Finite density QCD from an effective lattice theory

HIC for FAIR
Helmholtz International Center

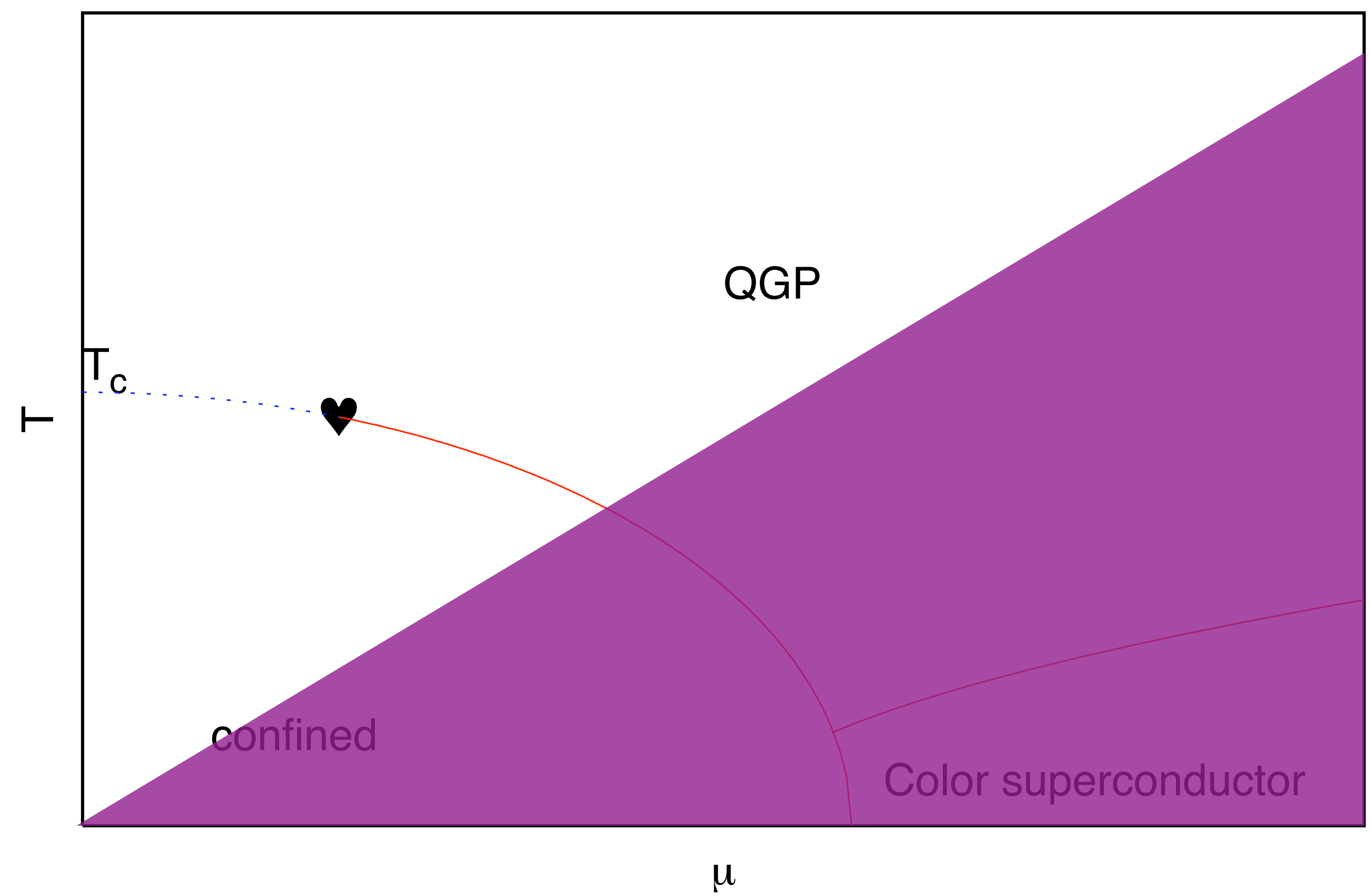
Owe Philipsen

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

in collaboration with M. Fromm, J. Langelage, S. Lottini, M. Neuman

- 3d effective lattice theories derived by strong coupling methods
- The deconfinement transition in Yang-Mills theory JHEP 1102 (2011) 057
- The deconfinement transition in QCD with heavy dynamical quarks JHEP 1201 (2012) 042
- Cold and dense QCD: transition to nuclear matter arXiv:1207.3005

The (lattice) calculable region of the phase diagram



- Sign problem prohibits direct simulation, circumvented by approximate methods: reweigthing, Taylor expansion, imaginary chem. pot., need $\mu/T \lesssim 1 \quad (\mu = \mu_B/3)$
- No critical point in the controllable region
- Flux representations + worm algorithm, complex Langevin: only particular models

Large densities on the lattice? Effective theories!

- E.w. phase transition: success with dimensional reduction!
- Scale "separation": $g^2T < gT < 2\pi T$

  Integrate hard scale perturbatively, treat eff. 3d theory on lattice,
  valid for sufficiently weak coupling

- Does not work for QCD, perturbative dim. red. breaks Z(3) of YM theory
- Bottom-up construction of Z(N)-invariant theory by matching:

  works for SU(2), unfinished for SU(3)
  Vuorinen, Yaffe; de Forcrand, Kurkela; Kurkela, Vuorinen; ....

- Here: solution for YM by strong coupling expansion!
- QCD, heavy fermions: sign problem of eff. theory milder for reweighting
- Sign problem of eff. theories curable by worm or complex Langevin!

Starting point: Wilson's lattice Yang-Mills action

Partition function; link variables as degrees of freedom

$$Z = \int \prod_{x,\mu} dU(x;\mu) \exp(-S_{YM}) \equiv \int DU \exp(-S_{YM})$$

Wilson's gauge action

$$S_W = -\frac{\beta}{N} \sum_p \mathrm{Re}\mathrm{Tr}(U_p) = \sum_p S_p \qquad \beta = \frac{2N}{g^2}$$

Plaquette:

$$\square \rightarrow 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \ldots$$

$U_\mu(x) = \mathrm{e}^{-iagA_\mu(x)}$

$$T = \frac{1}{aN_t}$$ continuum limit $a \rightarrow 0, N_t \rightarrow \infty$

Small $\beta(a) \Rightarrow$ small T

The effective theory, Yang-Mills

- Split temporal and spatial link integration and use character expansion ($a_r(\beta)$: expansion parameter of representation r)

$$
\begin{aligned}
Z &= \int [dW] \exp\left\{ \ln \int [dU_i] \prod_p \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right] \right\} \\
&\equiv \int [dW] \exp\left[-S_{eff} \right] \qquad W(\vec{x}) = \prod_{\tau=1}^{N_\tau} U_0(\tau, \vec{x})
\end{aligned}
$$

Expansion parameter: $u = a_f(\beta) = \beta/18 + \cdots$

$$
-S_{eff} = \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 + \ldots
$$

- S_n depend only on Polyakov loops

- Leading order graph in case of $N_\tau = 4$:

Figure: 4 plaquettes in fundamental representation lead to a 2 Polyakov loop interaction term

- Integration of spatial link variables leads to

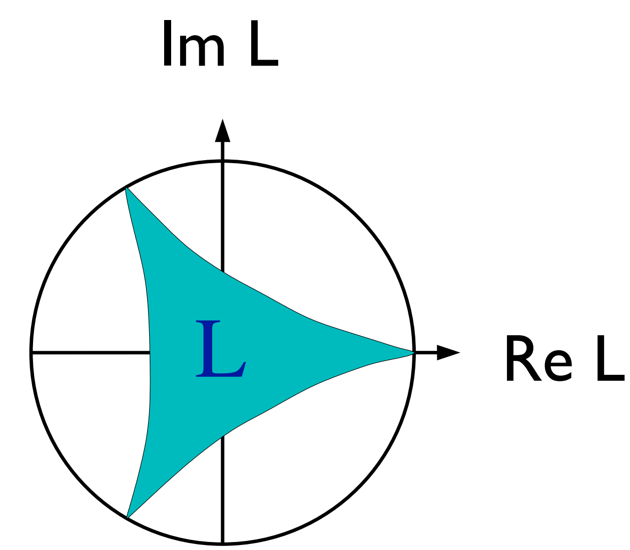
$$-S_1 = u^{N_\tau} \sum_{\langle ij \rangle} \mathrm{tr}\, W_i\, \mathrm{tr}\, W_j$$

- Possible generalizations: larger distance, higher dimensional representations, larger number of loops involved, ...
- *Here*: Decorate LO graph with additional spatial and temporal plaquettes

Effective one-coupling theory for SU(3) YM

(L= Tr W)

$$Z = \int [dL] \exp\left[-S_1 + V_{SU(3)}\right]$$
$$= \int [dL] \prod_{\langle ij \rangle} \left[1 + 2\lambda_1 \mathrm{Re}\left(L_i L_j^*\right)\right] *$$
$$* \prod_i \sqrt{27 - 18|L_i|^2 + 8\mathrm{Re}L_i^3 - |L_i|^4}$$



$$\lambda(u, N_\tau \geq 5) = u^{N_\tau} \exp\left[N_\tau\left(4u^4 + 12u^5 - 14u^6 - 36u^7 + \frac{295}{2}u^8 + \frac{1851}{10}u^9 + \frac{1055797}{5120}u^{10}\right)\right]$$

Subleading couplings

Subleading contributions for next-to-nearest neighbours:

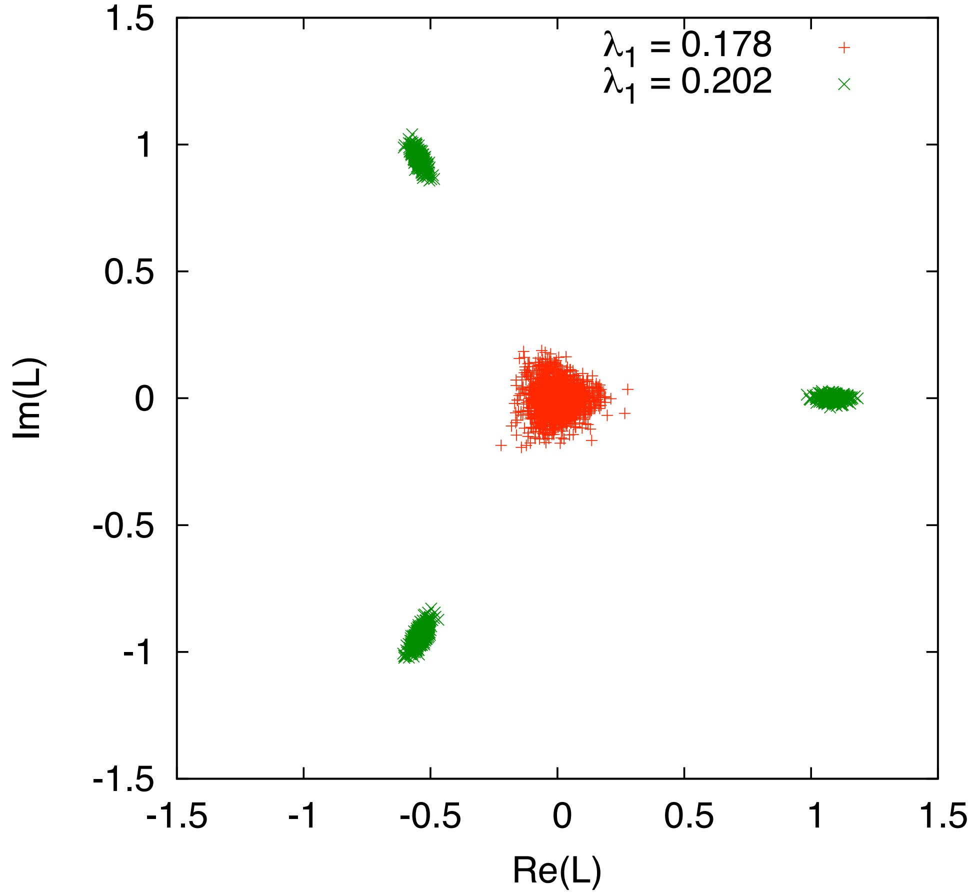
$$\lambda_2 S_2 \quad \propto \quad u^{2N_\tau+2} \sum_{[kl]}' 2\mathrm{Re}(L_k L_l^*) \quad \text{distance} \ = \sqrt{2}$$

$$\lambda_3 S_3 \quad \propto \quad u^{2N_\tau+6} \sum_{\{mn\}}'' 2\mathrm{Re}(L_m L_n^*) \quad \text{distance} \ = 2$$

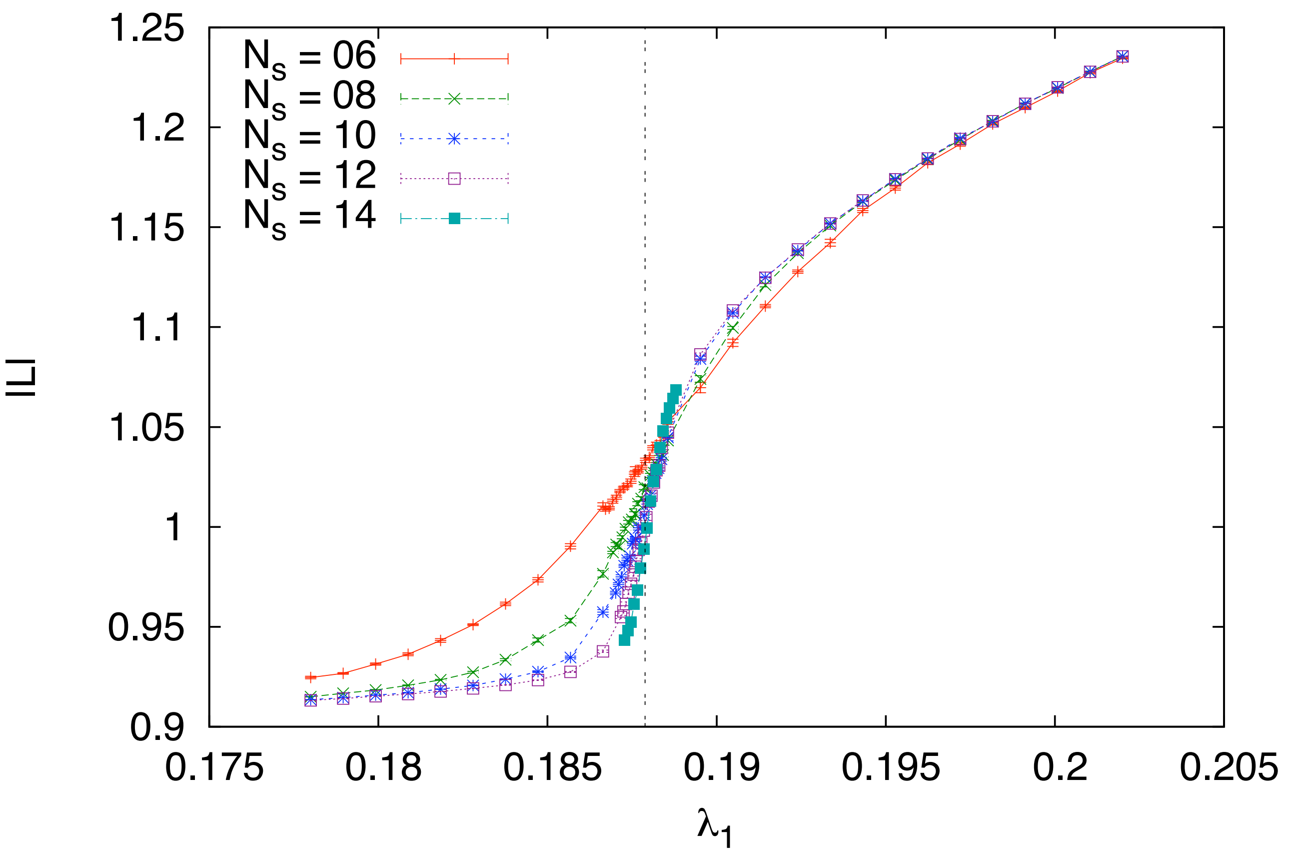
as well as terms from loops in the *adjoint* representation:

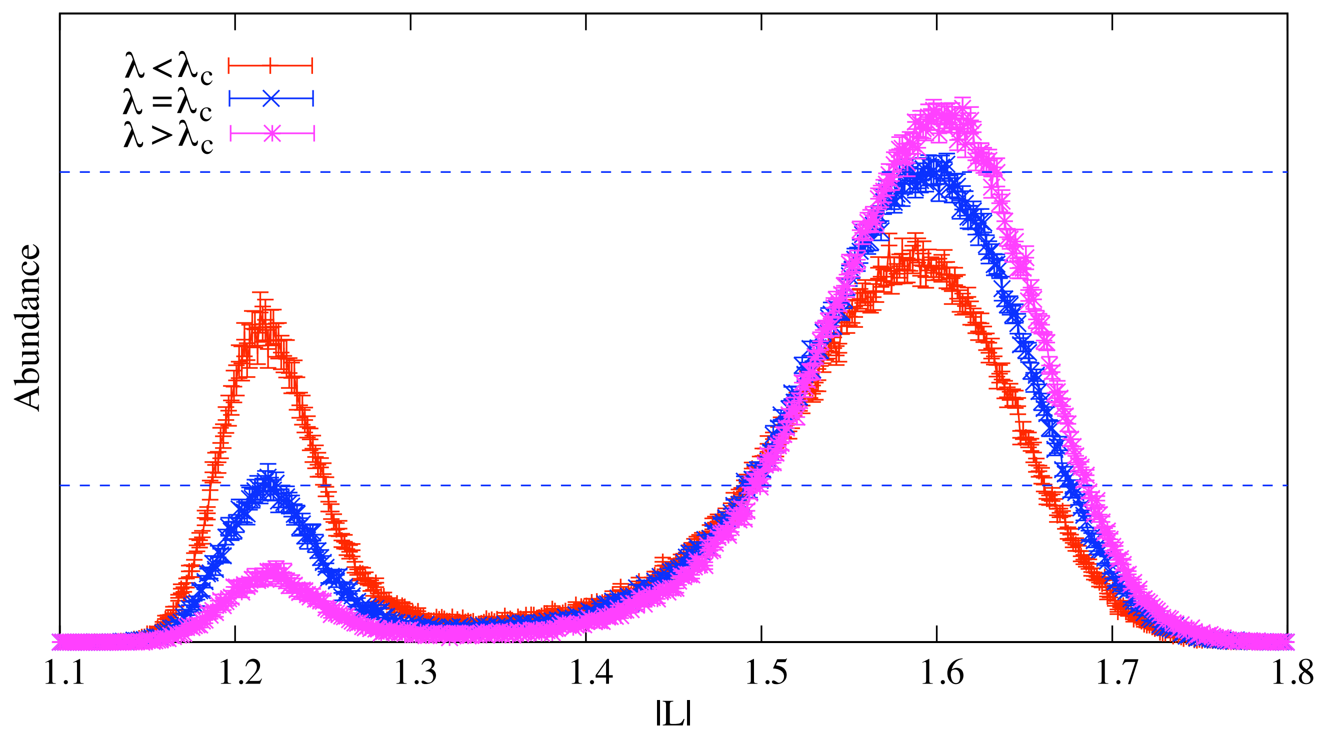
$$\lambda_a S_a \propto u^{2N_\tau} \sum_{<ij>} \mathrm{Tr}^{(a)} W_i \mathrm{Tr}^{(a)} W_j \quad ; \quad \mathrm{Tr}^{(a)} W = |L|^2 - 1$$

Numerical results for SU(3), one coupling



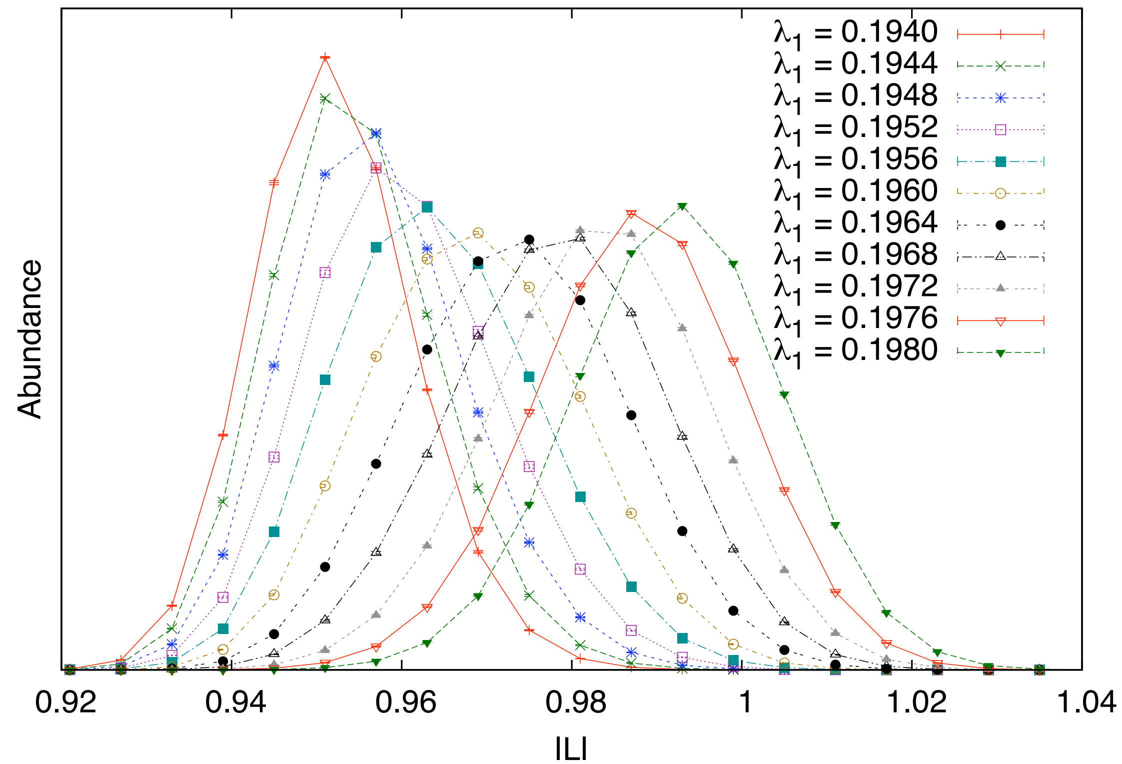
Order-disorder transition

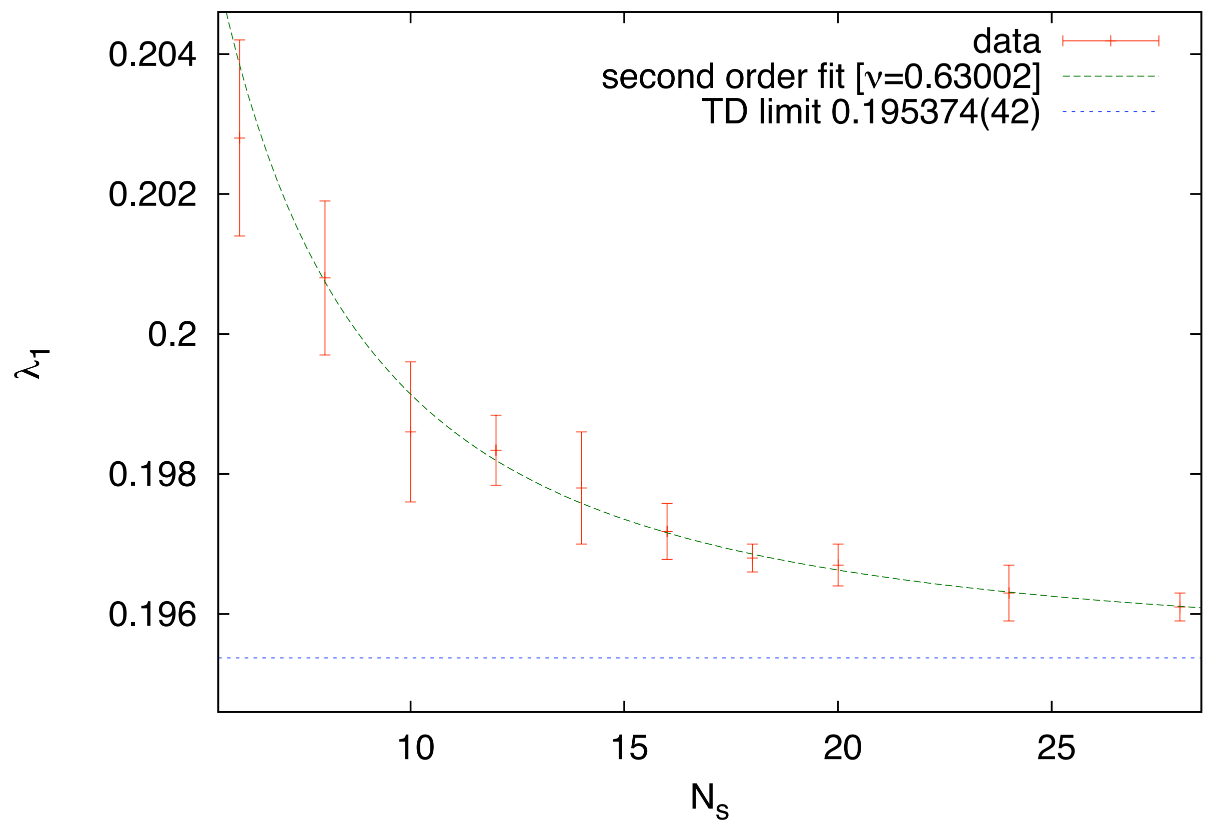




First order phase transition for SU(3) in the thermodynamic limit!

Numerical results for SU(2), one coupling

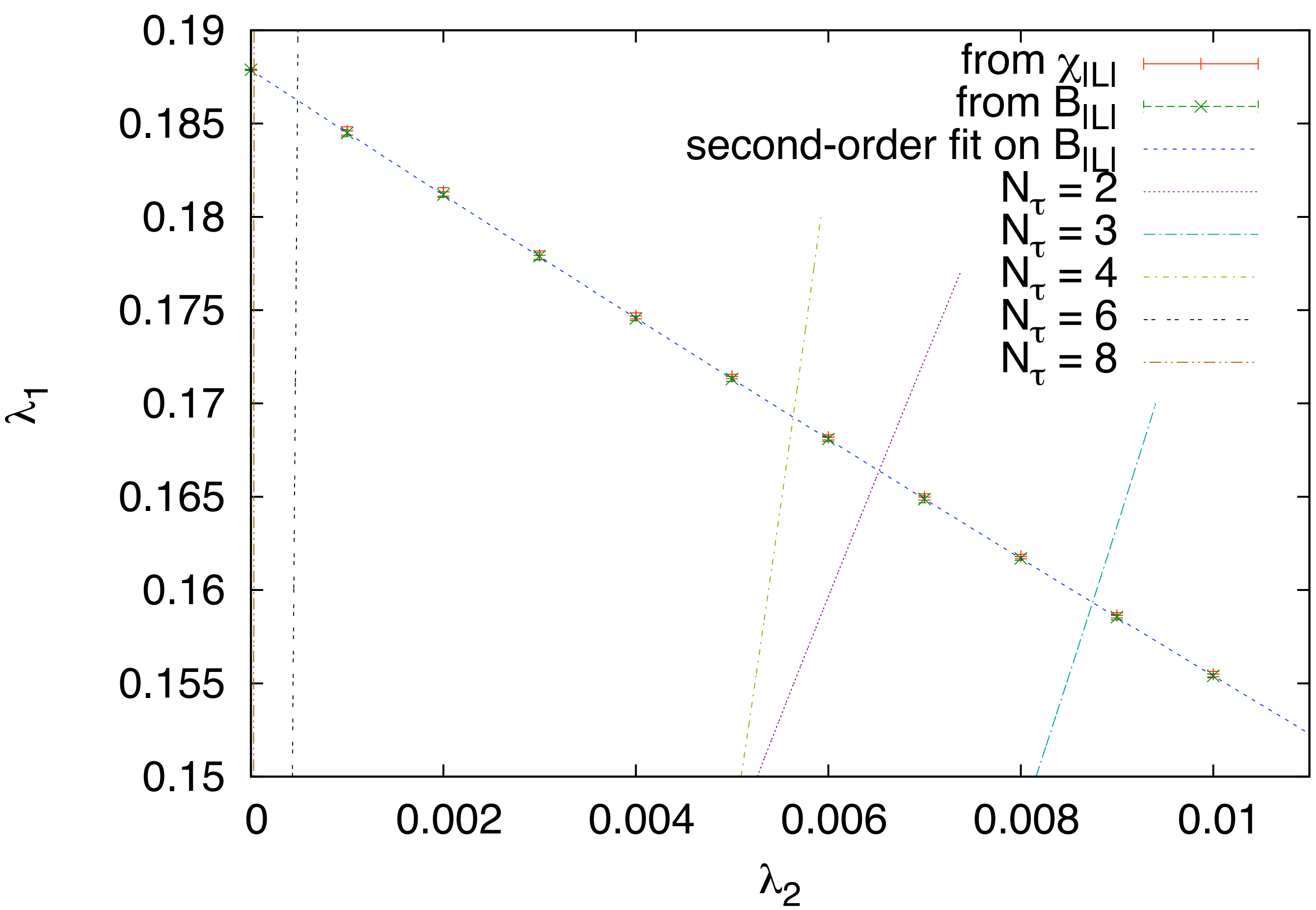


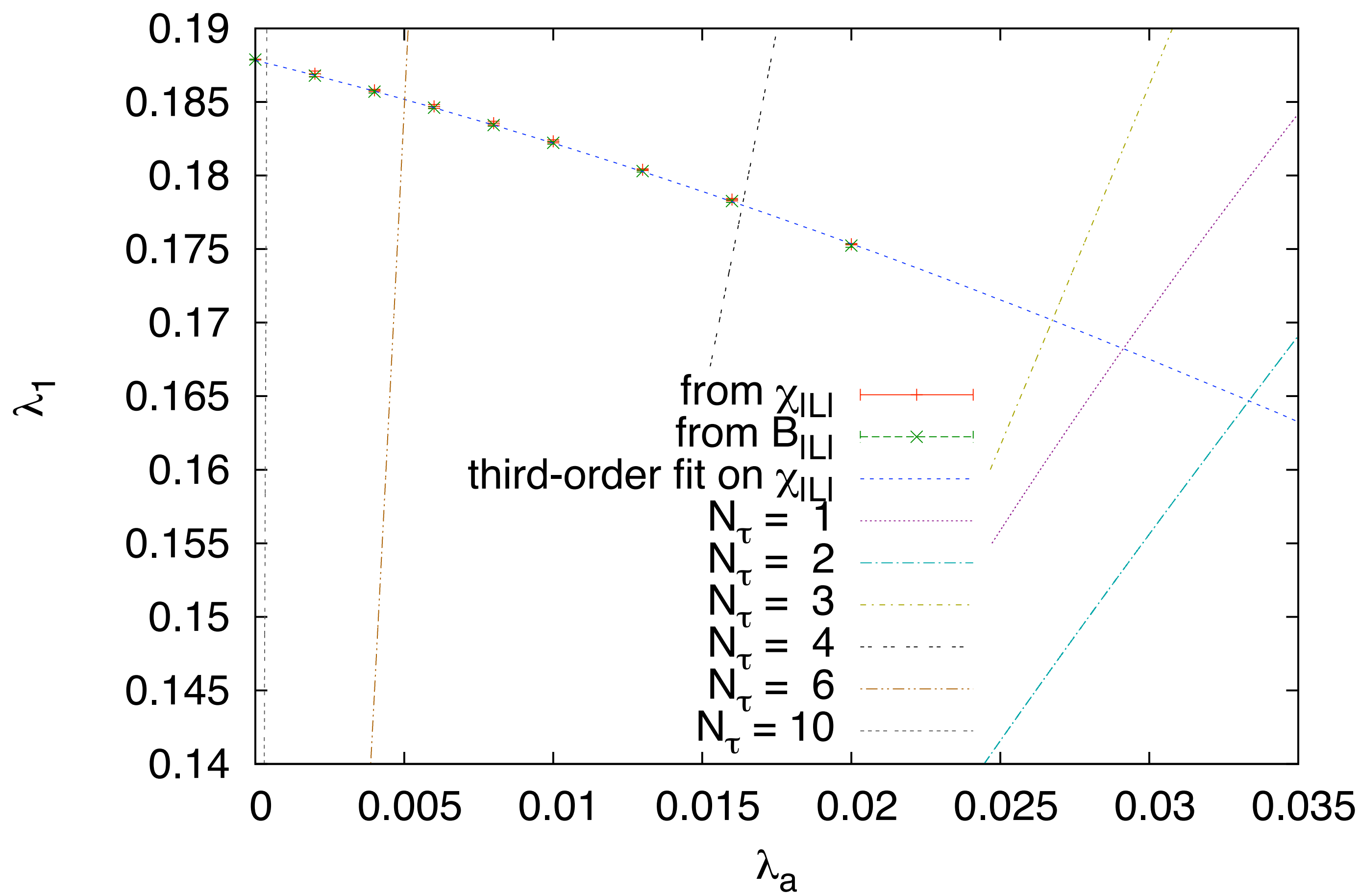


Second order (3d Ising) phase transition for SU(2) in the thermodynamic limit!

The influence of a second coupling

NLO-couplings: next-to-nearest neighbour, adjoint rep. loops





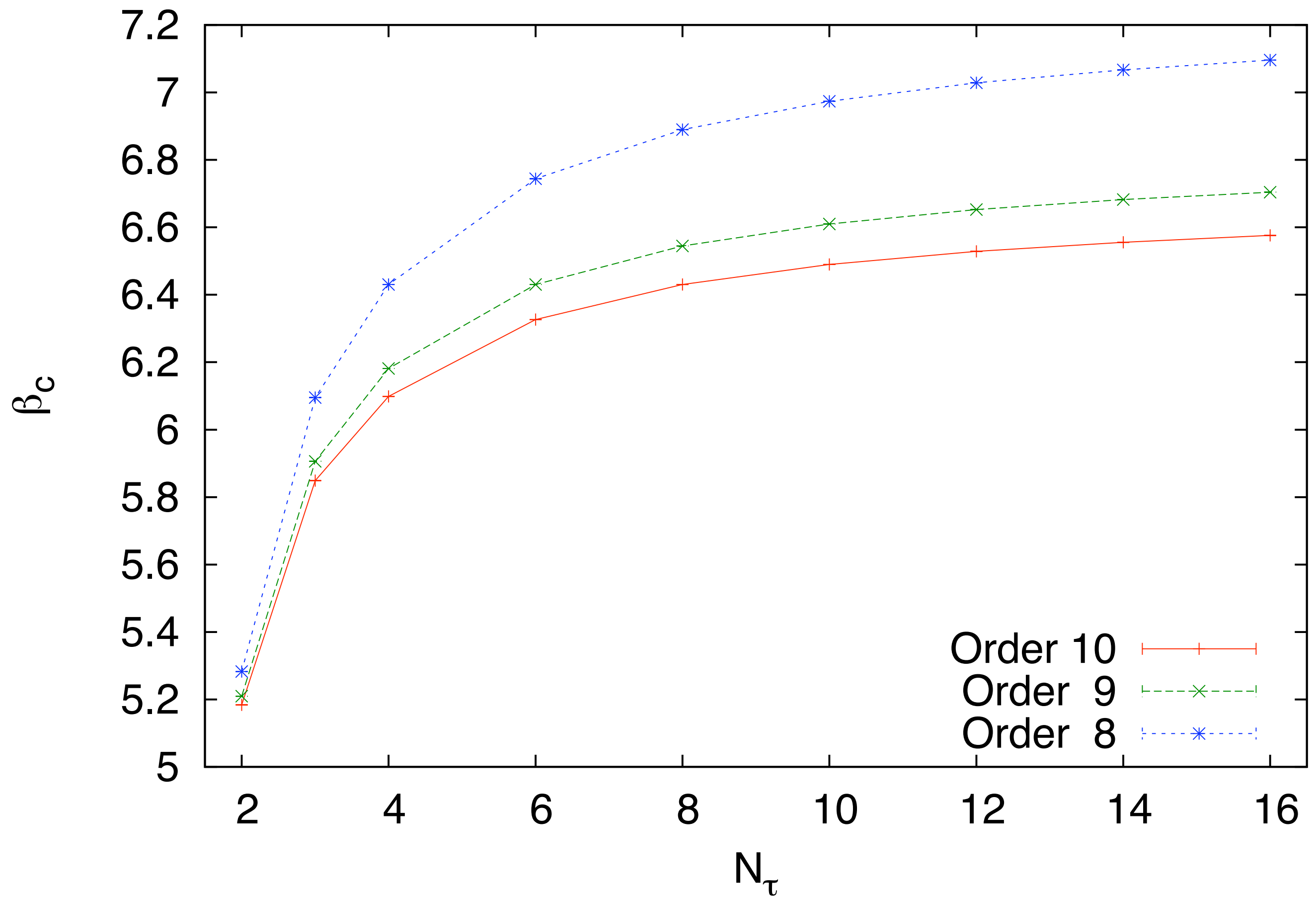
...gets very small for large N_τ !

Mapping back to 4d finite T Yang-Mills

Inverting

$$\lambda_1(N_\tau, \beta) \rightarrow \beta_c(\lambda_{1,c}, N_\tau)$$

...points at reasonable convergence

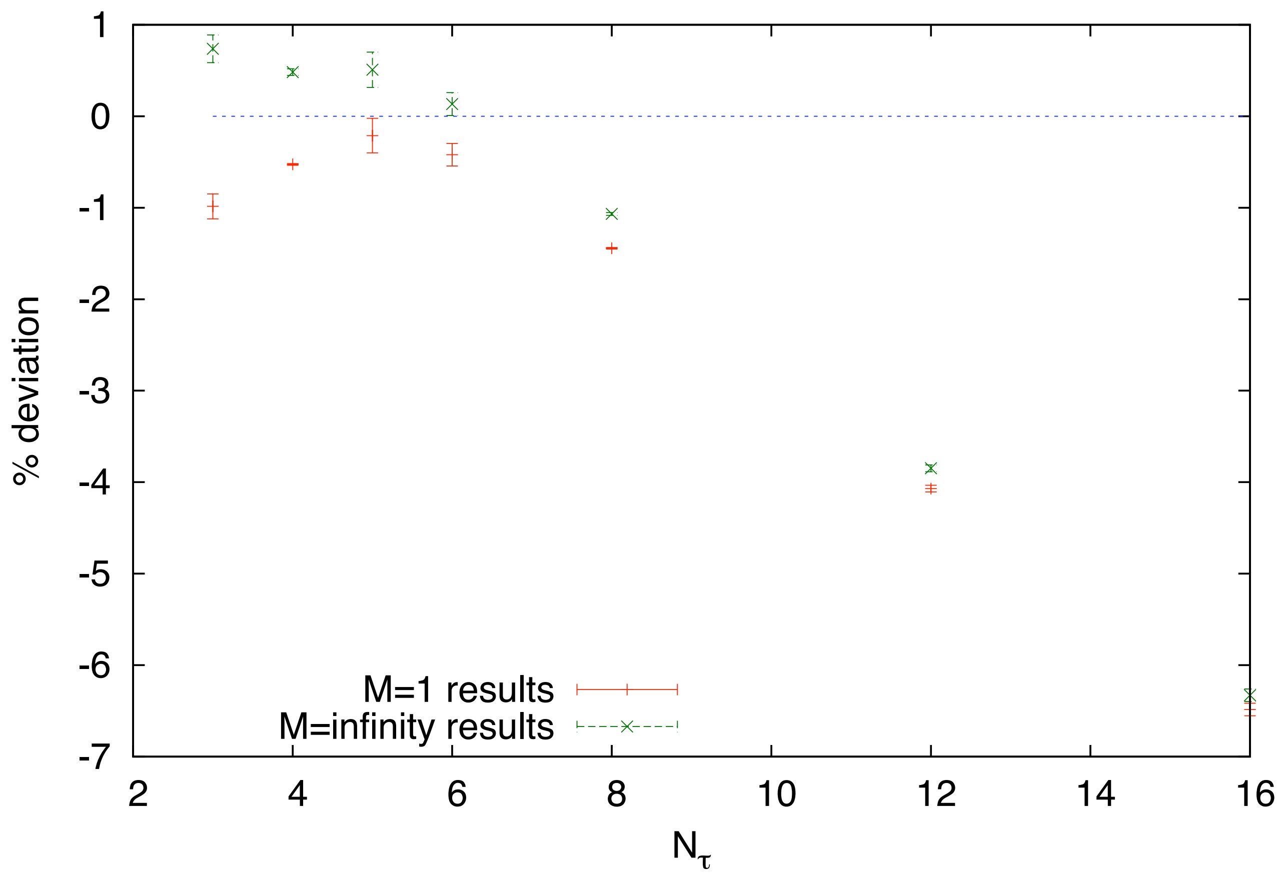


SU(3)

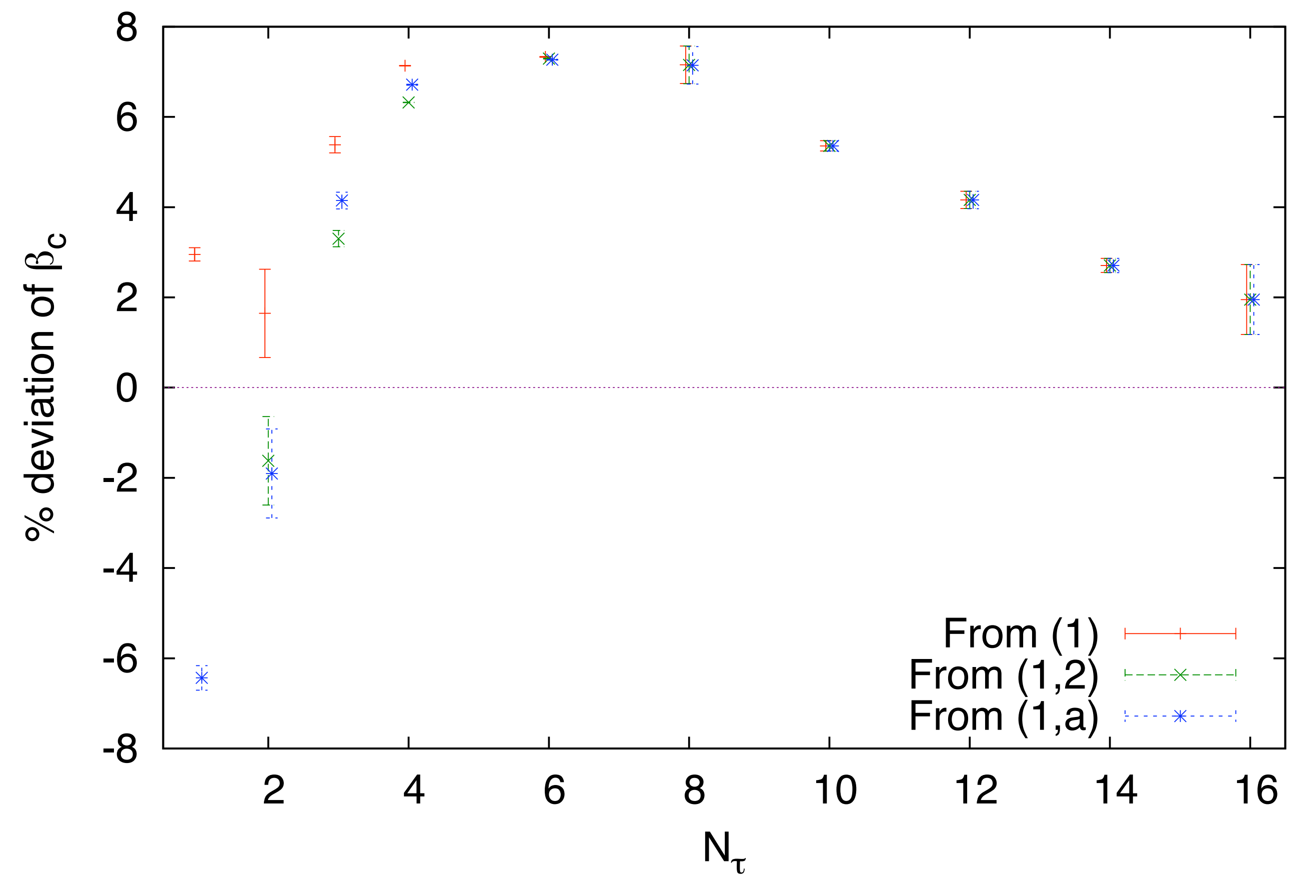
Comparison with 4d Monte Carlo

Relative accuracy for β_c compared to the full theory

SU(2)

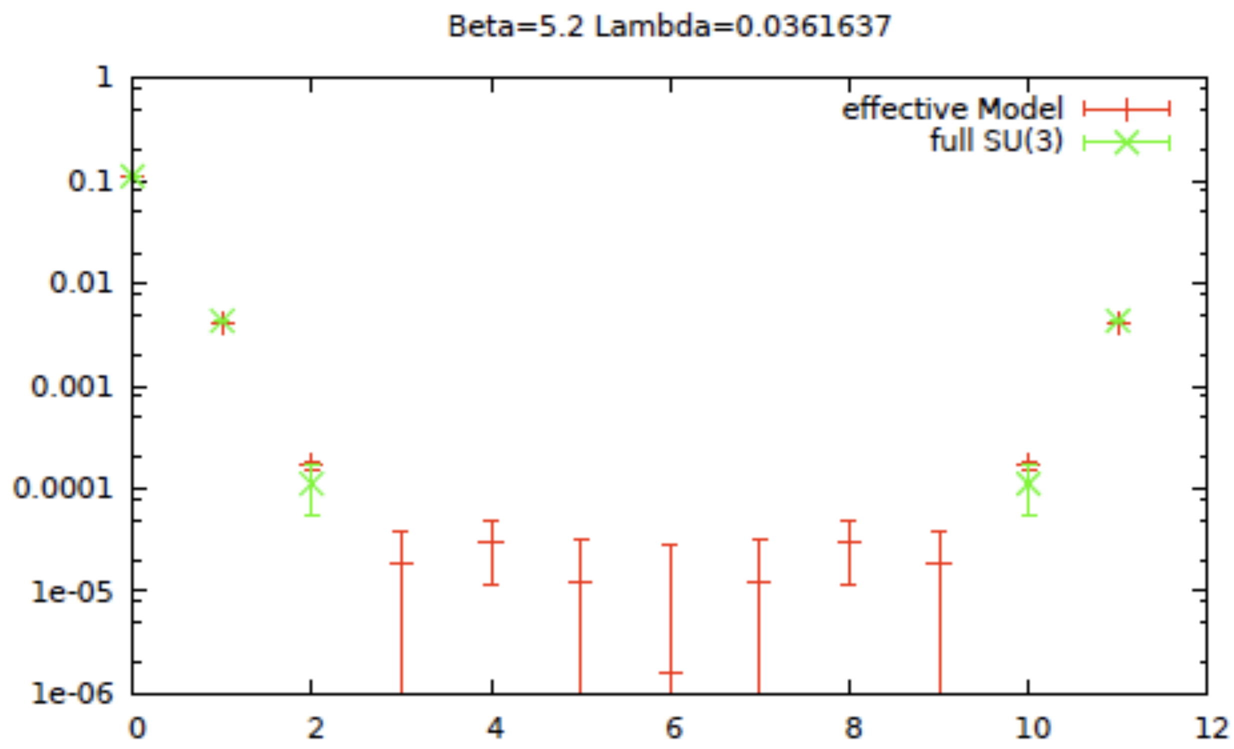


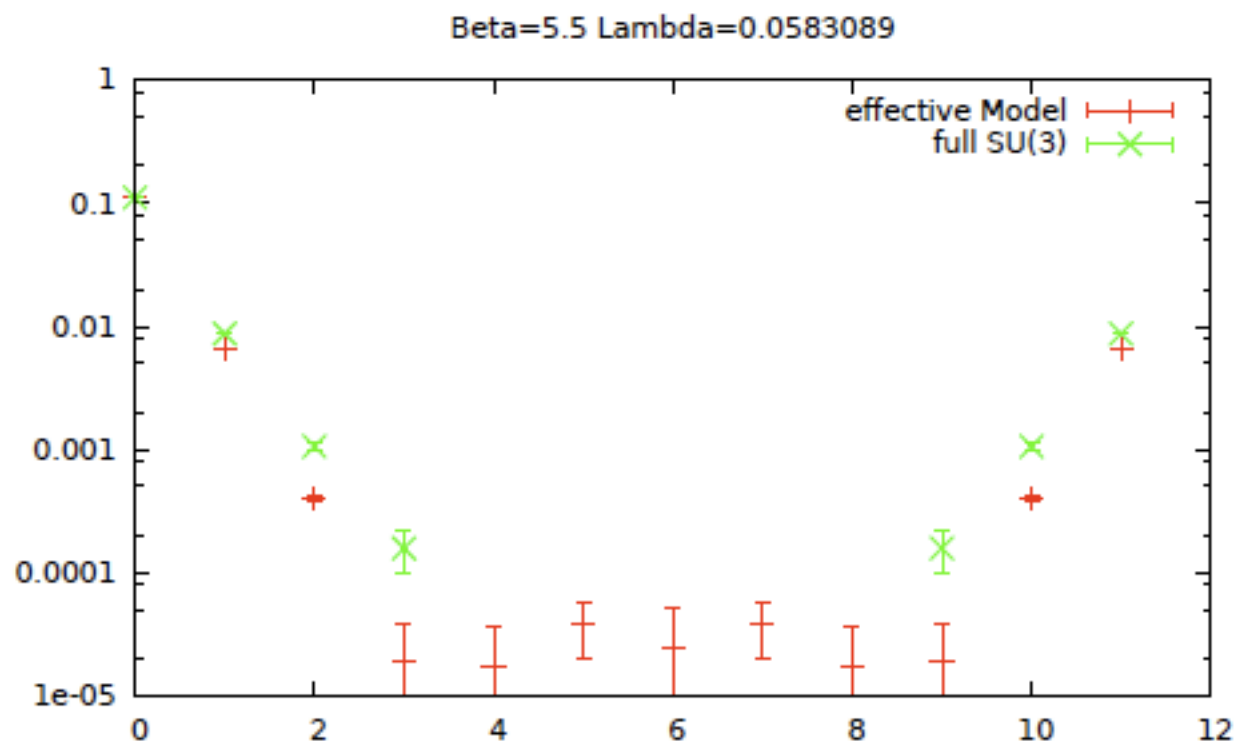
SU(3)



Note: influence of additional couplings checked explicitly!

Another check: correlation functions, SU(3)

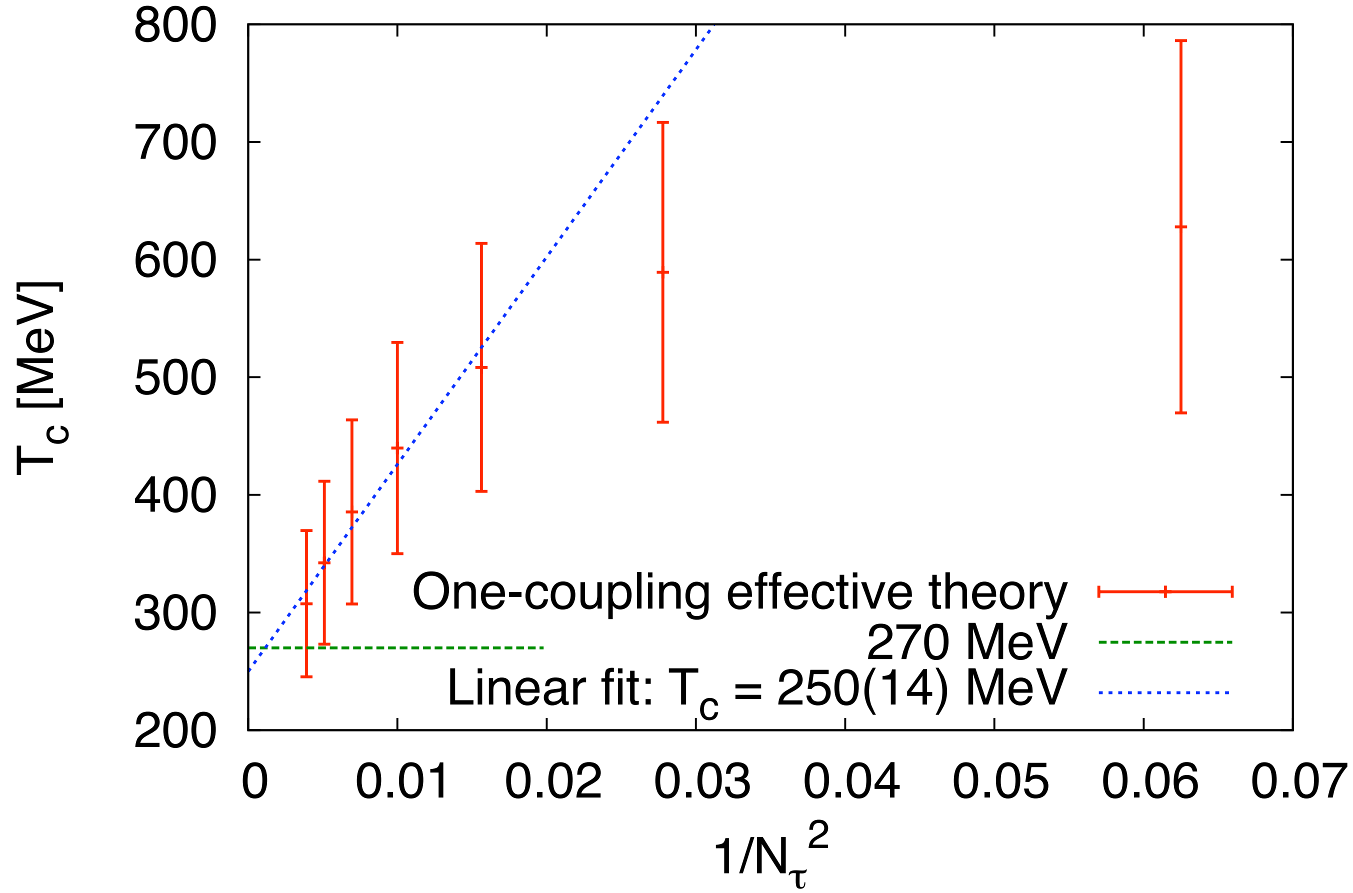




Nt=4:

β_c =5.69

Continuum limit feasible!



-error bars: difference between last two orders in strong coupling exp.

-using non-perturbative beta-function (4d T=0 lattice)

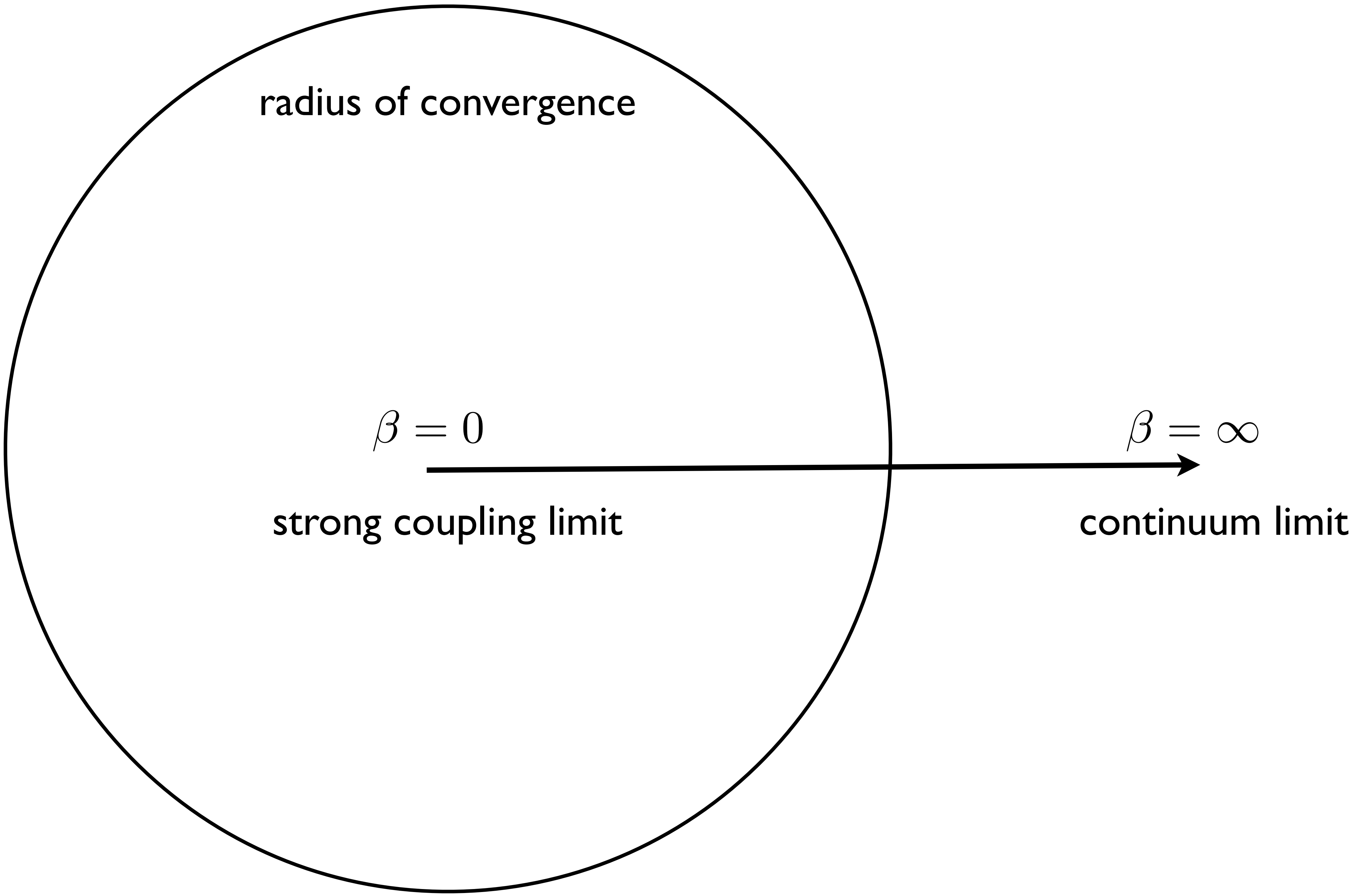
-all data points from one single 3d MC simulation!

How is this possible?

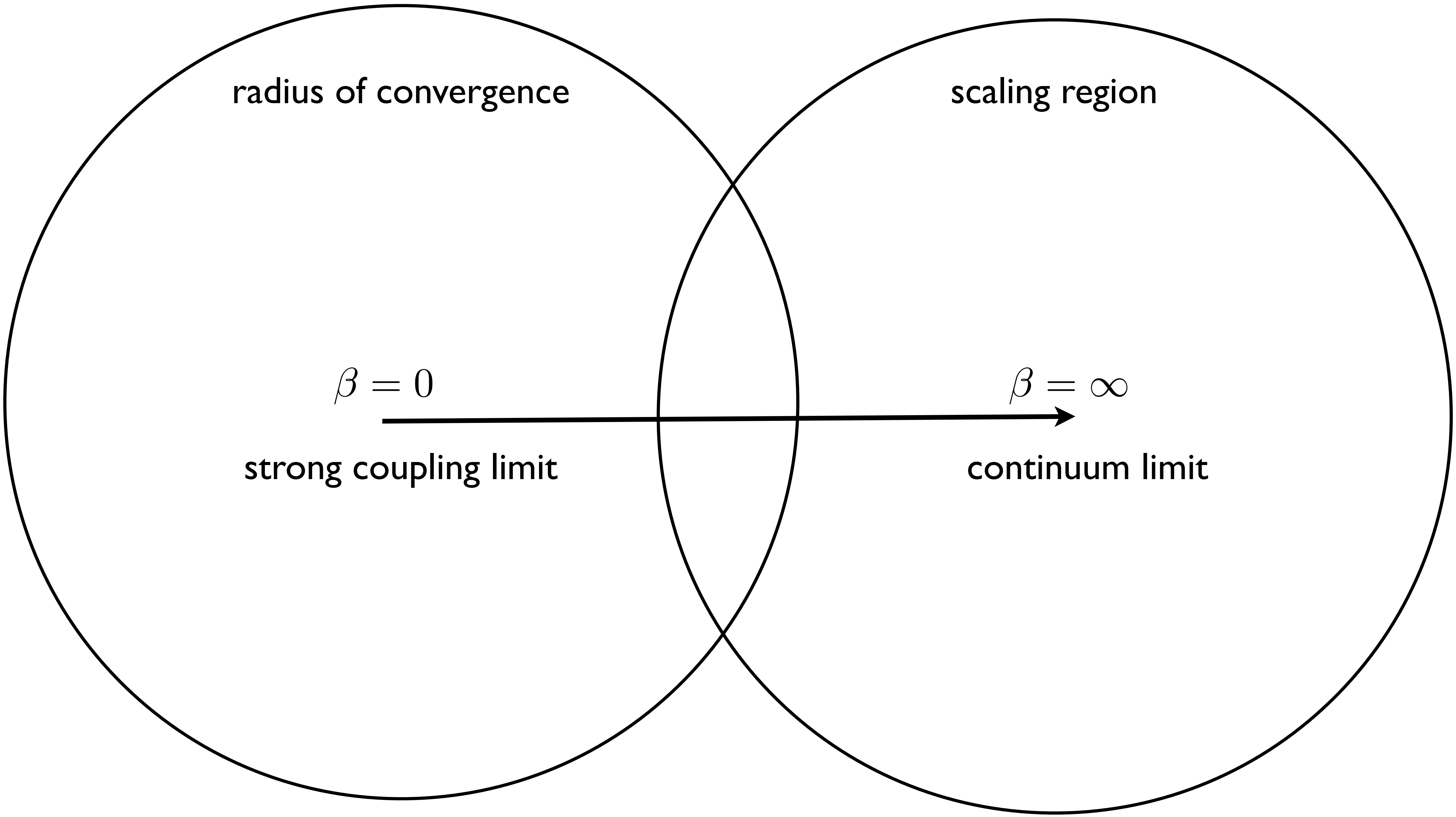
$\beta = 0$ $\beta = \infty$

strong coupling limit → continuum limit

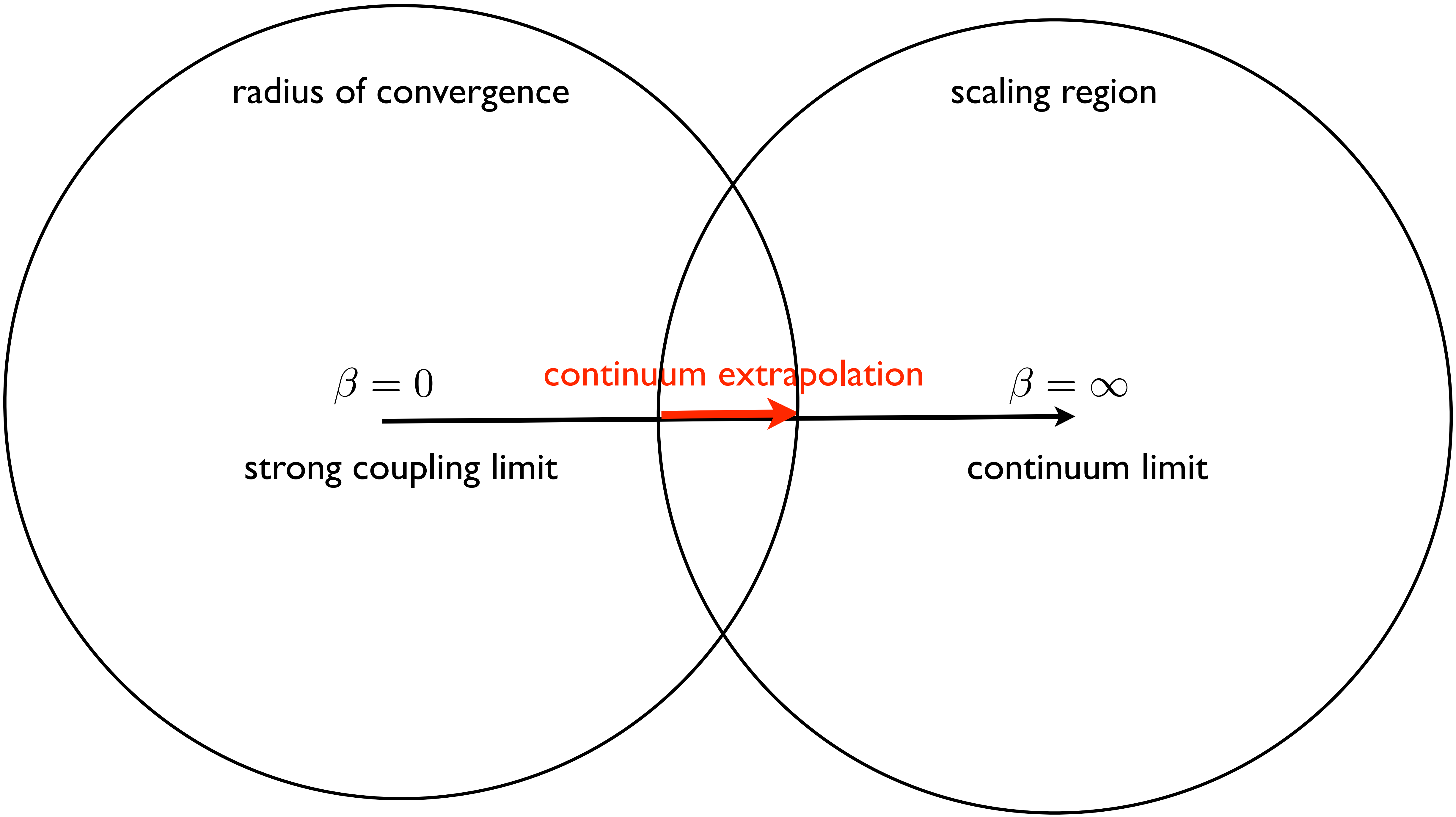
How is this possible?



How is this possible?



How is this possible?



Including heavy, dynamical Wilson fermions

N_f (degenerate) fermions $\Longrightarrow \quad S = S_{\text{gauge}} + S_q[U, \psi, \overline{\psi}]$

$$S_q = \sum_{x,y;f} \overline{\psi}_{f,y} \Big(\mathbb{1} - \kappa H[U] \Big)_{yx} \psi_{f,x} \quad , \quad H[U]_{yx} = \sum_{\pm\mu} \delta_{y,x+\hat{\mu}} (\mathbb{1} + \gamma_\mu) U_{x,\mu}$$

Integrate the Grassmann variables $\psi, \overline{\psi}$:

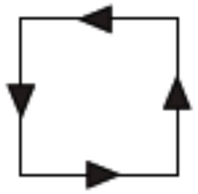
$$S = S_{\text{gauge}} - N_f \mathrm{Tr} \log(\mathbb{1} - \kappa H)$$

Expand in the *hopping parameter* $\kappa = 1/(2aM + 8)$: [*]

$$S = S_{\text{gauge}} + N_f \sum_{\ell=1}^{\infty} \frac{\kappa^\ell}{\ell} \mathrm{Tr} H[U]^\ell$$

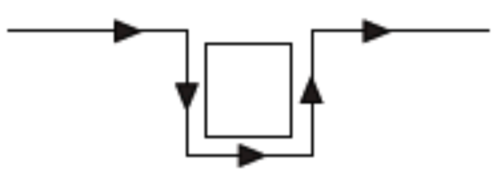
Similar to de Pietri, Feo, Seiler, Stamatescu 07, Aarts, Stamatescu 08 ...

Links along imaginary time gain $\exp(\pm\mu a)$



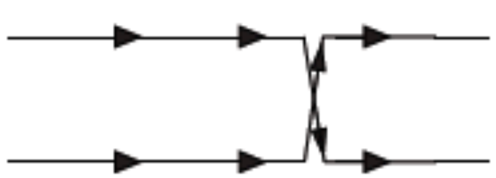
reabsorbed in gauge part: $\begin{cases} \beta \to \beta + \mathcal{O}(\kappa^4) \\ u(\beta) \to u(\beta,\kappa) \end{cases}$

LO Polyakov "magnetic" term $\sim \begin{cases} \underbrace{(2\kappa e^{+a\mu})^{N_\tau}}_{h_1} L \\ \underbrace{(2\kappa e^{-a\mu})^{N_\tau}}_{\bar{h}_1} L^* \end{cases}$



higher corrections to the above:

$$h_1 = (2\kappa e^{a\mu})^{N_\tau} \left[1 + \mathcal{O}(k^2) f(u) + \ldots \right]$$



other (suppressed) terms, such as $h_2(L_x L_{x+\hat{i}})$,

$$h_2 \sim (2\kappa e^{a\mu})^{2N_\tau} \kappa^2 \ldots$$

In general the model becomes (with $\overline{h}_i(\mu) = h_i(-\mu)$)

$$-S_{\rm eff} = \sum_i \lambda_i(u,\kappa,N_\tau) S_i^{\rm S} - 2N_f \sum_i \left[h_i(u,\kappa,\mu,N_\tau) S_i^{\rm A} + \overline{h}_i(u,\kappa,\mu,N_\tau) S_i^{\dagger{\rm A}} \right]$$

Now, keep only $\lambda_1 S_1^{\rm S}$ and $h_1 S_1^{\rm A} + \overline{h}_1 S_1^{\dagger{\rm A}}$ (now called just λ, h)

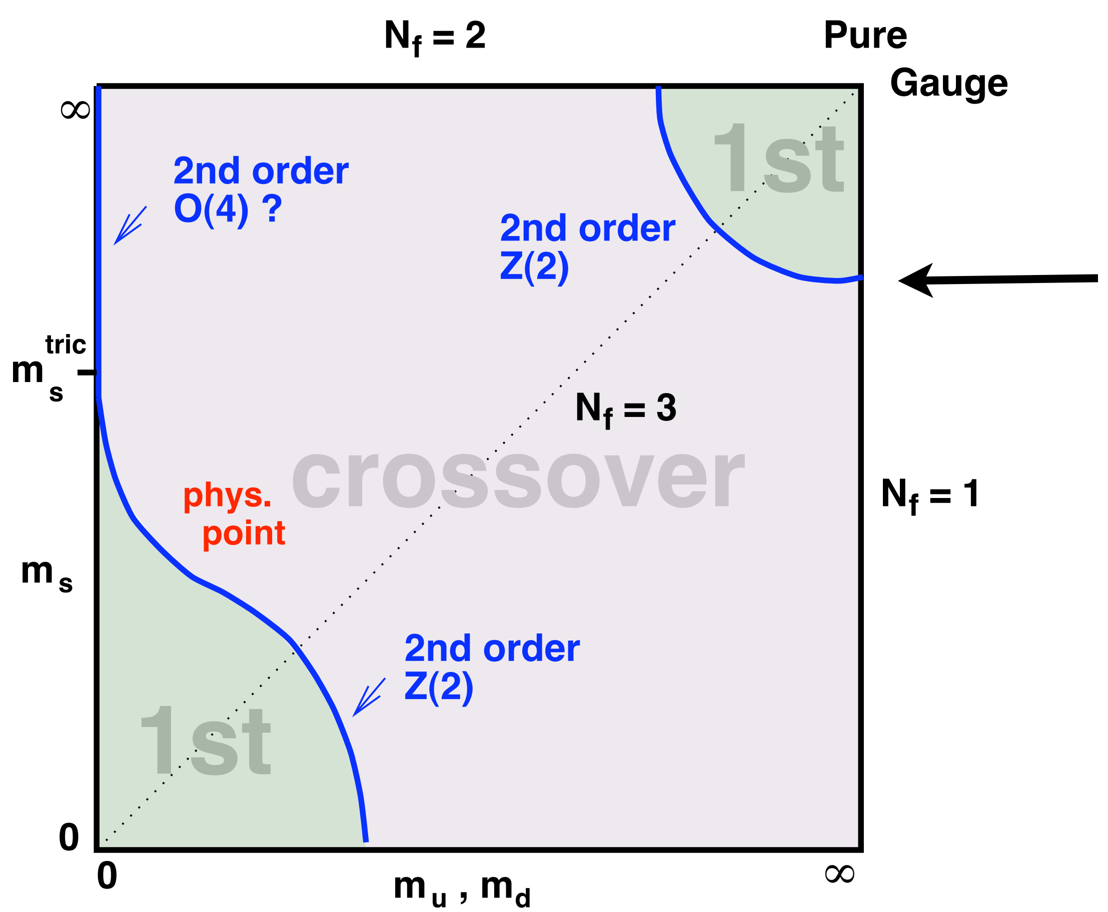
Higher powers of loops are resummed into a determinant:

$$Z_{\rm eff}(\lambda_1, h_1, \overline{h}_1; N_\tau) = \int [{\rm d}L] \Big(\prod_{<ij>} [1 + 2\lambda_1 {\rm Re} L_i L_j^*] \Big) \Big(\prod_x \underbrace{\det[(1 + h_1 W_x)(1 + \overline{h}_1 W_x^\dagger)]^{2N_f}}_{\equiv Q(L_x, L_x^*)^{N_f}} \Big)$$

No chemical potential ($h = \overline{h}$): the full (λ, h)-model has then

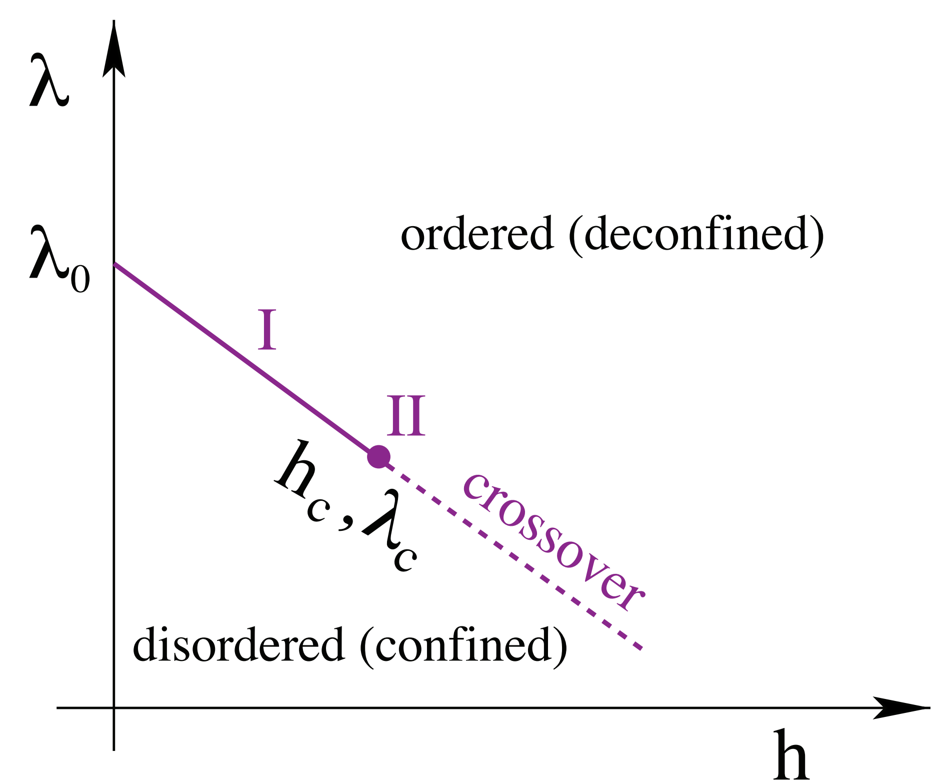
- a “spin-spin” interaction between neighbour Polyakov loops
- a “magnetic-field” term acting on sites

QCD: first order deconfinement transition region



deconfinement p.t.:
breaking of global $Z(3)$ symmetry;
explicitly broken by quark masses
transition weakens

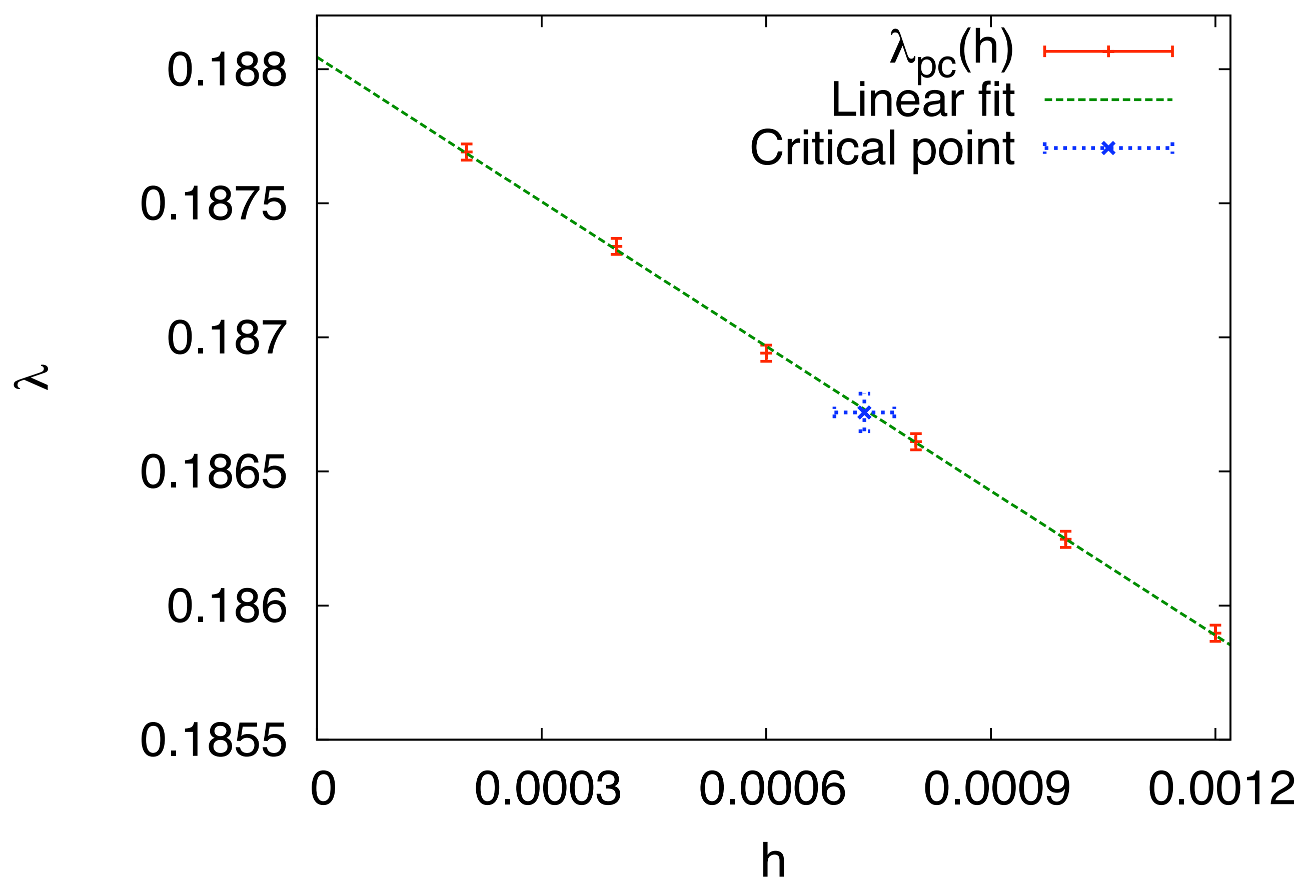
Phase diagram in eff. theory:



Phase boundary, numerically

To find $\lambda_{pc}(h)$, λ-scans were performed at various fixed h

- peak in $\chi_O = \langle O^2 \rangle - \langle O \rangle^2$
- dip in $B_O = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$

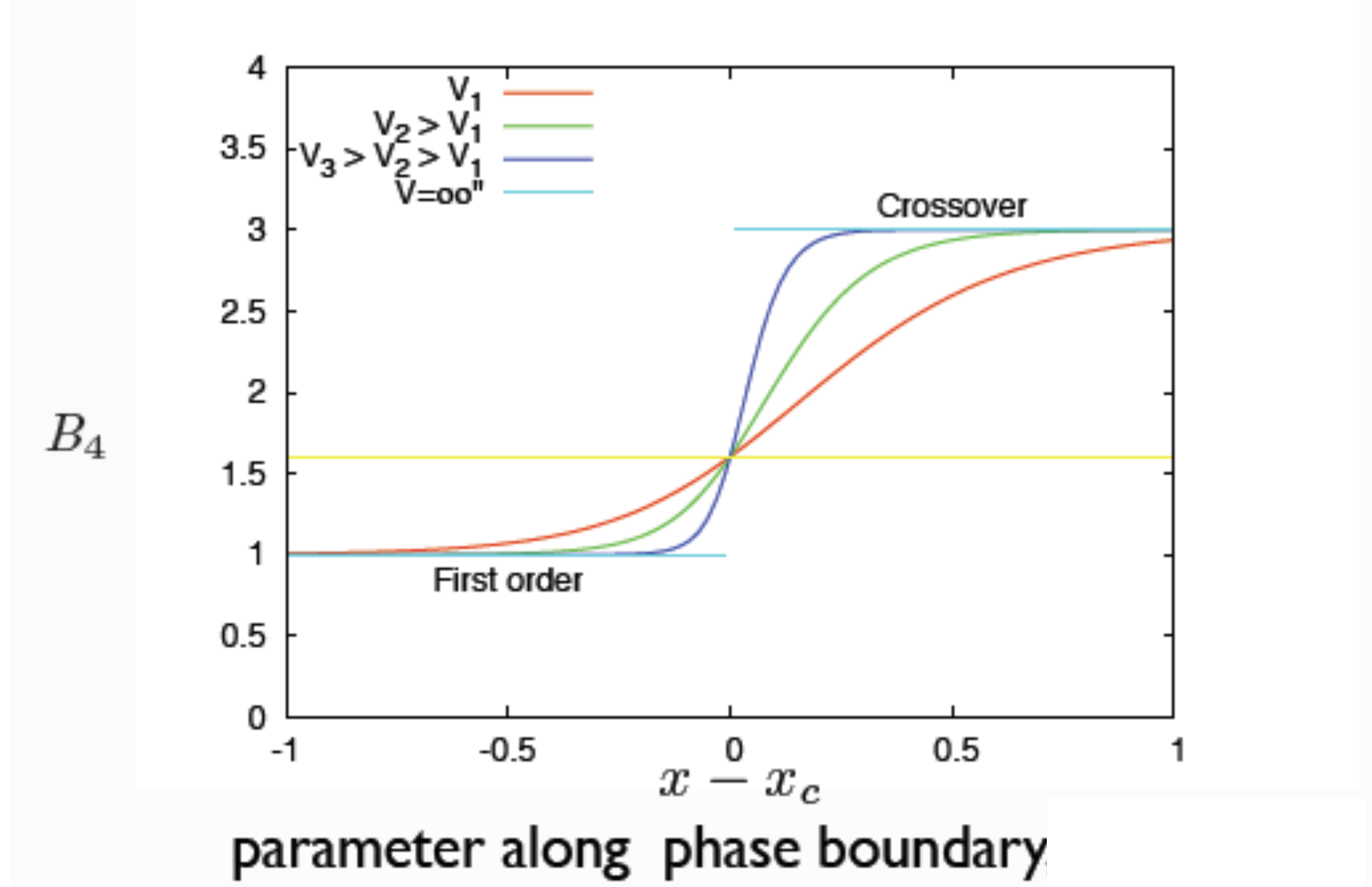


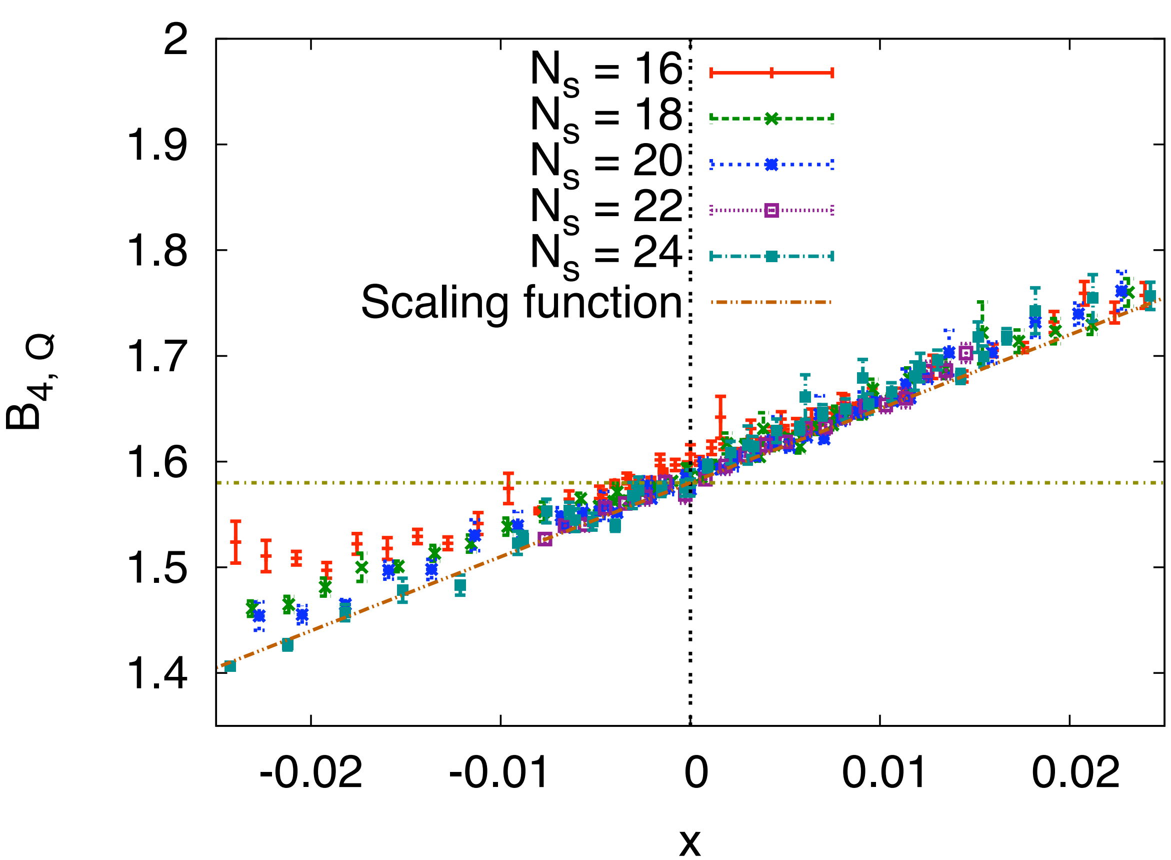
$$\lambda_{pc}(h) = 0.18805 - 1.797 \cdot h$$

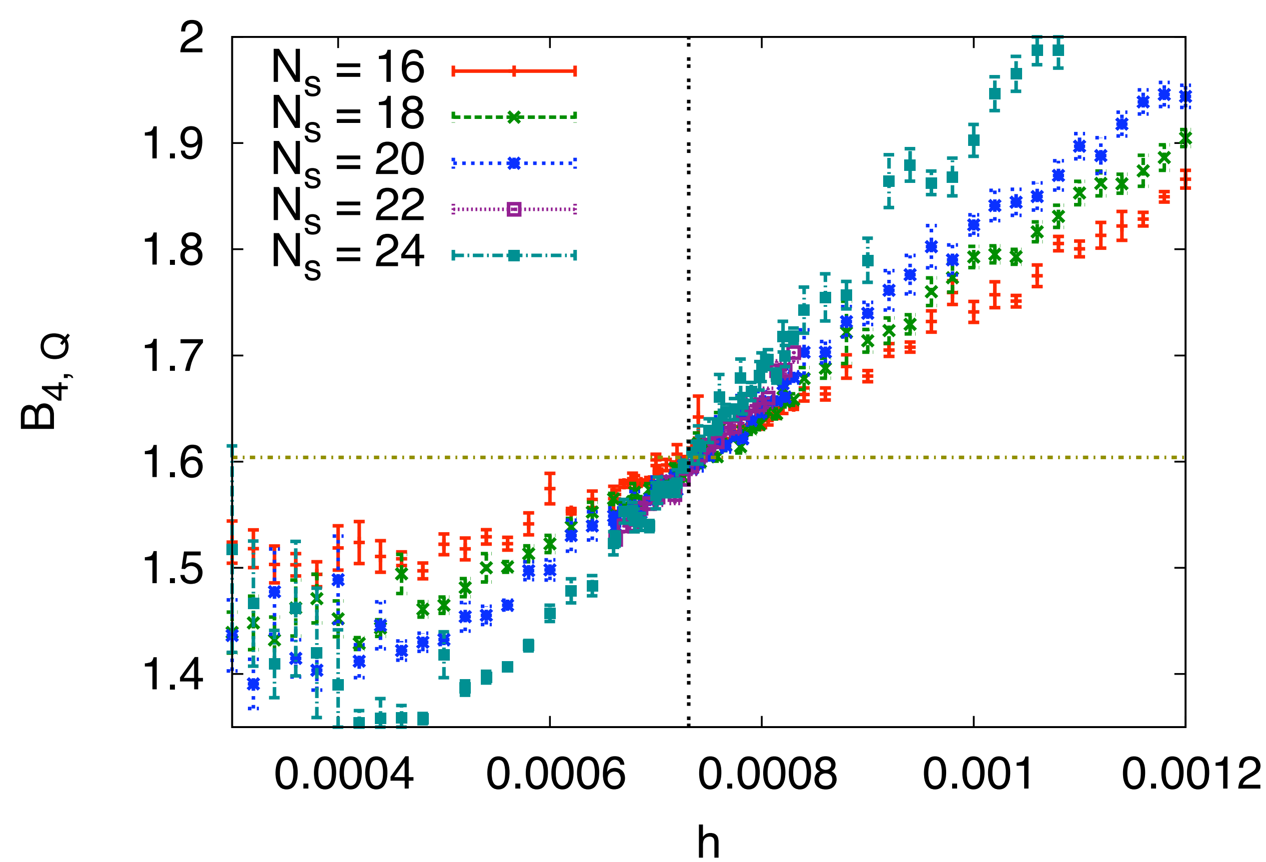
Observable to identify order of p.t.:

$$\delta B_Q = B_4(\delta Q) = \frac{\langle (\delta Q)^4 \rangle}{\langle (\delta Q)^2 \rangle^2}$$

$$B_4(x) = 1.604 + bL^{1/\nu}(x - x_c) + \ldots$$







The critical point

$$\lambda_c = 0.18672(7), h_c = 0.000731(40)$$

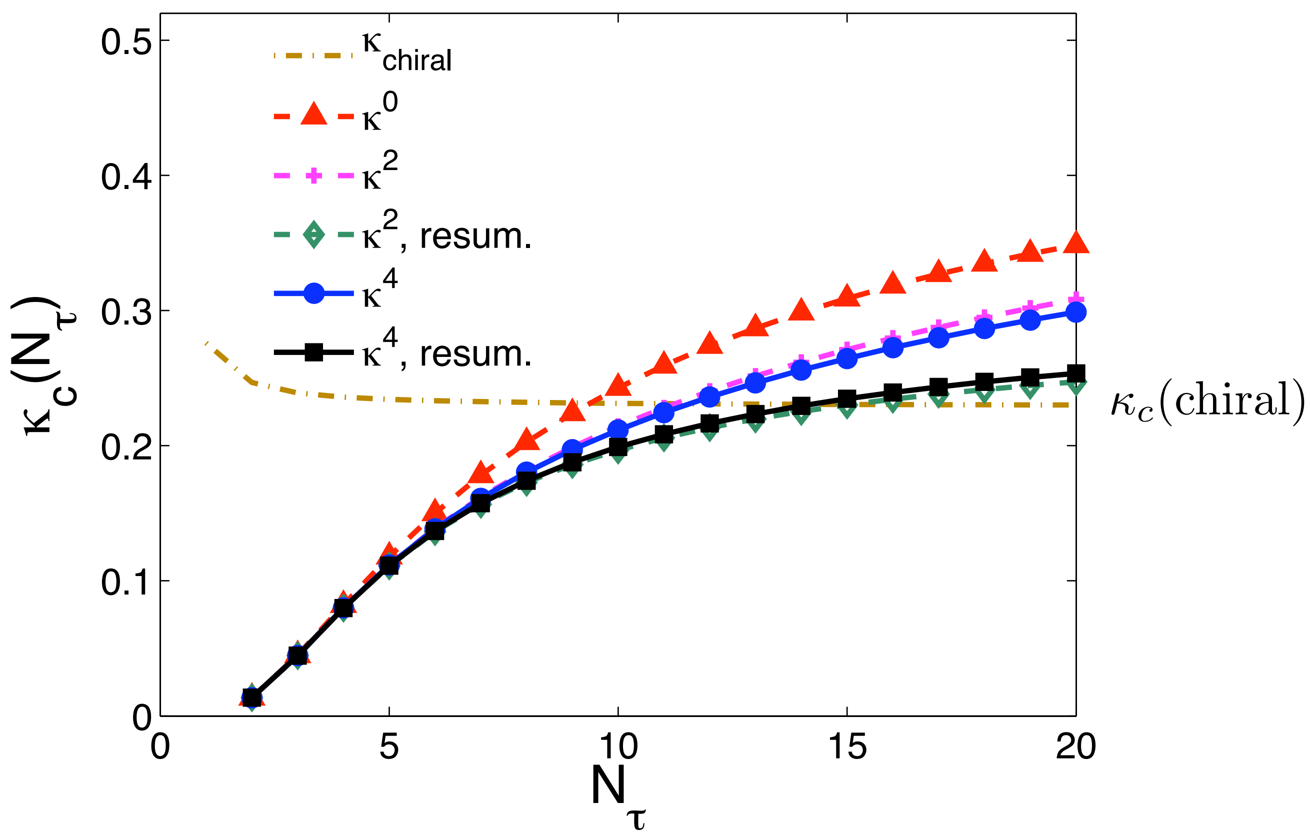
Mapping back to QCD:

		eff. theory	4d MC, WHOT	4d MC, de Forcrand et al
N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [23]	$\kappa_c(4)$, Ref. [22]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

$e^{-M/T} \simeq h/N_f$ [linear approximation in $h \ll 1 \ldots$]

Accuracy ~5%, predictions for Nt=6,8,... available!

Convergence properties:



Finite density: sign problem!

- Metropolis algorithm: Mild sign problem; $\frac{\mu}{T} \lesssim 3$
- Worm algorithm: No sign problem cf. Gattringer et al.

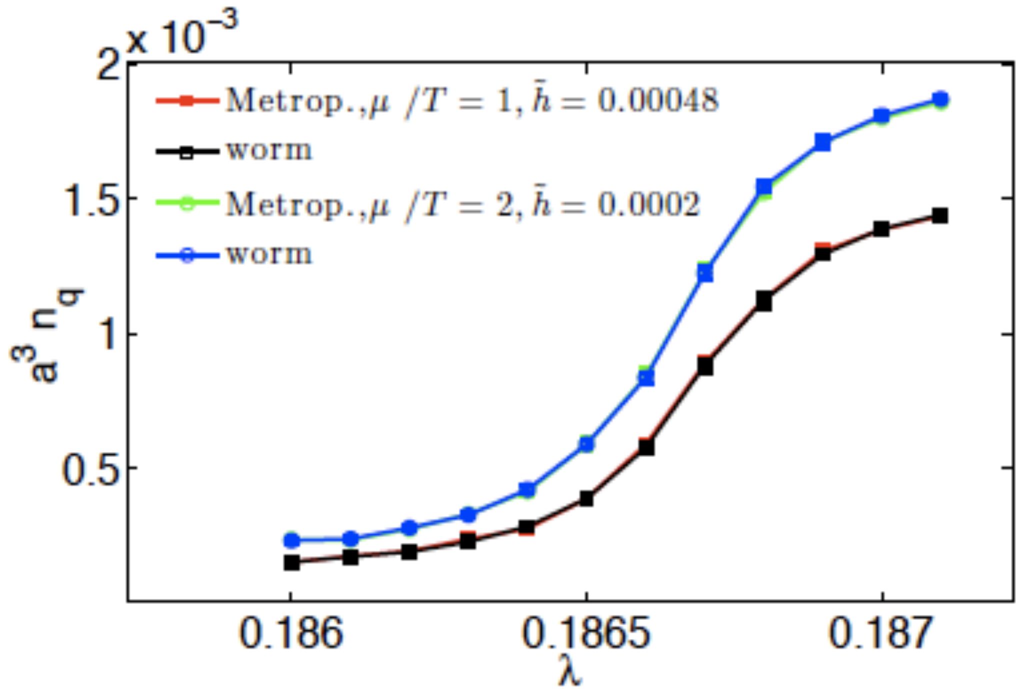
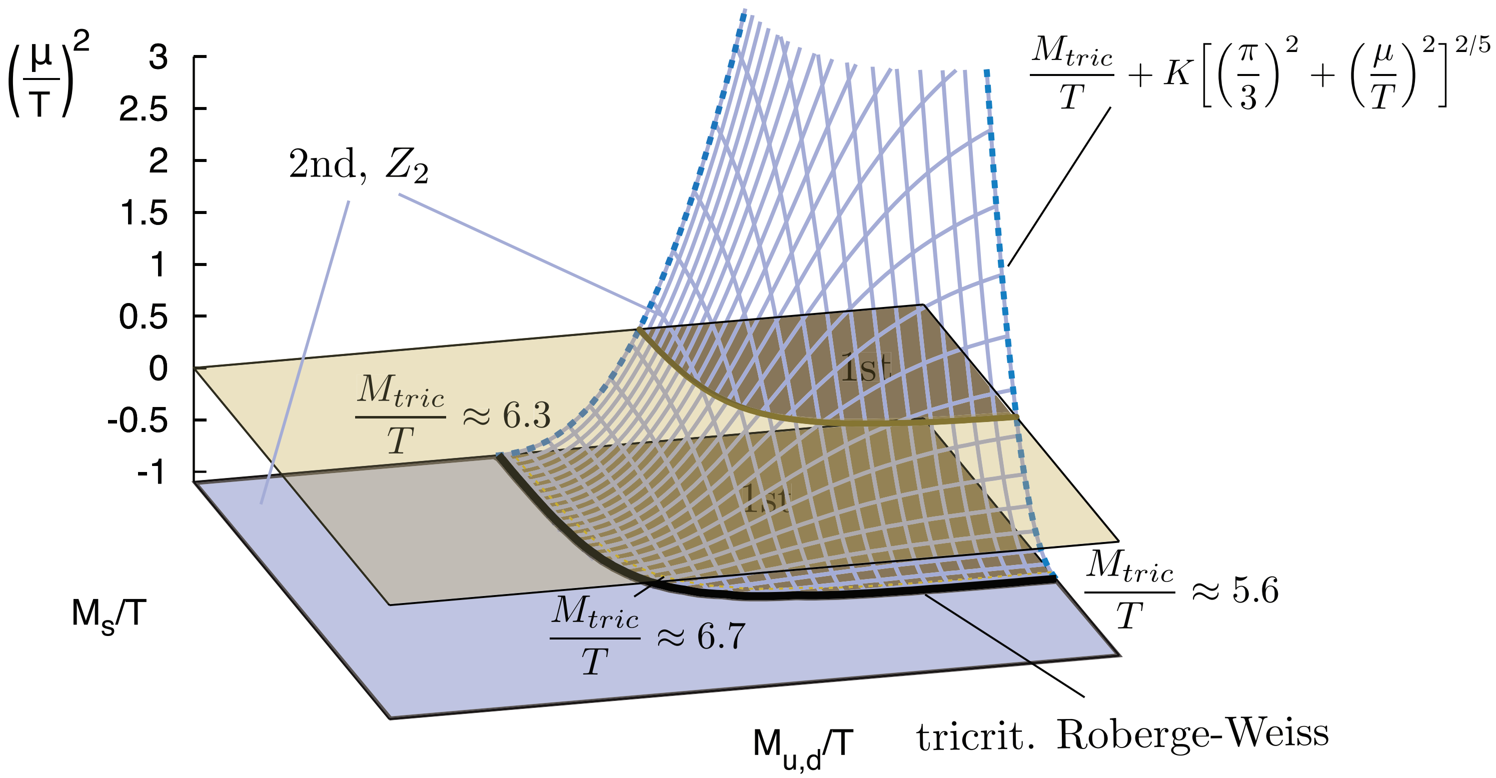


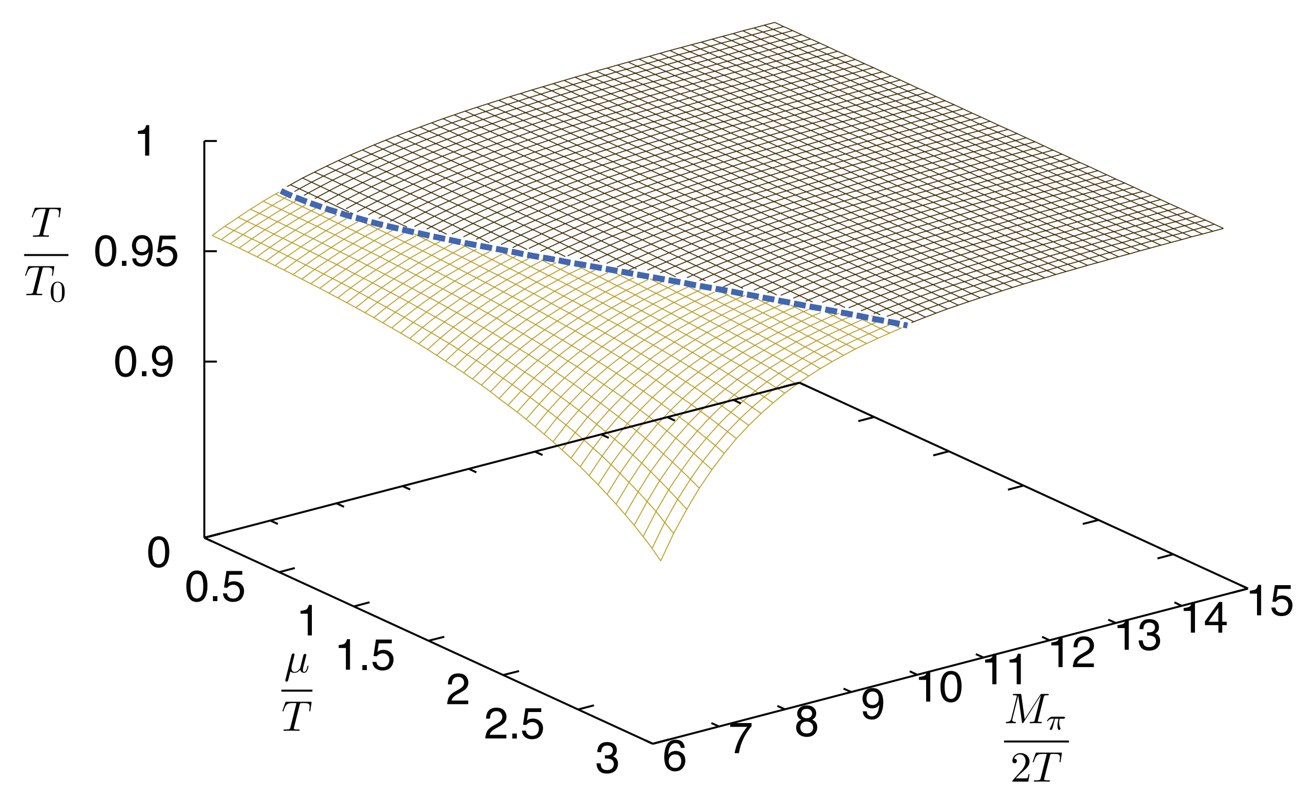
Figure: Quark density calculated with Z_{eff} from Metropolis or worm algorithm on 24^3 lattices for $\frac{\mu}{T} = 1$ and 2.

The fully calculated deconfinement transition



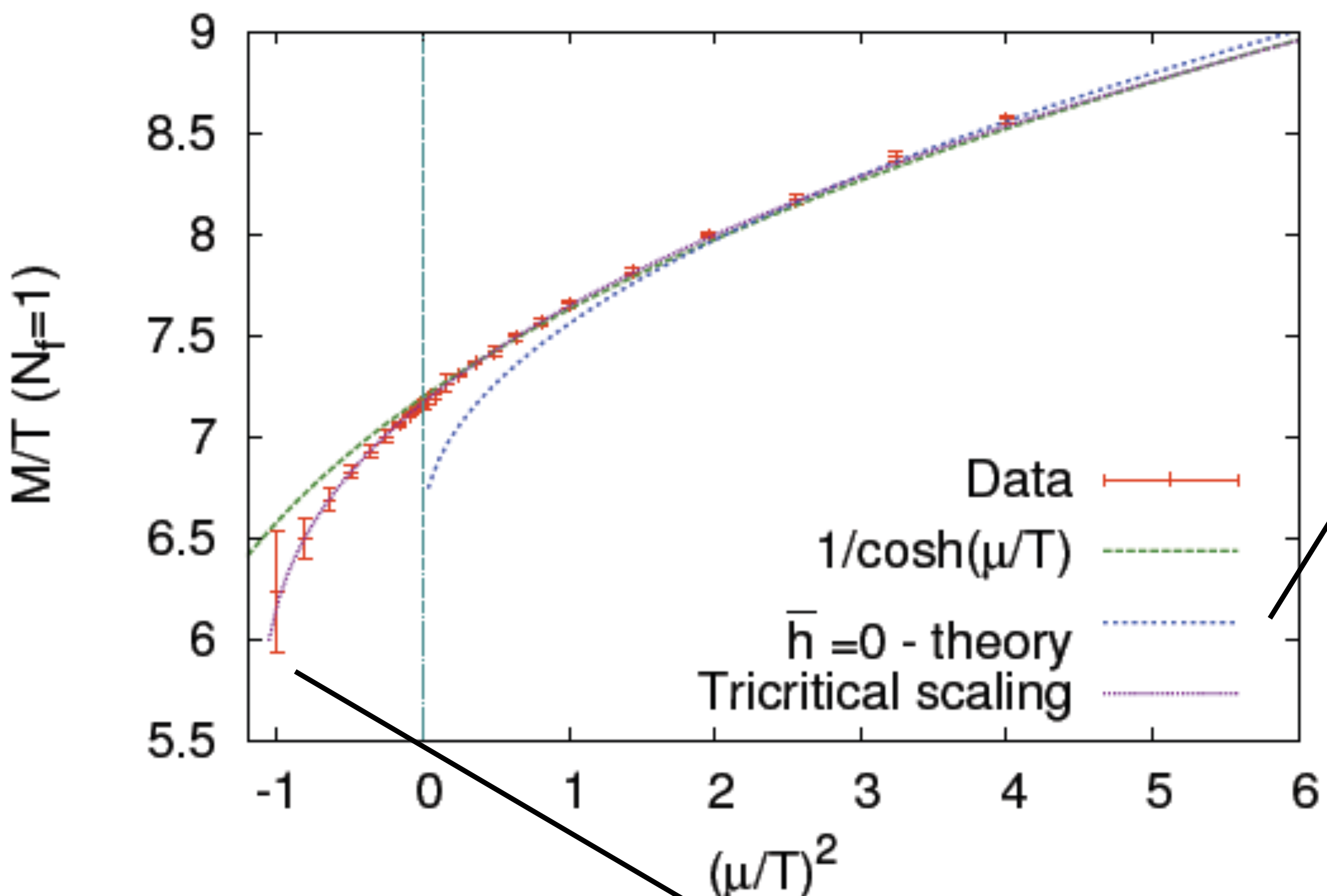
deconfinement critical surface

phase diagram for Nf=2, Nt=6



Critical quark mass as function of chemical potential

dense massive limit $\kappa \to 0, \mu \to \infty, \kappa e^{\mu/T} = \text{constant}$



de Forcrand, O.P. 10
D'Elia, Sanfilippo 10

- Roberge-Weiss (tricritical) endpoint at $\mu_i/T = \pi/3$
  ($\leftrightarrow$ boundary between Z_3-sectors)

Tricritical scaling works perfectly, even well into $\mu^2 > 0$!

$$\frac{M_c}{T} = \frac{M_{\text{tric}}}{T} + K\left[\left(\frac{\pi}{3}\right)^2 + \left(\frac{\mu}{T}\right)^2\right]^{2/5}$$

Cold and dense QCD I: static, strong coupling limit

For T=0 (at finite density) anti-fermions decouple $N_f = 1, h_1 = C, h_2 = 0$

$$C_f \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} = e^{(\mu_f - m_f)/T},\ \bar{C}_f(\mu_f) = C_f(-\mu_f)$$

$$Z(\beta=0) = \left[\prod_f \int \mathrm{d}W \left(1 + C_f L + C_f^2 L^* + C_f^3\right)^2 \right]^{N_s^3}$$

$$\xrightarrow{T\to 0} \left[1 + 4C^{N_c} + C^{2N_c}\right]^{N_s^3}$$

Free gas of baryons!

Quarkyonic?

$$n = \frac{T}{V}\frac{\partial}{\partial\mu}\ln Z = \frac{1}{a^3}\frac{4N_c C^{N_c} + 2N_c C^{2N_c}}{1 + 4C^{N_c} + C^{2N_c}}$$

$$\lim_{\mu\to\infty}(a^3 n) = 2N_c$$

Sivler blaze property + saturation!

$$\lim_{T\to 0} a^3 n = \begin{cases} 0, & \mu < m \\ 2N_c, & \mu > m \end{cases}$$

Cold and dense QCD II: interacting, dynamical

Z(3) breaking part (fermion determinant), including second coupling

$$e^{-S^a_{\rm eff}[W]} = \prod_n \Delta_n[W]$$

$$\Delta_1 = \prod_{f,i} \det[1 + h_{1f} W_i]^2 [1 + \bar{h}_{1f} W_i^\dagger]^2$$

includes qq interaction!

$$\Delta_2 = \prod_{f,<ij>} \left[1 - h_{2f} N_\tau \mathrm{Tr}_c \frac{W_i}{1 + C_f W_i} \mathrm{Tr}_c \frac{W_j}{1 + C_f W_j}\right]^2$$

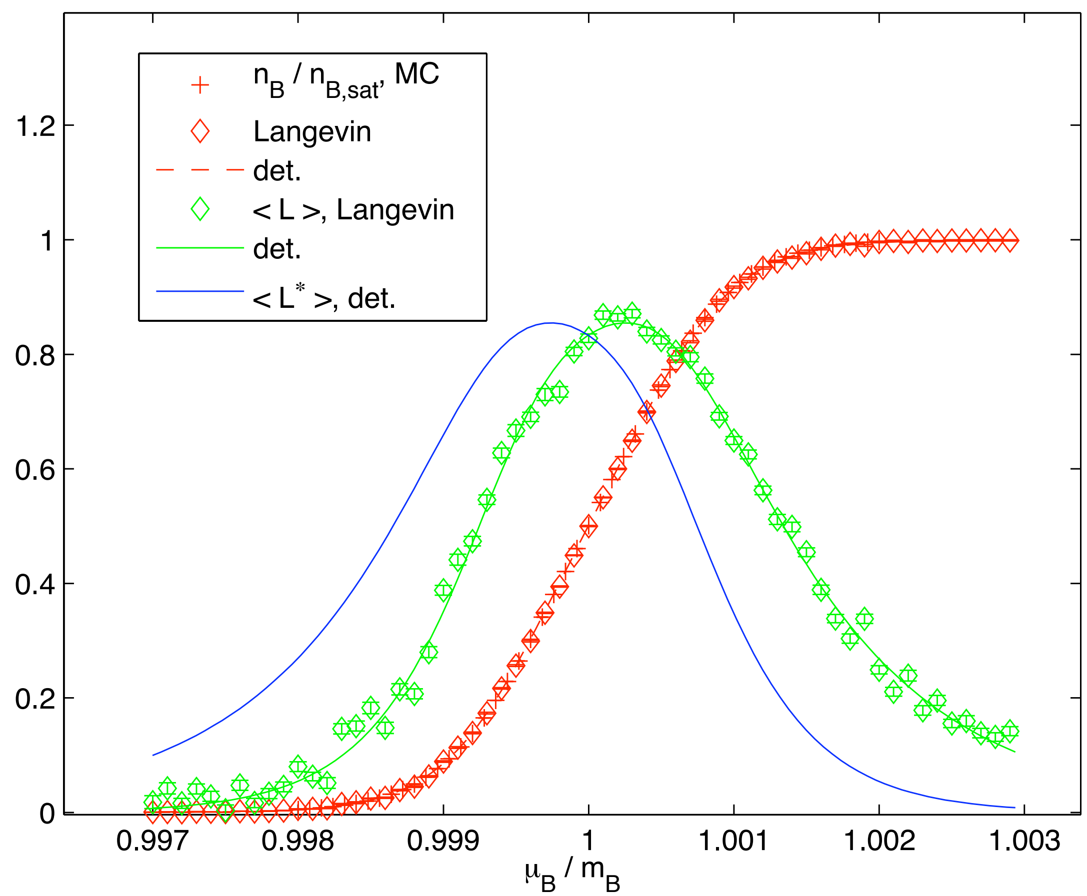
$$h_{1f} = C_f \left[1 + 6\kappa_f^2 N_\tau \frac{u - u^{N_\tau}}{1 - u} + \ldots\right] \qquad h_{2f} = C_f^2 \frac{\kappa_f^2}{N_c} \left[1 + 2\frac{u - u^{N_\tau}}{1 - u} + \ldots\right]$$

$$C_f \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} = e^{(\mu_f - m_f)/T}, \; \bar{C}_f(\mu_f) = C_f(-\mu_f)$$

Cold and dense QCD II: interacting, "dynamical"!

$m_\pi = 20\ \text{GeV}, T = 10\ \text{MeV}, a = 0.17\ \text{fm}$

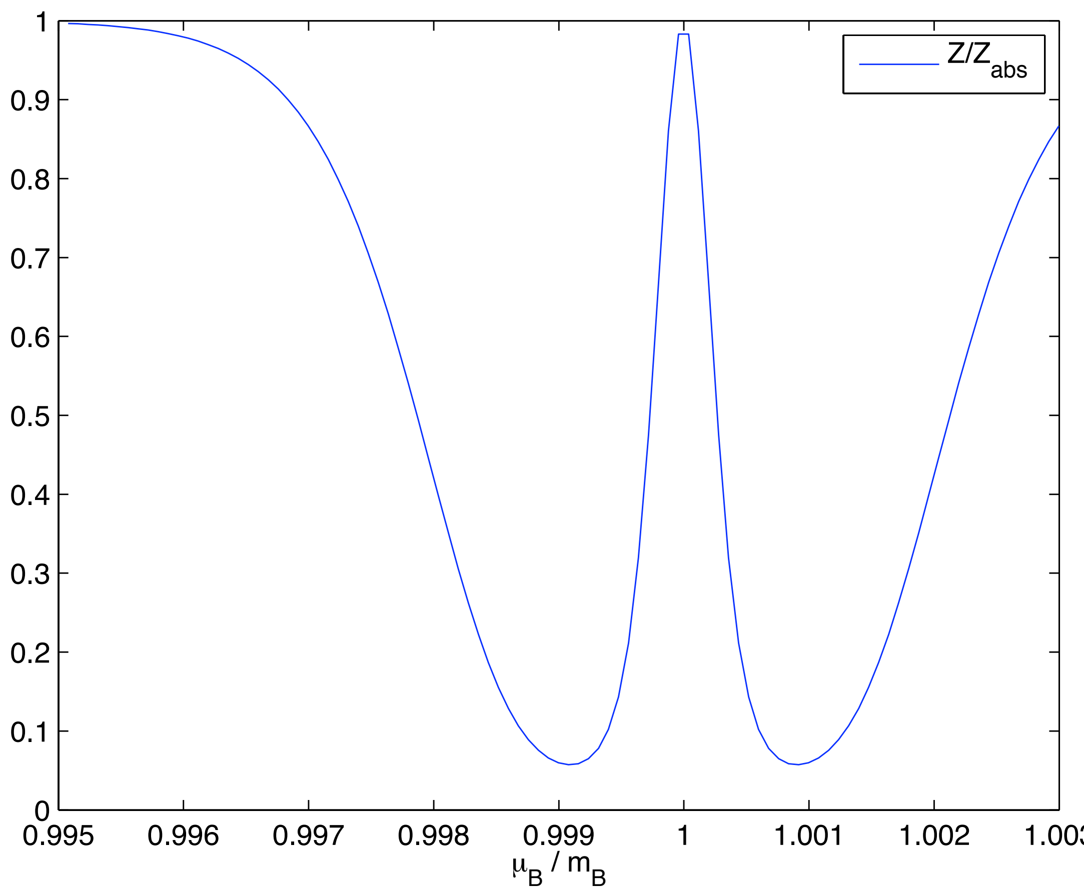
$\beta = 5.7, \kappa = 0.0000887, N_\tau = 116$



Analytic strong coupling soln. valid!

$\lambda(\beta = 5.7, N_\tau = 115) \sim 10^{-27}$

Metropolis, average sign:

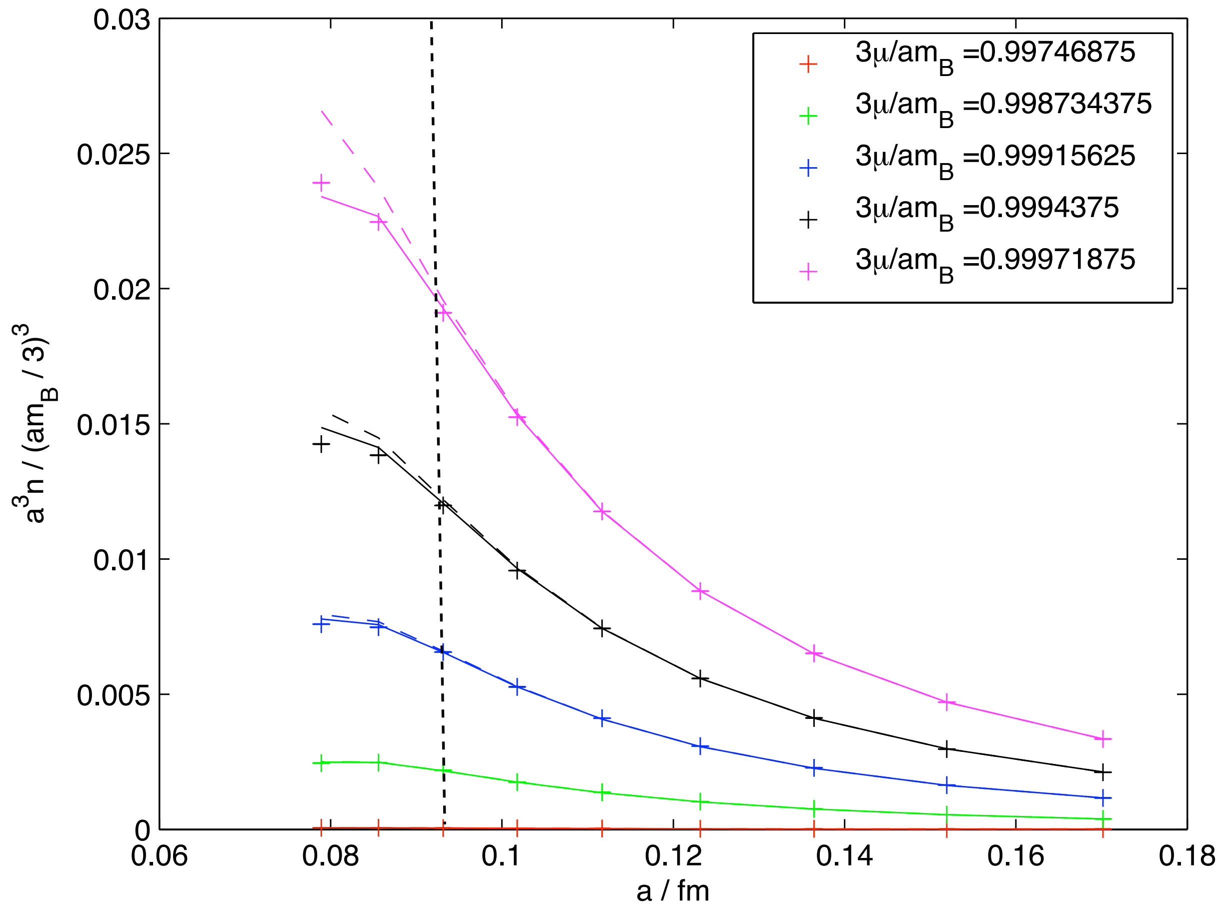


Complex Langevin:
convergence criteria satisfied,
cf. Seiler, Stamatescu, arts, James 12

Continuum extrapolation

Scaling with lattice spacing:

$$\frac{n_{\rm lat}(\mu)}{m_B^3} = \frac{n_{\rm cont}(\mu)}{m_B^3} + A(\mu)a + B(\mu)a^2 + \ldots$$

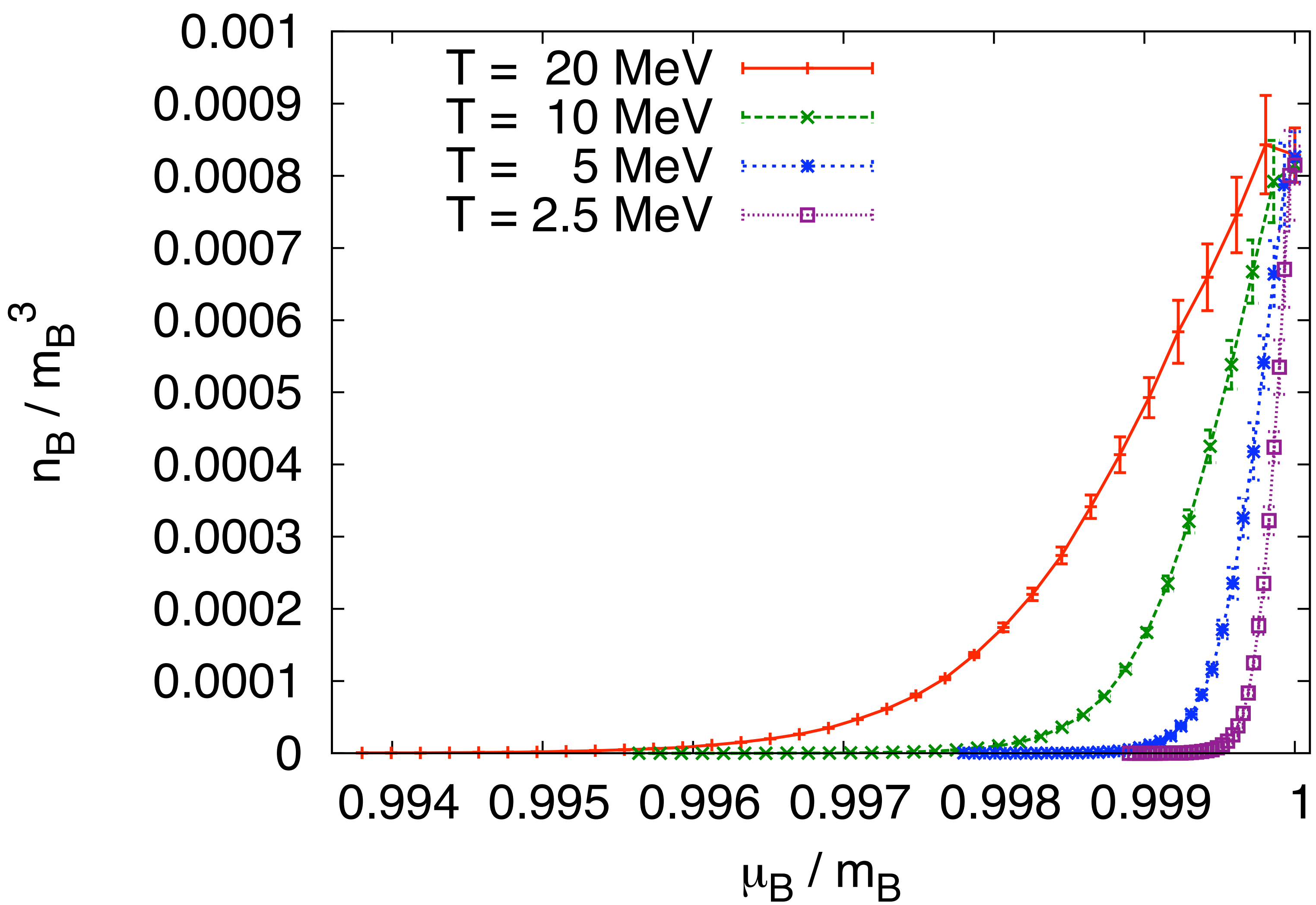


Solid/dashed lines: analytic strong coupling limit with/without $\mathcal{O}(\kappa^2)$:

Breakdown of hopping series!

The silver blaze property of QCD in the continuum

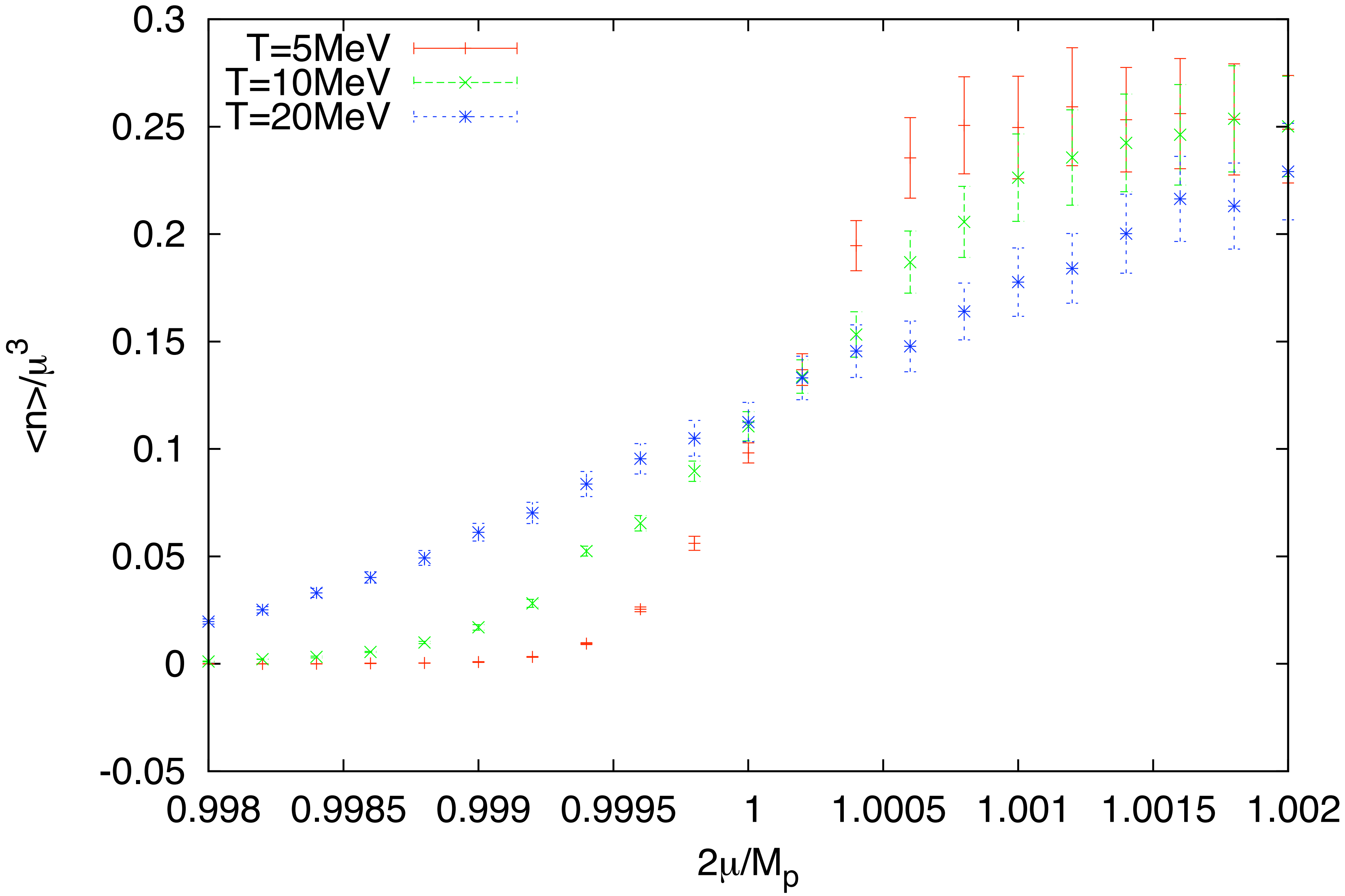
... with very heavy quarks



consistent with physical nuclear density

$$\sim 0.16\ \mathrm{fm}^{-3} \approx 0.15 \cdot 10^{-2}\ m_{\mathrm{proton}}^3$$

Different units: free gas behaviour at large density!



Conclusions

- Two-step treatment of QCD phase transitions:

  I. Derivation of effective action by strong coupling expansion
  II. Simulation of effective theory

- Z(N)-invariant effective theory for Yang-Mills, correct order of p.t., Tc with better than 10% accuracy in the continuum limit!

- Finite T deconf. transition for heavy fermions and all chemical potentials

- Silver blaze property + 1st. order phase transition to nuclear matter at T=0