Phase Structure of Causal Dynamical Triangulations

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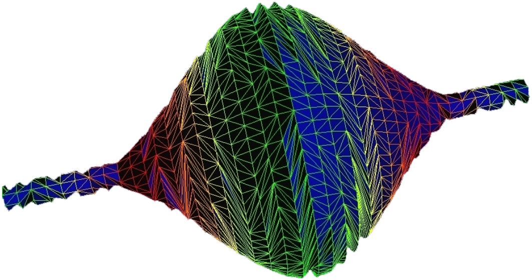
What is Causal Dynamical Triangulation?

Causal Dynamical Triangulation (CDT) is a background independent approach to quantum gravity.

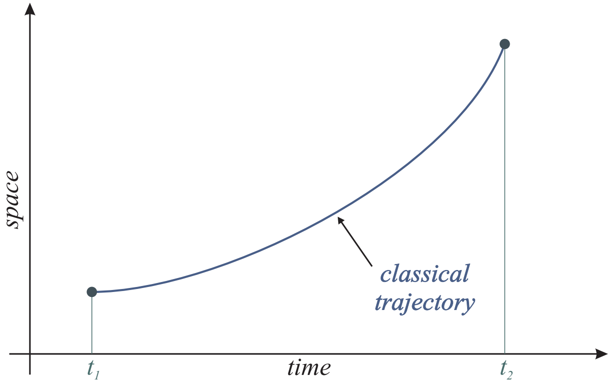
$$\int \mathrm{D}[g] e^{i S^{EH}[g]} \quad \rightarrow \quad \sum_{\mathcal{T}} e^{-S^R[\mathcal{T}]}$$

CDT provides a lattice regularization of the formal gravitational path integral via a sum over causal triangulations.



Path integral formulation of quantum mechanics

- A classical particle follows a unique trajectory.



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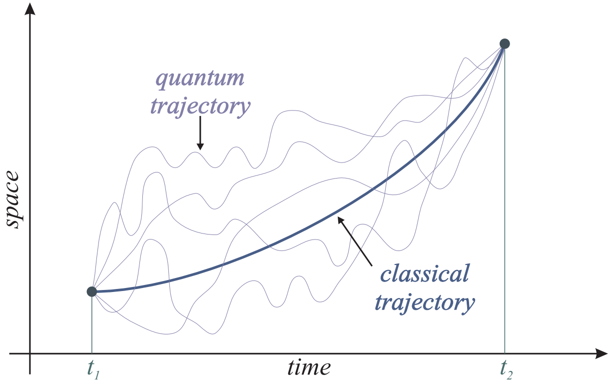
- *Quantum mechanics* can be described by *Path Integrals*: All possible trajectories contribute to the transition amplitude.



Path integral formulation of quantum mechanics

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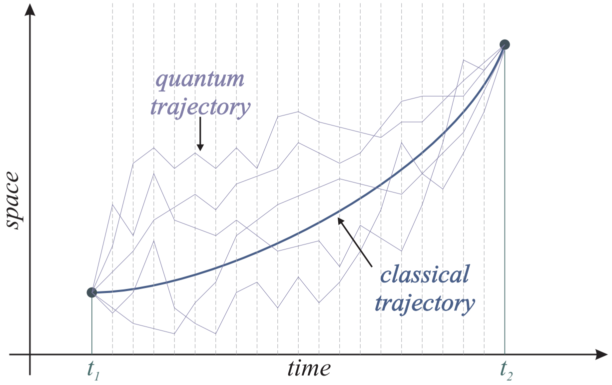
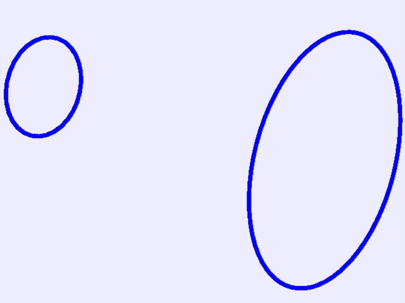
- *Quantum mechanics* can be described by *Path Integrals*: All possible trajectories contribute to the transition amplitude.
- To define the functional integral, we discretize the time coordinate and approximate any path by linear pieces.



Path integral formulation of quantum gravity

- General Relativity: gravity is encoded in space-time geometry.

1+1D Example: State of system: one-dimensional spatial geometry



Path integral formulation of quantum gravity

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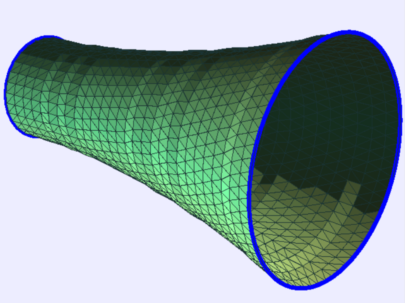
- The role of a trajectory plays now the geometry of four-dimensional space-time.

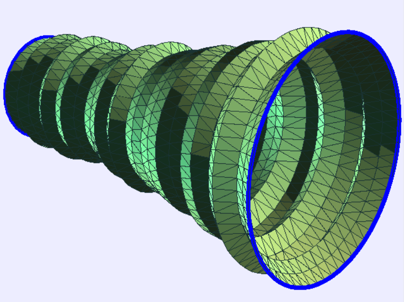
1+1D Example: Evolution of one-dimensional closed universe



Path integral formulation of quantum gravity

- General Relativity: gravity is encoded in space-time geometry.
- The role of a trajectory plays now the geometry of four-dimensional space-time.
- All space-time histories contribute to the transition amplitude.

Sum over all two-dimensional surfaces joining the in- and out-state



Transition amplitude

Our aim is to calculate the amplitude of a transition between two geometric states:

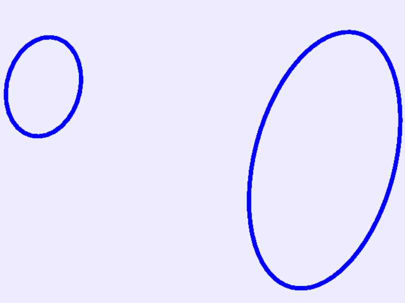
$$G(\mathbf{g}_i, \mathbf{g}_f, t) \equiv \int_{\mathbf{g}_i \to \mathbf{g}_f} \mathrm{D}[g] \mathrm{e}^{iS^{EH}[g]}$$

To define this path integral we have to specify the *measure* $\mathrm{D}[g]$ and the *domain of integration* - **a class of admissible space-time geometries** joining the in- and out- geometries.

Regularization by triangulation. Example in 2D

Dynamical **T**riangulations uses one of the standard regularizations in QFT: **discretization**.

1. One-dimensional **state** with a topology S^1 is built from *links* with length a.



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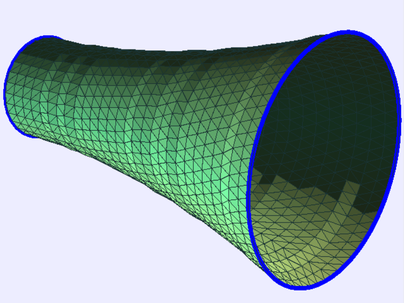

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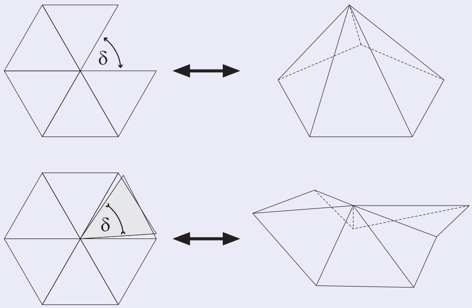
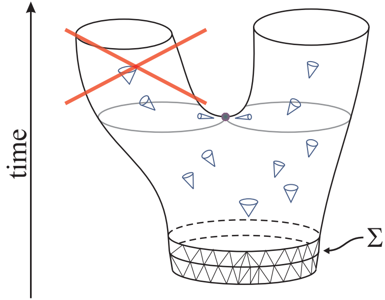
3. **Curvature** (angle defcit) is localized at vertices.



Causality

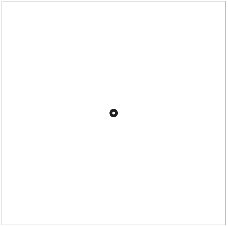
- **Causal D**ynamical **T**riangulations assume global proper-time foliation. Spatial slices (leaves) have fixed topology and are not allowed to split in time.
- Foliation distinguishes between time-like and spatial-like links.
- Such setup does not introduce causal singularities, which lead to creation of baby universes.
- CDT defines the class of admissible space-time geometries which contribute to the transition amplitude.



Fundamental building blocks of CDT

- d-dimensional simplicial manifold can be obtained by gluing pairs of d-simplices along their $(d-1)$-faces.
- Lengths of the time and space links are constant. Simplices have a fixed geometry.
- The metric is flat inside each d-simplex.
- The angle deficit (curvature) is localized at $(d-2)$-dimensional sub-simplices.

0D simplex - point



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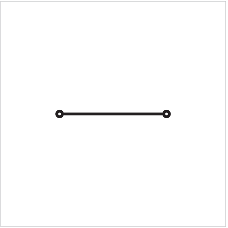
1D simplex - link



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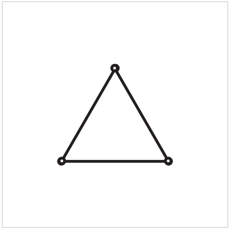
2D simplex - triangle



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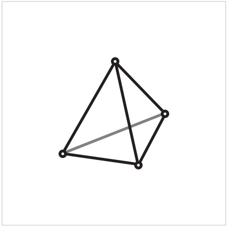
3D simplex - tetrahedron



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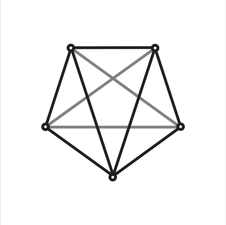
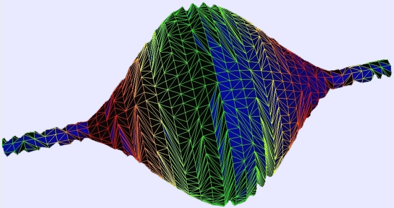
4D simplex - 4-simplex



Regularization by triangulation

- 4D simplicial manifold is obtained by gluing pairs of 4-simplices along their 3-faces.

4D space-time with topology $S^3 \times S^1$



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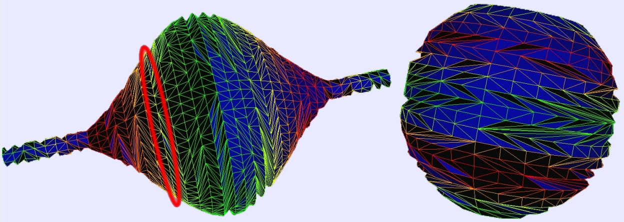
- 4D simplicial manifold is obtained by gluing pairs of 4-simplices along their 3-faces.
- Spatial states are $3\mathrm{D}$ geometries with a topology S^3. Discretized states are build from equilateral **tetrahedra**.

3D spatial slices with topology S^3



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- The metric is **flat** inside each 4-simplex.
- Length of time links a_t and space links a_s is constant.
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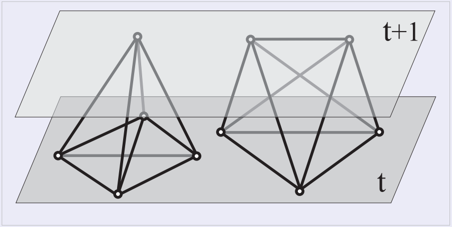
Fundamental building blocks of $4\mathrm{D}$ CDT - two types



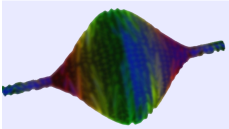
Regge action

The Einstein-Hilbert action has a natural realization on piecewise linear geometries called Regge action

$$S^E[g] = -\frac{1}{G}\int \mathrm{d}t \int \mathrm{d}^D x \sqrt{g}(R - 2\Lambda)$$

The partition function

$$\int \mathrm{D}[g] e^{iS^{EH}[g]} \quad \rightarrow \quad \sum_{\mathcal{T}} e^{-S^R[\mathcal{T}]}$$



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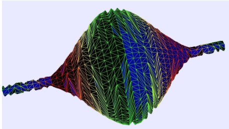
$$S^R[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$

N_0	number of vertices
N_4	number of simplices
N_{14}	number of simplices of type $\{1,4\}$
K_0 K_4 Δ	bare coupling constants $(G, \Lambda, a_t/a_s)$



Causal Dynamical Triangulations

- The partition function of quantum gravity is defined as a formal integral over all geometries weighted by the Einstein-Hilbert action.

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- To make sense of the gravitational path integral one uses the standard method of regularization - discretization.
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- The path integral is written as a nonperturbative sum over all causal triangulations $\mathcal{T}$.
- Wick rotation is well defined due to global proper-time foliation. $(a_t \to i a_t)$
- Using **Monte Carlo** techniques we can approximate expectation values of observables.

Numerical setup

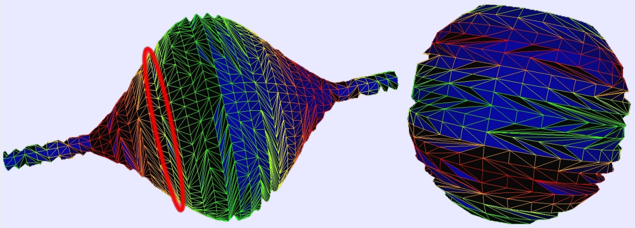
- **Monte Carlo** algorithm performs a random walk in the space of triangulations, it generates configurations with the probability $P[\mathcal{T}] = \frac{1}{Z} e^{-S[\mathcal{T}]}$.
- The walk consists of a series of 7 Pachner moves, which preserve topology and causality, are ergodic and fulfill the detailed balance condition. It is enough to know the probability functional $P(\mathcal{T})$ up to the normalization.
- To calculate the **expectation value of an observable**, we approximate the path integral by a sum over a finite set of Monte Carlo configurations

$$\begin{aligned} \langle \mathcal{O}[g] \rangle &= \frac{1}{Z} \int \mathcal{D}[g] \mathcal{O}[g] e^{-S[g]} \\ &\downarrow \\ \langle \mathcal{O}[\mathcal{T}] \rangle &= \frac{1}{Z} \sum_{\mathcal{T}} \mathcal{O}[\mathcal{T}] e^{-S[\mathcal{T}]} \approx \frac{1}{K} \sum_{i=1}^{K} \mathcal{O}[\mathcal{T}^{(i)}] \end{aligned}$$

Spatial slices

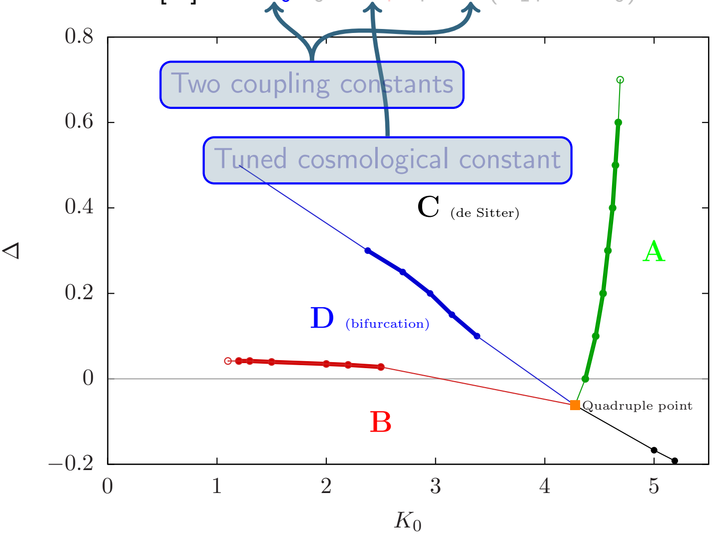
- The simplest observable giving information about the geometry, is the **spatial volume** $N(i)$ defined as a number of tetrahedra building a three-dimensional slice $i = 1 \dots T$.
- Restricting our considerations to the spatial volume $N(i)$ we reduce the problem to one-dimensional quantum mechanics.

3D spatial slices with topology S^3



Phase diagram

$$S[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



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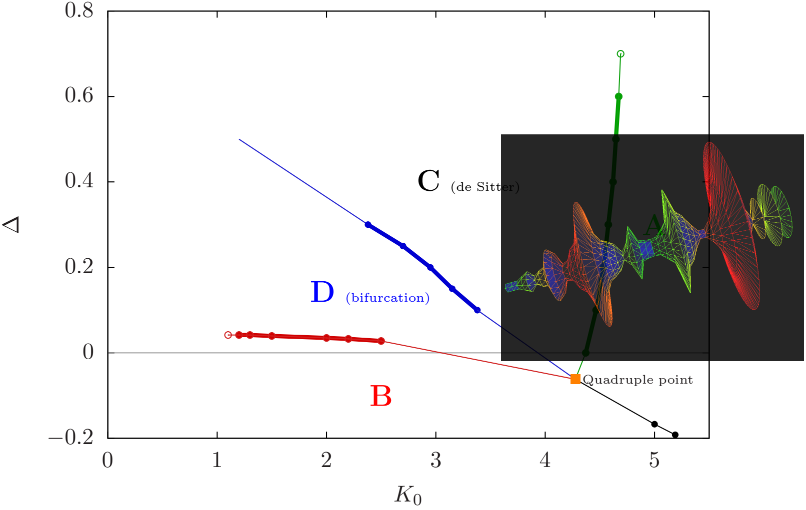

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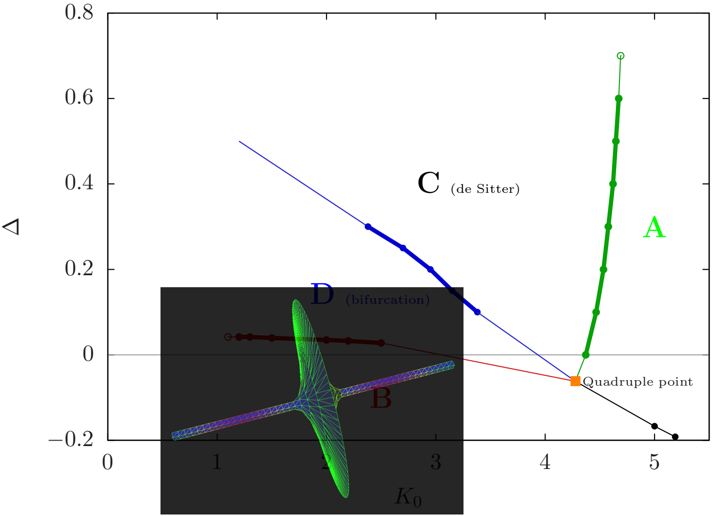

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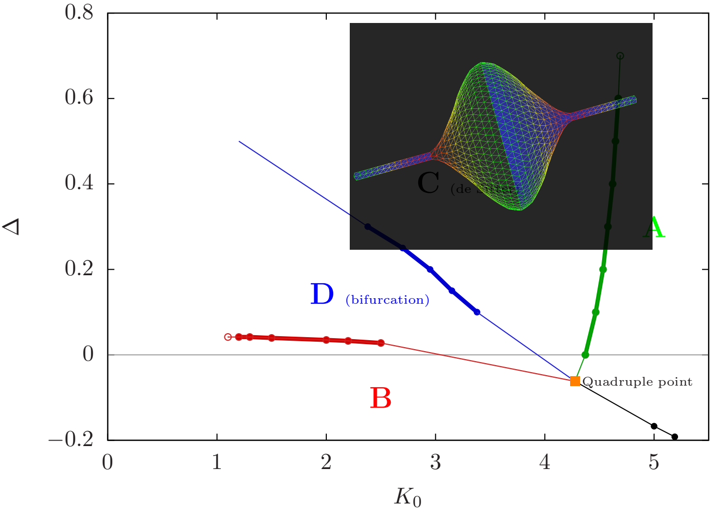

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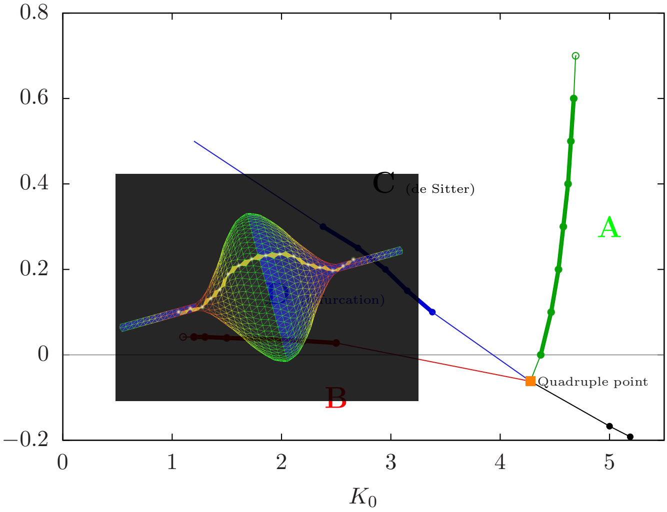
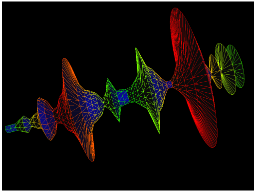
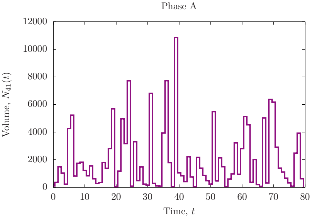
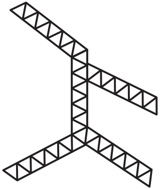


Phase A

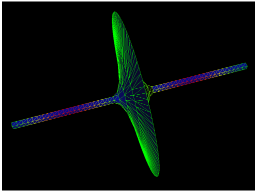


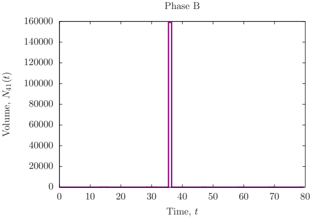


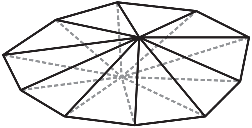


- Triangulations disintegrate into uncorrelated and irregular sequences of small "universes".
- This phase is related to the *branched polymers* phase which is present in Euclidean DT.
- The "geometry" *oscillates* in the time direction - analogy to the helicoidal phase of Lifshitz scalar model ($|\partial_t[g]| > 0$).

Phase B



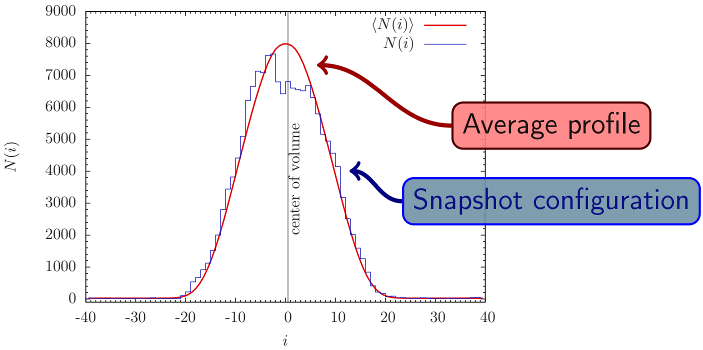




- Time dependence (of configurations) is reduced to a single time slice.
- The universe has neither time extension nor spatial extension. Hausdorff dimension $d_h = \infty$.
- Related to the *crumpled* phase in Euclidean DT.
- No geometry in classical sense - analogy to the paramagnetic phase of Lifshitz scalar model ($[g] = 0$).

De Sitter phase (C)

- In phase **C** the time translation symmetry is spontaneously broken and the three-volume profile $N(i)$ is bell-shaped.

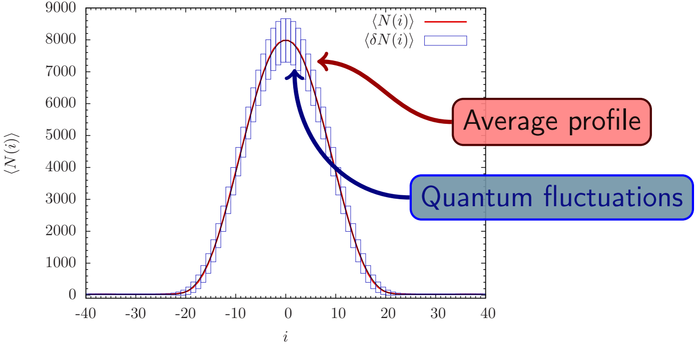


De Sitter phase (C)

- In phase **C** the time translation symmetry is spontaneously broken and the three-volume profile $N(i)$ is bell-shaped.
- The average volume $\langle N(i) \rangle$ is with high accuracy given by formula

$$\langle N(i) \rangle = H \cos^3 \left(\frac{i}{W} \right)$$

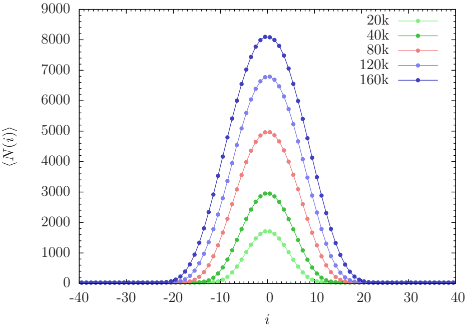
a classical **vacuum solution**.

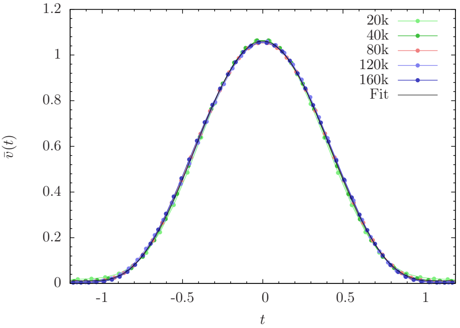


Hausdorff dimension

The time coordinate i and spatial volume $\langle N(i) \rangle$ scale with total volume N_4 as a genuine four-dimensional *Universe*,

$$t = N_4^{-1/4} i,$$
$$\bar{v}(t) = N_4^{-3/4} \langle N(i) \rangle = \frac{3}{4\omega} \cos^3 \left(\frac{t}{\omega} \right).$$





Effective action

- The average volume profile

$$v(t) = \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right)$$

corresponds to an Euclidean de Sitter space. It is a classical (maximally symmetric) solution of the minisuperspace action

$$S[v] = \frac{1}{G} \int \frac{\dot{v}^2}{v} + v^{1/3} - \lambda v \, \mathrm{d}t,$$

which is obtained from the Einstein-Hilbert action by „freezing" all degrees of freedom except the scale factor.

- Simulations inside phase C show that its discrete form

$$S[N] = \frac{1}{\Gamma} \sum_i \left(\frac{(N(i+1) - N(i))^2}{N(i+1) + N(i)} + \mu N(i)^{1/3} - \lambda N(i) \right)$$

also properly describes quantum fluctuations of the three-volume $N(i)$.

Effective action

Background space-time geometry

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Couples only adjacent slices

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Effective transfer matrix

- The effective action suggests the existence of an effective transfer matrix M labeled by the scale factor

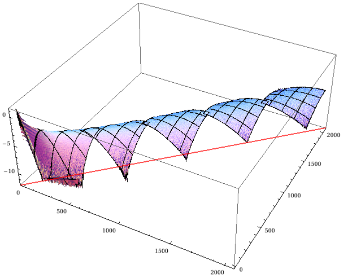
Directly measured

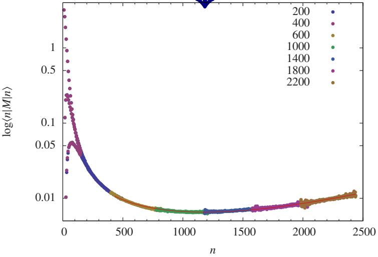
$$P(\{N(i)\}) = \frac{1}{Z} \underbrace{\langle N(1)|M|N(2)\rangle\langle N(2)|M|N(3)\rangle \cdots \langle N(T)|M|N(1)\rangle}_{e^{-S[N]}}$$

Product of matrix elements

$$\langle n|M|m\rangle = \mathcal{N} e^{-\frac{1}{\Gamma}\left[\frac{(n-m)^2}{n+m} + \mu\left(\frac{n+m}{2}\right)^{1/3} - \lambda\frac{n+m}{2}\right]}$$

Potential term





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- The effective action suggests the existence of an effective transfer matrix M labeled by the scale factor

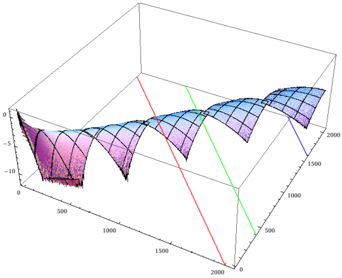
Directly measured

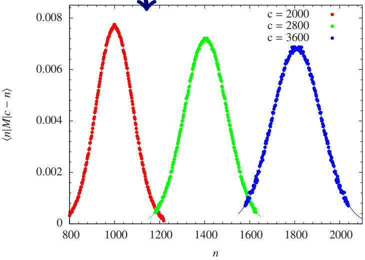
$$P(\{N(i)\}) = \frac{1}{Z} \underbrace{\langle N(1)|M|N(2)\rangle\langle N(2)|M|N(3)\rangle \cdots \langle N(T)|M|N(1)\rangle}_{e^{-S[N]}}$$

Product of matrix elements

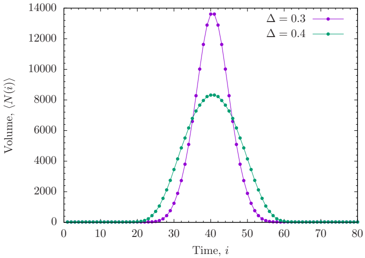
$$\langle n|M|m\rangle = \mathcal{N} e^{-\frac{1}{\Gamma}\left[\frac{(n-m)^2}{n+m} + \mu\left(\frac{n+m}{2}\right)^{1/3} - \lambda\frac{n+m}{2}\right]}$$

Kinetic term, gaussian



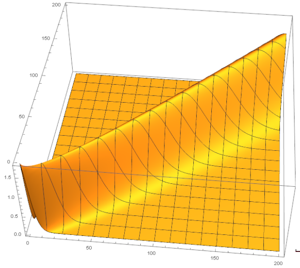


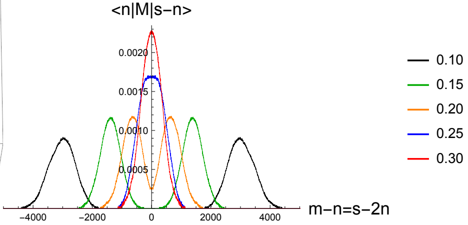
New bifurcation phase (D)



- The spatial volume profile $\langle N(i) \rangle$ is similar as in phase C.

New bifurcation phase (D)





- The spatial volume profile $\langle N(i) \rangle$ is similar as in phase C.
- However, the transfer matrix bifurcates into two branches. At some volume the kinetic term splits into a sum of two shifted Gausses.

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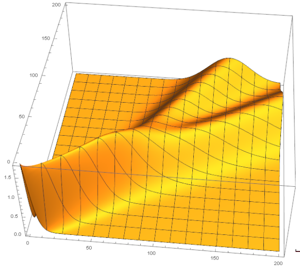

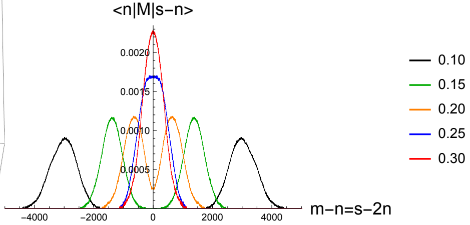

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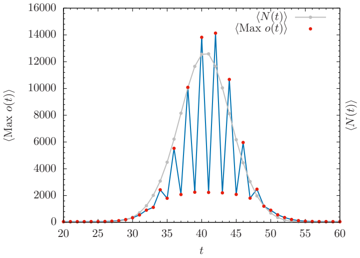

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- However, the transfer matrix bifurcates into two branches. At some volume the kinetic term splits into a sum of two shifted Gausses.
- In every second slice, there emerges a vertices of very high order.

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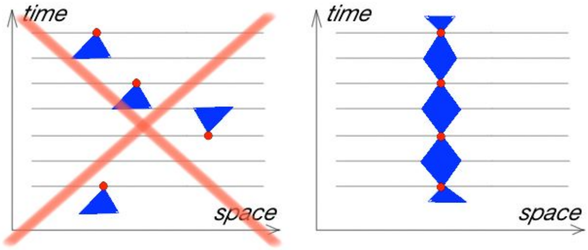
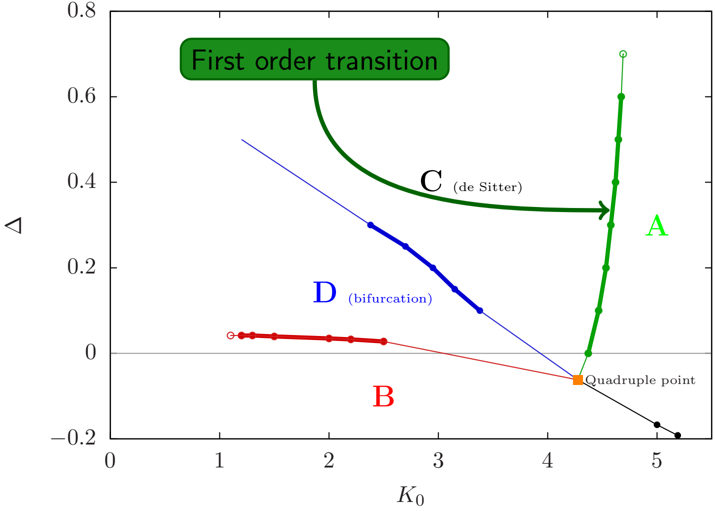


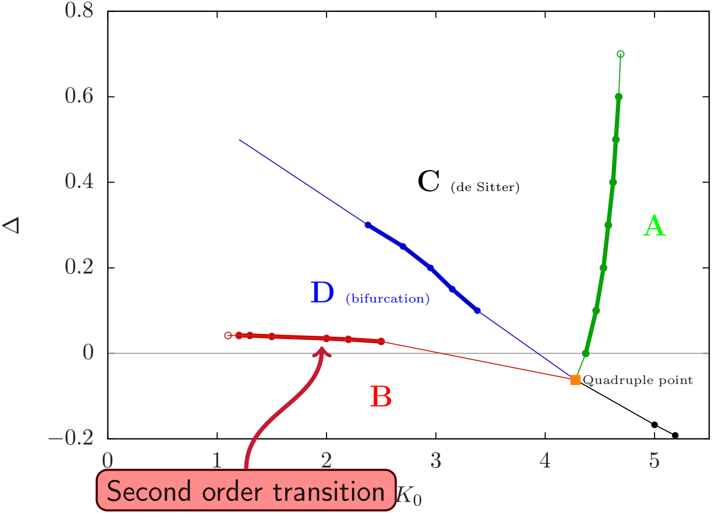
- The spatial volume profile $\langle N(i) \rangle$ is similar as in phase C.
- However, the transfer matrix bifurcates into two branches. At some volume the kinetic term splits into a sum of two shifted Gausses.
- In every second slice, there emerges a vertices of very high order.
- Periodic clusters of volume around singular vertices form a tube structure.
- Not captured by global properties of the triangulation.

Phase transition lines



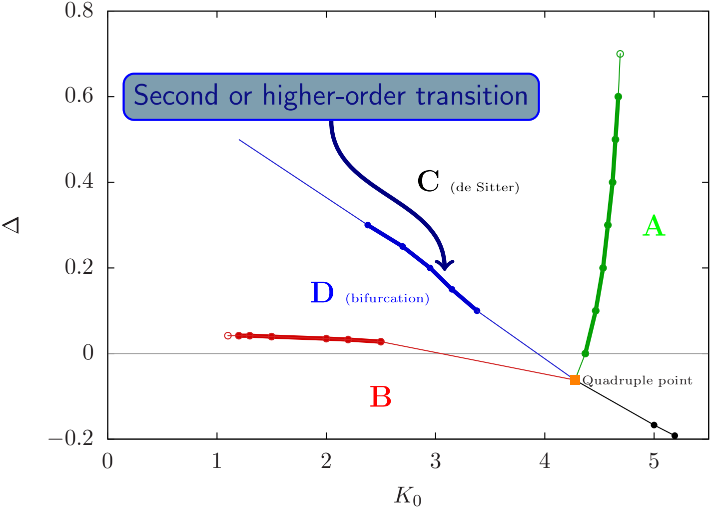
Order parameter: N_0

Phase transition lines



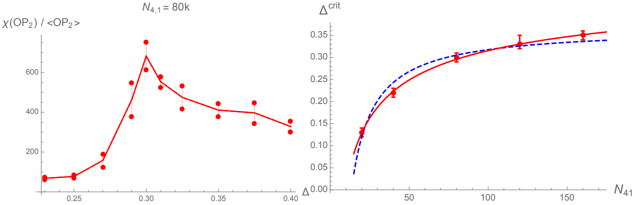
Order parameter: N_{32}

Phase transition lines



Order parameter: $\mathrm{Max}\ o(p)$

D/C transition



$$OP_2 = |\mathrm{Max}\ o_t - \mathrm{Max}\ o_{t+1}|$$

- Peak of the susceptibility $\chi(OP_2)$ gives a clear signal of the phase transition.
- The details of the microscopic geometry play important role in the bifurcation transition.
- Preliminary measurements of the position of critical point Δ^{crit} for different total volumes N_4, suggest a second or higher-order transition. (estimate of the cricial exponent $\nu \approx 2.6 \pm 0.6$)

$$\Delta^{crit}(N_4) = \Delta^{crit}(\infty) - \alpha \cdot N_4^{-1/\nu}$$

Summary

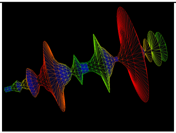
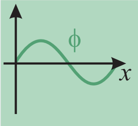
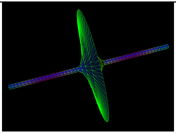
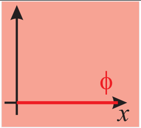
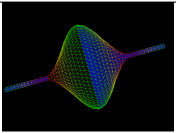
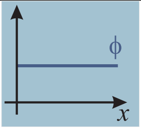
- The model of **C**ausal **D**ynamical **T**riangulations is a lattice approach to quantum gravity.
- In phase C a **four-dimensional** background geometry emerges dynamically. It corresponds to the Euclidean **de Sitter** space, i.e. **classical solution** of the minisuperspace model.
- The superimposed **quantum fluctuations** of the scale factor are described by the **minisuperspace model**.
- We have presented an up-to-date phase diagram. It includes a recently discovered **bifurcation** phase.
- The new phase is characterized by a bifurcation in the transfer matrix, vertices of high order and a propagating structure of clusters.
- Importance of the microscopic details of the geometry explaining why the transition went unnoticed.

Thank You!

Comparison of Lifshitz scalar and CDT phases

Landau free-energy-density:

$$F[\phi(x)] = a_2\phi^2 + a_4\phi^4 + c_2(\partial_\alpha\phi)^2 + d_2(\partial_\beta\phi)^2 + e_2(\partial_\beta^2\phi)^2$$

		A Helicoidal	$m = 0$	$a_2 > 0$	$\phi = 0$
		B Para	$m > 0$	$d_2 < 0$	$\lvert\partial_t\phi\rvert > 0$
		C Ferro	$m = 0$	$a_2 < 0$	$\lvert\phi\rvert > 0$