Renormalization Group Flows in Tensorial Group Field Theories

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Quantum gravity from random geometries

Quantum gravity from "lattice/discrete" models
(causal dynamical triangulations, Regge calculus, spin foams/group field theories ...)
$\Rightarrow$ problem of continuum limit

Important aspects:

- Phase diagram
- Order of phase transitions
- Small vs large scale properties

$\Rightarrow$ it sounds like a big call for the **Renormalization Group**

This talk:

an FRG perspective on TGFT

Disclaimer:
not much to say about QG in this talk $\Rightarrow$ a field theory divertissement

Overview

- Introduction: a different perspective on the $O(N)$ model
- FRG approach to critical behavior in tensorial group field theories
  [with J. Ben Geloun and D. Oriti, arXiv:1411.3180; and with V. Lahoche, arXiv:1508.06384]

Non-locality in QFT

Locality is a fundamental postulate of quantum field theory.

What happens if we abandon it?

It depends ...

Many ways to get away from locality, of course leading to different situations:

- Strings, holography...
- Non-locality in the effective action (IR physics)
- Non-locality in the propagator as an UV regulator (to be removed or to be viewed as fundamental, e.g. non-local gravity or non-commutative field theory)
- **The GFT way**

The $O(N)$ model in d dimensions

- Fields:
  $\phi_a(x) : \mathbb{R}^d \to \mathbb{R}, \quad a = 1 \dots N$
- Vector-like transformation:
  $\phi_a(x) \to \sum_b R_{ab}\phi_b(x), \quad R_{ab} \in O(N)$
- Invariants:
  $\phi^2(x) \equiv \frac{1}{2}\sum_a \phi_a(x)\phi_a(x)$, and similar but with derivatives
- Effective action:
  $\Gamma[\phi] = \int d^dx \left[V(\phi^2) + \frac{1}{2}Z(\phi^2)\partial_\mu\phi_a\partial^\mu\phi_a + \mathcal{O}(\partial^4)\right]$
- FRG equation, e.g. in the Local Potential Approximation:

$$k\partial_k\tilde{V}_k(\tilde{\phi}^2)+d\,\tilde{V}_k(\tilde{\phi}^2)-(d-2)\tilde{\phi}^2\tilde{V}_k'(\tilde{\phi}^2) = \frac{(N-1)c_d}{1+\tilde{V}_k'(\tilde{\phi}^2)} + \frac{c_d}{1+\tilde{V}_k'(\tilde{\phi}^2)+2\tilde{\phi}^2\tilde{V}_k''(\tilde{\phi}^2)}$$

The $O(N)$ model in $d = 0$

1. Mass-induced flow

Formal limit $d \to 0$ in the FRG equation
$=$ flow of ordinary N-dim integral wrt k from "mass term" $k^2\phi^2$

(physical interpretation: effective description of IR flow for the model on a compact space, or in dS) [DB '14; Guilleux,Serreau '15]

$$e^{W_k[J]} = \int \left(\prod_{a=1}^{N} d\phi_a \right) e^{-V(\phi^2) - k^2\phi^2 + \sum_a J_a \phi_a}$$

The $O(N)$ model in $d = 0$

1. Mass-induced flow
2. Wilsonian flow directly at $d = 0$

   Start from $d = 0$ and perform the N integrals one at a time, in the spirit of Wilsonian RG

   [Higuchi,Itoi,Sakai '93; Zinn-Justin '14]

   Same idea originally proposed for Matrix Models by Brezin and Zinn-Justin

   [Brezin,Zinn-Justin '92; Eichhorn,Koslowski '13-'14]

$$e^{-S_{N'}} = \int \left(\prod_{a=N'}^{N} d\phi_a \right) e^{-V(\phi^2)}$$

A different perspective on the vector index

- Complex case: $\phi_a \in \mathbb{C}$ and
  $\phi_a \to \sum_b U_{ab}\phi_b, \quad \bar{\phi}_a \to \sum_b \bar{U}_{ab}\bar{\phi}_b \qquad U_{ab} \in U(N)$
  $\Rightarrow \phi^2 \equiv \sum_a \bar{\phi}_a \phi_a$

- Start from $d = 0$, and map $a = 1, \ldots N \;\; \to \;\; p = -N', \ldots N'$.
  $\Rightarrow$ Theory on S^1 with UV cutoff N'

- Non-compact limit: $\quad S^1 \to \mathbb{R} \quad \Rightarrow \quad \phi_a \to \phi_p$ with continuous momentum p

$$\Rightarrow \quad \phi^2 = \int dp\, \bar{\phi}_p \phi_p = \int dx\, \bar{\varphi}(x)\varphi(x)$$

  where $\varphi(x) = \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} \phi_p$ is the Fourier transform

- ϕ^2 is the only possible local Fourier-invariant potential

- Note: invariance under $\lim_{N\to\infty} U(N)$ is larger than Fourier invariance, but the latter is sufficient for restricting the action

A Fourier-invariant field theory and its soft breaking

- Give up locality: generic $V(\phi^2) = V(\int dp\, \bar{\phi}_p \phi_p)$
- In order to introduce a dynamics and launch a perturbative RG flow, break it with a kinetic term

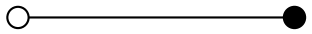
$$\int dp\, \bar{\phi}_p p^2 \phi_p + V(\phi^2)$$

  similar to soft breaking of (ultra-)locality by the kinetic term in standard scalar field theory
- Note: symmetry $x \leftrightarrow p$ can be restored by adding a term like $\int dp\, \bar{\phi}_p \frac{\partial^2}{\partial p^2} \phi_p = \int dx\, \bar{\varphi}(x) x^2 \varphi(x)$ (harmonic oscillator potential as in non-commutative scalar QFT [Grosse,Wulkenhaar '03])
- Apply your favorite machinery (e.g. multiscale loop vertex expansion [Gurau,Rivasseau '15])

Higher dimensions

Fourier invariance $\Rightarrow$ main constraint: pairwise identification of points

In one dimension it is quite boring: only one invariant, $\bar{\phi}_a \phi_a$



Higher dimensions

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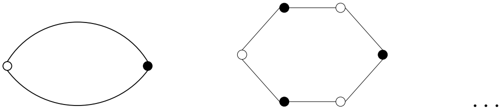
In two dimensions we have matrix model type of interactions,
$\mathrm{Tr}[(M\bar{M})^n] = M_{a_1 a_2} \bar{M}_{a_2 a_3} M_{a_3 a_4} \ldots \bar{M}_{a_n a_1}$



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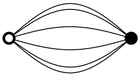
In higher dimensions we get many more possible interactions:

$$\int d^6x\, \psi(x_1, x_2, x_3, x_4, x_5, x_6)\, \bar{\psi}(x_1, x_2, x_3, x_4, x_5, x_6)$$



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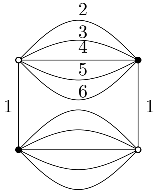
$$\int d^6x \, d^6x' \, \psi(x_1, x_2, x_3, x_4, x_5, x_6) \, \bar{\psi}(x'_1, x_2, x_3, x_4, x_5, x_6) \\ \times \psi(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6) \, \bar{\psi}(x_1, x'_2, x'_3, x'_4, x'_5, x'_6)$$



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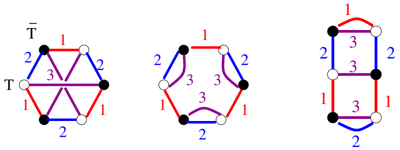


⇒ Tensorial Group Field Theories (TGFTs)

Tensorial Group Field Theories

- GFT is a field theory on $G^{\otimes D}$, for some group G
- TGFT is a GFT with same combinatorial structure as tensor models
- Tensor models are TGFTs with $G = U(1)$, a sharp UV cutoff N on momenta ($p_i \in \mathbb{Z}$), and a trivial (ultralocal) propagator (just a mass term)
- A first step towards richer TGFTs: nontrivial propagator, $\Delta = \sum_i |p_i|^\alpha$
- Propagator softly breaks the invariance of the action under $U(N)^{\otimes D}$.
  Similar to how propagator breaks the ultralocal nature of interactions in typical (scalar) field theories.
- This breaking naturally induces a renormalization group flow of the theory
- Non-compactness is in general problematic, it needs regularization (e.g. go back from $G = \mathbb{R}$ to $S^1 \simeq U(1)$)
  [Ben Geloun, Martini, Oriti '15]

TGFT - the typical actions

$$S[\phi] = Z\,\mathrm{Tr}_2(\bar{\phi}\cdot K\cdot\phi) + m\mathrm{Tr}_2(\bar{\phi}\phi) + S_{\mathrm{int}}[\phi]$$

$$\mathrm{Tr}_2(\bar{\phi}\cdot K\cdot\phi) = \sum_{\mathbf{P},\,\mathbf{P}'} \bar{\phi}_{\mathbf{P}}\,K(\mathbf{P},\,\mathbf{P}')\,\phi_{\mathbf{P}'}\,, \qquad \mathrm{Tr}_2(\phi^2) = \sum_{\mathbf{P}} \bar{\phi}_{\mathbf{P}}\phi_{\mathbf{P}}\,,$$

$$S_{\mathrm{int}}[\phi] = \sum_{n_b} \lambda_{n_b}\mathrm{Tr}_{n_b}(\mathcal{V}_{n_b}\cdot\phi^{n_b})\,,$$

e.g. rank-3 model [DB, Ben Geloun, Oriti '14]

$$K(\mathbf{P},\,\mathbf{P}') = \delta_{\mathbf{P},\mathbf{P}'}(\frac{1}{3}\sum_{i=1}^{3}|p_i|)\,, \qquad \delta_{\mathbf{P},\mathbf{P}'} := \prod_{i=1}^{3}\delta_{p_i,p_i'}\,, \qquad \mathrm{Tr}_2(\phi^2) = \sum_{p_i\in\mathbb{Z}}\phi_{123}^2\,,$$

$$S_{\mathrm{int}}[\phi] = \frac{\lambda}{4}\mathrm{Tr}_4(\phi^4) := \frac{\lambda}{4}\Big(\sum_{p_i,p_i'\in\mathbb{Z}}\phi_{123}\,\phi_{1'23}\,\phi_{1'2'3'}\,\phi_{12'3'} + \mathrm{Sym}(1\to 2\to 3)\Big)$$

(proven to be renormalizable [Ben Geloun, Samary '12])

RG flow in tensorial group field theories

1. Perturbative approach [Ben Geloun; Carrozza; Lahoche; Oriti; Rivasseau; etc 2012-2015]

   $\Rightarrow$ Existence of renormalizable models
   ($\Rightarrow$ breaking of unitary invariance does not lead to disaster!)

   $\Rightarrow$ Asymptotic freedom is quite a general feature of (renormalizable) TGFTs
   (Note: wave function renormalization already at one loop, crucial to AF)

2. FRG [DB; Ben Geloun; Oriti; Lahoche; Martini; '14-'15]

   $\Rightarrow$ Wilson-Fisher-type fixed point is also rather common in TGFTs

Very interesting situation from a field theory point of view:

asymptotic freedom and IR fixed point!

FRG for TGFT

Usual FRG approach (truncation ansatz for the Wetterich equation, expand the rhs, identify operators, obtain beta functions)

After a lengthy calculation (using Litim's cutoff; $t = \ln N$):

$$\eta \equiv \partial_t \ln Z_N = \frac{54\lambda_N N(2N+1)}{3(N+m_N)^2 - 2\lambda_N(27N^2+18N+5)}$$

$$\partial_t m_N = \frac{-\lambda_N N}{(N+m_N)^2}\Big[\eta\,(18N^2+9N+4)+(54N^2+36N+9)\Big] - \eta\, m_N$$

$$\partial_t \lambda_N = \frac{2\lambda_N^2 N}{(N+m_N)^3}\Big(\frac{1}{3}\eta\,(18N^2+45N+25)+18(N+1)^2\Big) - 2\,\eta\,\lambda_N$$

Due to the polynomials in N in the beta functions, it is impossible to find a rescaling of the couplings with N that would lead to an autonomous system

This is a consequence of hidden external scale: radius of S^1
($N = kL$, if L is the radius of S^1 and k the usual FRG scale)

$\Rightarrow$ As in FRG at finite temperature, or on a sphere, we obtain a non-autonomous system

β-equations at large N

At large N, polynomials reduce to leading monomials
$\Rightarrow$ it is possible to obtain an autonomous system
Define a rescaled mass via

$$m_N = N\bar{m}_N$$

(no rescaling for λ_N, which is therefore a "dimensionless" coupling)
$\Rightarrow$ obtain the following autonomous system:

$$\eta = \frac{36\lambda_N}{(1+\bar{m}_N)^2 - 18\lambda_N}$$

$$\partial_t \bar{m}_N = -18\frac{\lambda_N}{(1+\bar{m}_N)^2}(\eta+3) - (1+\eta)\,\bar{m}_N$$

$$\partial_t \lambda_N = 12\frac{\lambda_N^2}{(1+\bar{m}_N)^3}(\eta+3) - 2\,\eta\,\lambda_N$$

Fixed points at large N

We find the usual Gaussian fixed point (GFP) at $\bar{m}_N = \lambda_N = 0$, plus two non-Gaussian fixed points (NGFPs) at

$$\bar{m}^*_\pm = -\frac{14 \pm \sqrt{7}}{21} \simeq \begin{cases} -0.7926 \\ -0.5407 \end{cases}, \quad \lambda^*_\pm = -\frac{\bar{m}^*_\pm}{189} \simeq 10^{-3} \times \begin{cases} 4.193 \\ 2.861 \end{cases}$$

At quadratic order in the couplings near the GFP we have

$$\partial_t \bar{m}_N \simeq -\bar{m}_N - 54\lambda_N + 72\bar{m}_N\lambda_N - 648\lambda_N^2 \,,$$

$$\partial_t \lambda_N \simeq -36\lambda_N^2 \,.$$

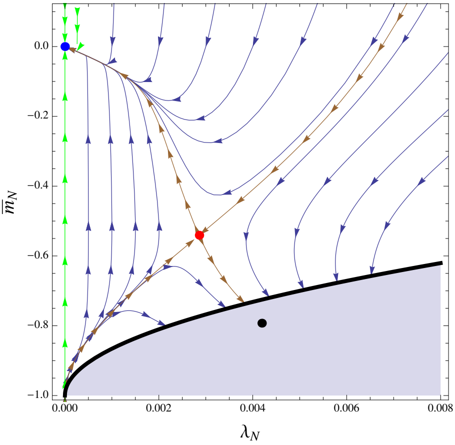
⇒ asymptotic freedom

For the NGFP at $\{\bar{m}^*_-, \lambda^*_-\}$ the critical exponents are

$$\theta^{(-)}_\pm = \frac{1}{4}\left(17\sqrt{7} - 47 \pm \sqrt{2776 - 1038\sqrt{7}}\right) \simeq \begin{cases} 0.8571 \\ -1.868 \end{cases},$$

⇒ one relevant and one irrelevant eigen-perturbation.

Flow diagram at large N



(Arrows point towards the UV, i.e. growing N)

β-equations at small N

Same rescaling for the mass, plus

$$\lambda_N = N^2 \bar{\lambda}_N$$

from which we obtain

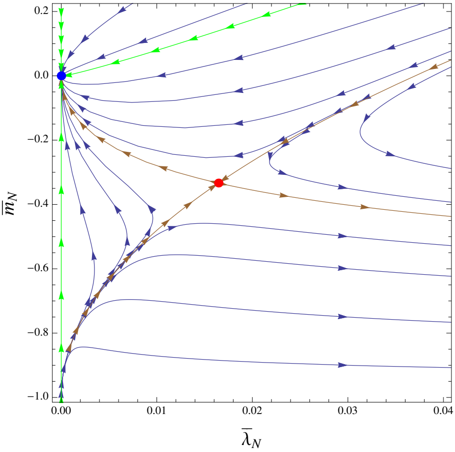
$$\partial_t \bar{m}_N = -\bar{m}_N - 9\frac{\bar{\lambda}_N}{(1+\bar{m}_N)^2}\,,$$

$$\partial_t \bar{\lambda}_N = -2\bar{\lambda}_N + 36\,\frac{\bar{\lambda}_N^2}{(1+\bar{m}_N)^3}\,.$$

$\Rightarrow$ scaling dimensions of a zero-dimensional theory

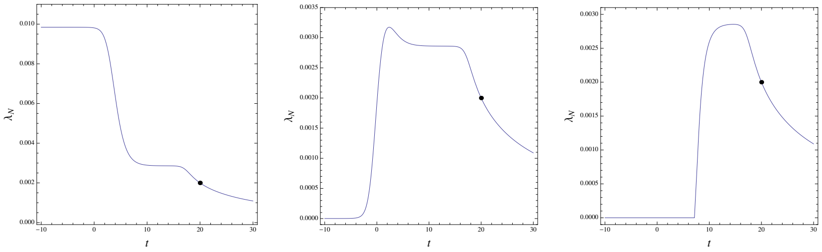
(typical phenomenon on compact spaces)

Flow diagram at small N



Flow trajectories at general N

unrescaled coupling



Flow trajectories for λ_N versus t, near the critical line. The three curves are for initial conditions (black dot) at $t_0 = 20$ with $\lambda_{N_0} = 0.002$ and (from left to right) $\bar{m}_{N_0} = -0.22133$, $\bar{m}_{N_0} = -0.221337043$ and $\bar{m}_{N_0} = -0.2214$. The curve in the middle is at criticality (the renormalised mass reaches zero with high accuracy). The flat plateau at $\lambda_N = 0$ in the last curve is instead due to the mass hitting the singularity, and it is a manifestation of the bad parametrization in the broken phase.

Other results and outlook

Almost identical picture in very different models:

- rank-6 model with closure constraint (imposing an additional gauge invariance)
  [DB, Lahoche '15]
- non compact rank-3 model with different propagator
  [Ben Geloun, Martini, Oriti '15]

$\Rightarrow$ Quite general feature of TGFTs ?

But it is hard to systematically extend the truncation
$\Rightarrow$ can we bring the WF fixed point close to the GFP and study it in perturbation theory?

Idea: dimensional continuation, but with propagator $\sim p^{-n(d)}$, in order to preserve asymptotic freedom, and look for d_c at which NGFP $\to$ GFP
$\beta(\lambda) = -b_1(d)\lambda^2 + b_2(d)\lambda^3 \Rightarrow \lambda^* = b_1/b_2$,
if $\exists d_c$ s.t. $b_1(d_c) = 0$ and $b_2(d_c) > 0$, then expand in $\epsilon = d - d_c$
[DB, Lahoche -work in progress]

Back to quantum gravity?

- The non-trivial IR fixed point has one relevant perturbation, whose sign determines whether we are in a symmetric ($\langle\phi\rangle = 0$) or broken ($\langle\phi\rangle \neq 0$) phase
- Geometrogenesis?
  Symmetric phase = no extended geometry
  Broken phase = extended (classical?) geometry
- Relevant to Cosmology from GFT? [Gielen, Oriti, Sindoni - 2013]
- Non-compact limit needed for true phase transition:
  a motivation for Lorentz group in GFT?

Conclusions

Main messages:

- Tensor models are non-local field theories whose Feynman diagrams have the interpretation of random simplicial manifolds
- Phase transitions play a crucial role in the search of continuum random geometries in tensor models and TGFT
- They can be studied by functional RG methods
- In TGFT we find asymptotic freedom, and an IR fixed point associated to a phase transition between a symmetric and a broken phase

Outlook:

- More in depth study of Wilson-Fisher fixed point in TGFT
- Geometric interpretation of broken phase in TGFT?