/3$ the result becomes quite simple :

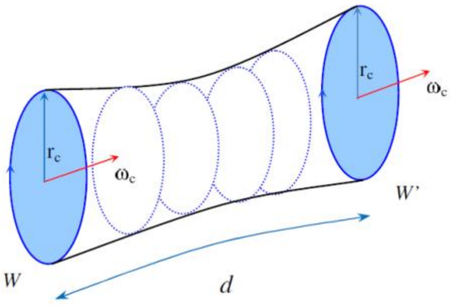
$$n \;=\; d \;-\; 1/\nu$$

$$< \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y| - d) >_c \underset{d \ll \xi}{\sim} \frac{A_0}{a^2\, d^2}$$

$$N_R \;\equiv\; \sqrt{A_0} = 0.335(20)$$

If the two parallel transport loops are *not* infinitesimal :

$$< \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y| - d) >_c \underset{d \ll \xi}{\sim} \frac{A_1}{\xi^2\, d^2}$$



… Related to Matter Density Correlations

The *classical* field equations relate the local curvature to the local matter density

$$R(x) \simeq 8\pi G \, \rho(x)$$

For the macroscopic matter density contrast one then obtains

$$G_\rho(r) \; = < \delta\rho(r) \; \delta\rho(0) > \; = \; \left(\frac{r_0}{r} \right)^\gamma$$

From the lattice one computes :

$$\gamma = 2$$

$$r_0 \; = \; \frac{1}{8\pi \, G \, \rho_0} \cdot \frac{\sqrt{A_1}}{\xi} \; \approx \; 0.380 \, \xi$$

Astrophysical measurements of G(r) are roughly consistent with

$$\gamma \approx 1.8 \pm 0.3$$

The End