A Supersymmetric Lattice Theory: $\mathcal{N} = 4$ YM

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November 29, 2012

Lattice SUSY - the problems and how to dodge them

$\mathcal{N} = 4$ Super Yang-Mills: new formulation

Non-perturbative study: phase diagram

Barriers to Lattice Supersymmetry

- $\{Q, \overline{Q}\} = \gamma_\mu p_\mu$. No generators of infinitessimal translations on lattice. Equivalently: no Leibniz rule for difference ops on lattice: $\Delta(AB) \neq \Delta AB + A\Delta B$.
- Classical SUSY breaking leads to (many) SUSY violating ops via quantum corrections. Couplings must be adjusted with cut-off $(1/a)$ to achieve SUSY in continuum limit -fine tuning.
- Discretization of Dirac equation: Lattice theories contain additional fermions (doublers) which do not decouple in continuum limit. Consequence: no. fermions $\neq$ no. bosons
- Lattice gauge fields live on lattice links and take values in group. Fermions live on lattice sites and (for adjoint fields) live in algebra ....

Putting SUSY on a lattice

Goals of any successful SUSY lattice formulation:

- Reduce/eliminate fine tuning. In particular scalar masses.
- More symmetrical treatment of bosons and fermions - particularly for gauge theories.
- Keep exact gauge invariance. Lesson of lattice QCD (Wilson)
- Avoid fermion doubling...
- Avoid sign problems. After integration over fermions is effective bosonic action real ? Monte Carlo simulation requires this ...

$\boxed{\text{New formulations exist with all these features}}$

New ideas - twisting

- Rewrite continuum theory in twisted variables.
- Exposes a single scalar supersymmetry $\mathcal{Q}$ whose algebra is simple: $\mathcal{Q}^2 = 0$. Furthermore $S =\sim \mathcal{Q}\Lambda$.
- Key: this SUSY can be retained on discretization: easy to build invariant lattice action.
- Fine tuning reduced (eliminated ?):

$$\textbf{Exact hypercubic symmetry} \quad \stackrel{a\to 0}{\to} \quad \textbf{Full Poincare invariance}$$
$$\textbf{Exact } \mathcal{Q} \textbf{ symmetry} \quad \to \quad \textbf{Full SUSY}$$

- See that $\boxed{\text{all}}$ fields will live on links and take values in algebra.
- Structure of fermionic action dictated by exact SUSY - would doublers will be physical

Most interesting application: $\mathcal{N} = 4$ SYM

Many lattice SUSY theories in $D < 4$.
However in $D = 4$ they single out a unique theory: $\mathcal{N} = 4$ YM

- Fascinating QFT - finite but non-trivial. A lattice formulation gives a non-perturbative definition of theory (like lattice QCD for QCD)
- Heart of AdS/CFT correspondence. Equivalence between string theory in AdS_5 and $\mathcal{N} = 4$ SYM on boundary. Lattice formulation allows us to verify and extend holographic ideas: compute classical and quantum string corrections ... (expansion in $1/N$ and $1/\lambda$)
- Possible connection to low energy physics: Prototype CFT (SU(2) adj ..). Higgs as a dilaton arising from scalar fluctuations along flat directions ?

Twisting - basic idea

- Consider (extended) SUSY theories possessing additional flavor (R) symmetries.
- If flavor contains an appropriate subgroup can Twist: decompose fields under $G = \mathrm{Diag}(SO_{\mathrm{Lorentz}}(D) \times SO_{\mathrm{R}}(D))$
- Fermions: spinors under both factors – become $\boxed{\text{integer}}$ spin after twisting.
- Scalars transform as vectors under R-symmetry – $\boxed{\text{vectors}}$ after twisting.
- Gauge fields remain vectors – combine with scalars to make $\boxed{\text{complex}}$ gauge fields. Still just $U(N)$ gauge symmetry...

$\boxed{\text{Important: flat space: just a change of variable}}$

Twisted (Lattice) Fields for $\mathcal{N}=4$

Usual fields	Twisted fields
$A_\mu, \mu = 1\ldots4 \quad \phi_i, i = 1\ldots6$	$\mathcal{U}_a, a = 1\ldots5$
$\Psi^i, i = 1\ldots4$	$\eta, \psi_a, \chi_{ab}, a, b = 1\ldots5$

- Scalars appear as $\mathrm{Im}\,\mathcal{U}_a$! (miracle of twisting...)
- Fermions appear as anticommuting antisymmetric tensors ...
- All Lattice fields live on links.
- Lattice is determined: 5 (complex) gauge fields $\rightarrow$ lattice with (equal) 5 basis vectors. 4D implies $\sum_{a=1}^{5} \mathbf{e}^a = 0$. $\boxed{A_4^*}$
- All fields take values in $U(N)$ algebra.
- Fields transform like links: $\psi_a \rightarrow G(x)\psi_a(x)G^\dagger(x+a)$

Twisted supersymmetry

Scalar $\mathcal{Q}$ arises after transforming to twisted variables: decompose fields under twisted rotation group

$$SO(4)' = \mathrm{diag}\left(SO_{\mathrm{rot}}(4) \times SO_R(4)\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{U}_a &= \psi_a \\
\mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\overline{\mathcal{U}}_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

field strength: $\mathcal{F}_{ab} = \mathcal{U}_a(x)\mathcal{U}_b(x+a) - \mathcal{U}_b(x)\mathcal{U}_a(x+b)$

$\mathcal{Q}$-exact form of action

$$S = \frac{N}{\lambda}(S_1 + S_2)$$

$$S_1 = \mathcal{Q} \sum_x \mathrm{Tr} \left(\chi_{ab}(x)\mathcal{F}_{ab}(x) + \eta[\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) + \frac{1}{2}\eta d \right)$$

where $\mathcal{F}_{ab} = \mathcal{D}_a^{(+)}U_b(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+a) - \mathcal{U}_b(x)\mathcal{U}_a(x+b)$

$$S_2 = \sum_x \mathrm{Tr}\ \epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c^{(-)}\chi_{de}$$

$\mathcal{Q}^2 = 0$ guarantees S_1 invariant. $\mathcal{Q}\chi_{ab} = \overline{\mathcal{F}}_{ab}$ plus Bianchi yield $\mathcal{Q}S_2 = 0$

Derivatives

Derivatives replaced with covariant differences compatible with lattice G.I eg.

$$\mathcal{D}_a^{(+)}\mathcal{U}_b(x) = \mathcal{F}_{ab}(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+a) - \mathcal{U}_b(x)\mathcal{U}_a(x+b)$$

Transforms like link $x \rightarrow x + a + b$
Contract with $\chi_{ab}(x)$ to form gauge invariant loop ...

$$\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) = \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-a)\mathcal{U}_a(x-a)$$

Transforms like site - contract with η

Lattice action

$\boxed{\text{Twisting=change of variables in flat space}}$

$$
\begin{aligned}
S_1 &= \sum_{\mathbf{x}} \mathrm{Tr}\left(\mathcal{F}^{\dagger}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a \mathcal{U}_a\right)^2 \right. \\
&- \left. \chi_{ab}\mathcal{D}^{(+)}_{[a}\psi_{\,b]} - \eta\overline{\mathcal{D}}^{(-)}_a\psi_a\right) \\
S_2 &= -\frac{1}{2}\sum_{\mathbf{x}} \mathrm{Tr}\epsilon_{abcde}\chi_{de}(\mathbf{x}+\mu_{\mathbf{a}}+\mu_{\mathbf{b}}+\mu_{\mathbf{c}})\overline{\mathcal{D}}^{(-)}_{\mathbf{c}}\chi(\mathbf{x}+\mu_{\mathbf{c}})
\end{aligned}
$$

- Bosonic action collapses to $\boxed{\text{Wilson plaquette}}$ if $\mathcal{U}^{\dagger}_a\mathcal{U}_a = 1$.
- Fermions: Kähler-Dirac action $\equiv$ (reduced) staggered fermions Describes 4 (Majorana) fermions in continuum limit.

Gauge invariance, doublers and all that

- All terms local, correspond to closed loops and hence are lattice gauge invariant
- $\mathcal{U}_a$'s non compact! $\mathcal{U}_a = \sum_B T^B \mathcal{U}_a^B$ – flat measure $\int \prod D\mathcal{U}_a D\overline{\mathcal{U}}_a$. Nevertheless, still gauge invariant - Jacobians resulting from gauge transformation of $\mathcal{U}$ and $\overline{\mathcal{U}}$ cancel.
- Bigger question: how to generate correct naive continuum limit requires that can expand (suitable gauge) $\mathcal{U}_a = I + \mathcal{A}_a(x) + \ldots\ldots$?

Naive continuum limit

- Need $\mathcal{U}_a = I + \mathcal{A}_a(x) + \ldots$. Here, unlike lattice QCD, unit matrix necessary for generating kinetic terms arises from the vev of a dynamical field! - trace piece of imaginary part (scalar) of the gauge field
- Ensure by adding gauge invariant potential

$$\delta S = \mu^2 \sum_{x,a} \left(\frac{1}{N} \mathrm{Tr}\, (\mathcal{U}_a^\dagger \mathcal{U}_a) - 1 \right)^2$$

To leading order: If $\mathcal{U}_a = e^{A_a + iB_a}$ then $\mathrm{Tr}\, B_a = 0$.

- Breaks $\mathcal{Q}$ SUSY softly. All breaking terms must vanish for $\mu \to 0$ (exact $\mathcal{Q}$).

Quantum corrections ...

Can show:

- Lattice theory renormalizable: only counterterms allowed by exact symmetries correspond to terms in original action
- Effective potential (formally) vanishes to all orders in p. theory. No scalar mass terms!
- At one loop:
  - No fine tuning: common wavefunction renormalization
  - Vanishing beta function: Divergence structure matches continuum ....
- Need to go beyond p. theory. Phase diagram of lattice theory

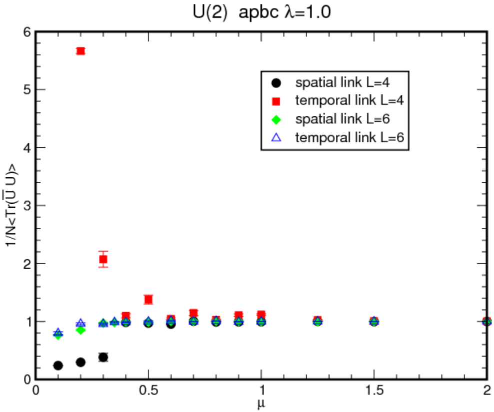
Sketch of $\Gamma_{\rm eff}=0$

- Classical vacua $\boxed{\text{constant commuting complex matrices } \mathcal{U}_\mu}$
- Expand to quadratic order about generic vacuum $\mathcal{U}_b(x) = I + \mathcal{A}^c_b + a_b(x)$. Integrate
- Bosons $\det^{-5}\left(\overline{\mathcal{D}}^{(-)}_a \mathcal{D}^{(+)}_a\right)$
- Ghosts+Fermions: $\det\left(\overline{\mathcal{D}}^{(-)}_a \mathcal{D}^{(+)}_a\right) + \left(Pf(M_F) \overset{Maple}{=} \det^4\left(\overline{\mathcal{D}}^{(-)}_a \mathcal{D}^{(+)}_a\right)\right)$
- Thus $Z_{\rm pbc}=1$ at 1-loop. $\mathcal{Q}$-exact structure – result good to all orders! $\boxed{\text{Exact quantum moduli space}}$
- Witten index: all states cancel except vacua. Counting indep of g.

Simulations

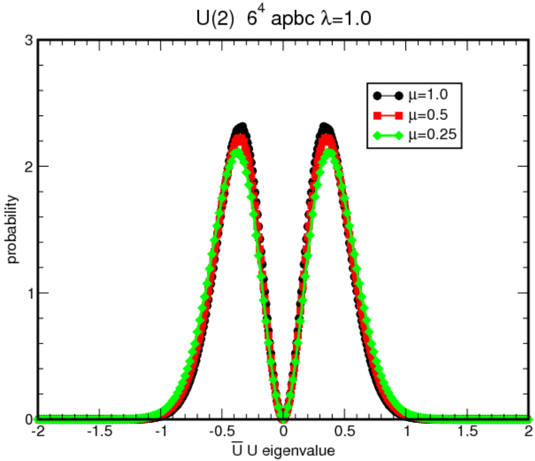
- Integrate fermions $\rightarrow \mathrm{Pf}(M)$. Realize as $\det\left(M^{\dagger}M\right)^{-\frac{1}{4}}$
- Standard lattice QCD algs may be used: RHMC with Omelyan, multi time step evolution. GPU acceleration for inverter (speedup: 5-10 over single core code for $L = 8^3 \times 16$)
- Phase quenched approximation should be ok: analytical argument, numerical evidence ...
- First step: phase structure .... $U(2), L^4, apbc, \; L = 4, 6, 8$
  - Fix the unit matrix vev ? Instabilities from flat directions ?
  - Supersymmetry realized ?
  - String tension, chiral symmetry breaking ?
  - Phase transitions ?

Setting the vev



Classical vev stable $< \mathcal{U}_a \mathcal{U}_a >= I$ for $\mu > \mu_*$ with μ_* decreasing for increasing L

$SU(2)$ Flat directions - I

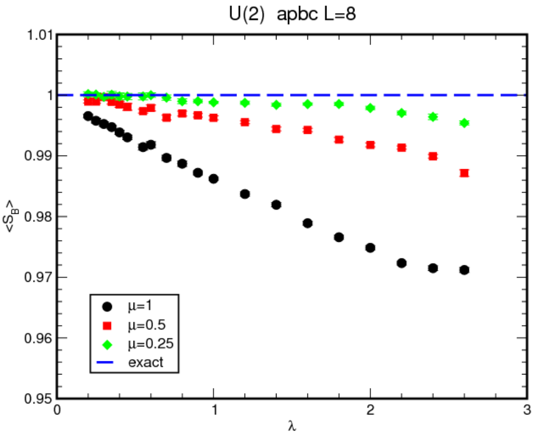


Distribution of scalars insensitive to μ

Comments

- Common statement: "Moduli space is not lifted in $\mathcal{N} = 4$ by quantum corrections ..."
  Why is scalar distribution not flat as $\mu \to 0$?
- Pfaffian vanishes on flat directions. Formally this zero cancels against infinity from boson zero modes but latter are lifted at non-zero μ.
- Thus configurations corresponding to flat directions make no contribution to lattice path integral.
- Small fluctuations around flat directions cost increasing action as move away from origin in field space - large scalar eigenvalues suppressed.

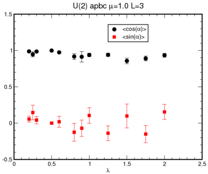
Test of exact supersymmetry



For $\mu \to 0$ S_B given by simple $\mathcal{Q}$ Ward identity.

Sign problem ?

- Integrate fermions: complex Pfaffian. But observed phase small in phase quenched simulations..



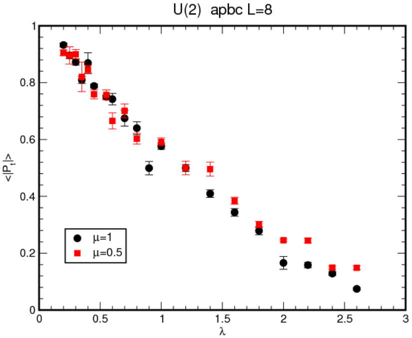
Why ? For $\mu = 0$ and pbc one can show that $Z^{1-loop}_{\text{lattice}} = 1$ indep of λ! No phase appears!

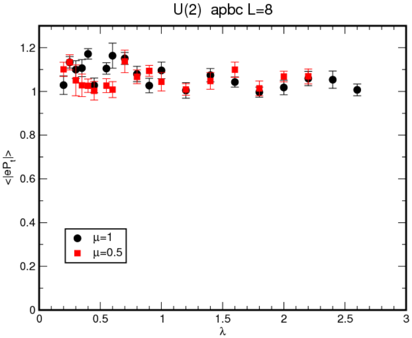
$\boxed{\text{Exact } \mathcal{Q} \text{ symmetry -(formally) true to all orders in p theory!}}$

Phase structure - Polyakov lines

$$P = \prod \mathcal{U}_t$$

$$P = \prod \exp\ (\mathcal{U}_t - \mathrm{Tr}\,\mathcal{U}_t)$$

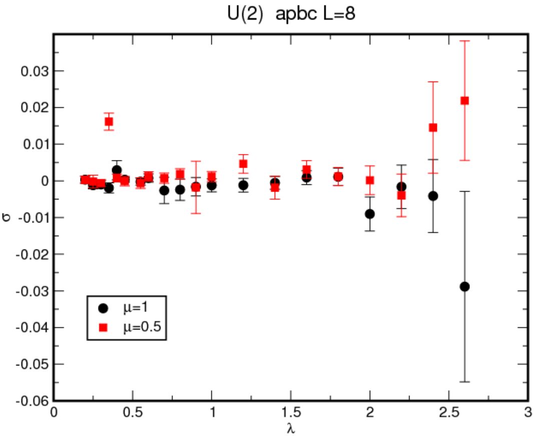




Traceless modes yield λ-indep Polyakov line

Consistent with SUSY: $\mathcal{Q}\overline{\mathcal{U}} = 0$

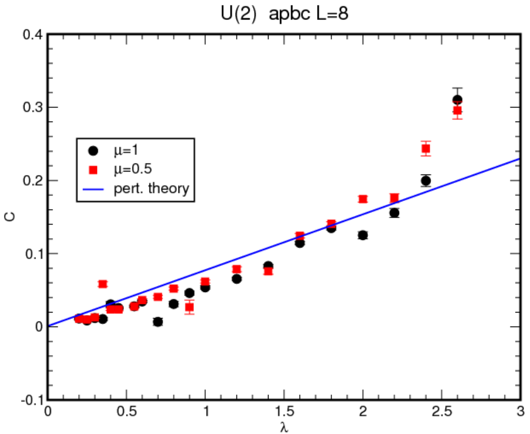
Phase structure - String tension



Extract by fitting $W(R, T)$ to $e^{-V(R)T}$ and extract σ

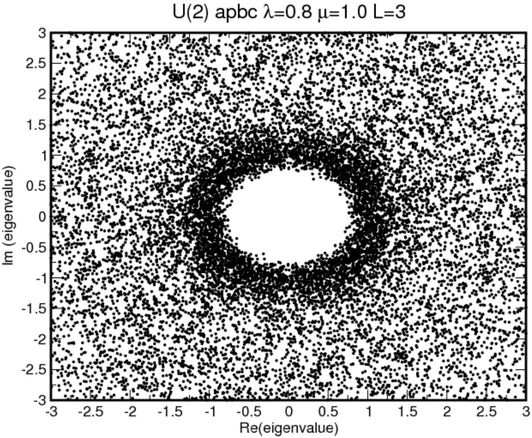
Vanishing for all λ

Coulomb fits



Fits are good and consistent with p theory .. no sign of Maldacena $\sqrt{\lambda}$ behavior

Chiral symmetry breaking - or lack of it ..



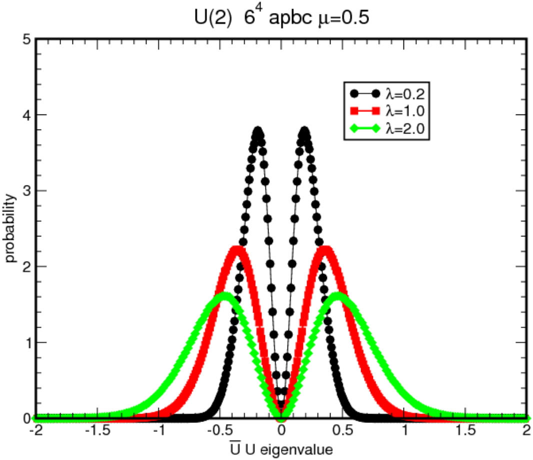
Eigenvalues excluded from origin: insensitive to μ and λ

Conclusions

- Simulations of $\mathcal{N} = 4$ YM look promising: gauge invariance and (some) SUSY can be preserved. No instabilities from flat directions, no sign problem.
- Prelim investigations show no sign of any phase transitions as vary λ. String tension small and static quark potential best fit with simple Coulomb term. Evidence for single, deconfined phase.
- Consistent with pert theory: 1 loop calc shows $\beta_{\text{latt}}(\lambda) = 0$
- In addition to tests of AdS/CFT theory may serve as test bed for lattice theories with IRFPs

The end

$SU(2)$ Flat directions - II



Localized distribution for all λ

frame

- Establish phase structure definitively ... using large lattices (better quark potentials), push to stronger λ, smaller μ
- Examine the spectrum: evidence of SUSY, anomalous dims. Compare to known results in $\mathcal{N}=4$
- Check restoration of full SUSY: study broken SUSY Ward identities, determine how much fine tuning needed.
- Need to understand how to take continuum limit; in QCD send $\beta \to \infty$ with ever increasing L. In CFT g is parameter - does not determine lattice spacing. Continuum physics by increasing L. But how to tune μ ?

Exciting time - lots to do !!