Lattice field theories with dual variables

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Euclidean path integral, complex action problem and dual representation

- Vacuum expectation values with Feynman's path integral:

$$\langle O \rangle \; = \; \frac{1}{Z} \int D[\psi] \; e^{\,-\,S[\psi]} \; O[\psi]$$

- In a Monte Carlo simulation observables are computed as averages over field configurations ψ distributed according to

$$P[\psi] \; = \; \frac{1}{Z} \, e^{-S[\psi]}$$

- For finite chemical potential μ the action $S[\psi]$ is complex and the Boltzmann factor cannot be used as probability weight in a stochastic process.

Rewriting a system in terms of new variables where only real and positive terms appear in the partition sum could overcome the complex action problem.

I will discuss two examples

- Relativistic Bose gas = charged ϕ^4 field with chemical potential.

- Scalar electrodynamics with 2 flavors and chemical potential.

Y. Delgado Mercado, H.G. Evertz, C. Gattringer, Phys. Rev. Lett., 2011

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Y. Delgado Mercado, A. Schmidt, C. Gattringer, in preparation

C. Gattringer, T. Kloiber, in preparation

Example for a dual representation: Charged ϕ^4 field

- Continuum action:

$$S \;=\; \int d^4x \Big[\phi(x)^\star [m^2 - \mu^2 - \triangle] \phi(x) \;+\; \lambda\, |\phi(x)|^4 \Big] \;+\; i\mu N$$

- Action on the lattice:

$$S \;=\; \sum_x \Big[\, \kappa \, |\, \phi_x \,|^2 \;+ \lambda \, |\, \phi_x \,|^4 - \sum_{j=1}^{3} \Big(\phi_x^\star \, \phi_{x+\widehat{j}} \;+\; \phi_x^\star \, \phi_{x-\widehat{j}} \Big) \\ \qquad - \; \phi_x^\star \, e^{-\mu} \, \phi_{x+\widehat{4}} \;+\; \phi_x^\star \, e^{\mu} \, \phi_{x-\widehat{4}} \Big]$$

C. Gattringer, T. Kloiber, arXiv:1206.2954

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Dual representation – I

- Expand the individual nearest neighbor terms:

$$e^{\,e^{-\mu\,\delta_{\nu,4}}\,\phi_x^\star\,\phi_{x+\widehat{\nu}}} \;=\; \sum_{j_{x,\nu}=0}^{\infty} \frac{\left(e^{-\mu\,\delta_{\nu,4}}\right)^{j_{x,\nu}}}{(j_{x,\nu})!}\;\left(\phi_x\right)^{j_{x,\nu}}\;\left(\phi^\star_{x+\widehat{\nu}}\right)^{j_{x,\nu}}$$

$$e^{\,e^{\mu\,\delta_{\nu,4}}\,\phi_x^\star\,\phi_{x-\widehat{\nu}}} \;=\; \sum_{\overline{j}_{x,\nu}=0}^{\infty} \frac{\left(e^{\mu\,\delta_{\nu,4}}\right)^{\overline{j}_{x,\nu}}}{(\overline{j}_{x,\nu})!}\;\left(\phi_x\right)^{\overline{j}_{x,\nu}}\;\left(\phi^\star_{x-\widehat{\nu}}\right)^{\overline{j}_{x,\nu}}$$

- Idea: Use the $j_{x,\nu}$ and $\overline{j}_{x,\nu}$ as the new degrees of freedom.

- Remaining ϕ-integrals at a site x :

$$\int_{\mathbb{C}} d\,\phi_x\;\; e^{-\kappa|\phi_x|^2-\lambda|\phi_x|^4}\;\left(\phi_x\right)^{F(j,\overline{j})}\left(\phi_x^\star\right)^{\overline{F}(j,\overline{j})}$$

$F_x(j,\overline{j})$, $\overline{F}_x(j,\overline{j}) \in \mathbb{N}_0$ are linear combinations of the j and $\overline{j}$ variables attached to the site x. They correspond to the total j, $\overline{j}$-flux at x.

Dual representation – II

- Using $\phi_x = r\, e^{i\theta}$ the integrals at a site x read:

$$\int_{\mathbb{C}} d\,\phi_x \;\; e^{-\kappa|\phi_x|^2 - \lambda|\phi_x|^4} \, (\phi_x)^{F(j,\overline{j})} \, (\phi_x^\star)^{\overline{F}(j,\overline{j})} \; = $$

$$\int_0^\infty dr \; r^{\,F_x + \overline{F}_x + 1} \; e^{-\kappa r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta \; e^{\,i\theta\,[F_x - \overline{F}_x]} \; = \; \mathcal{I}(F_x + \overline{F}_x)\; \delta(F_x - \overline{F}_x)$$

- At every site there is a weight factor $\mathcal{I}(F_x + \overline{F}_x)$ and a constraint.

- The constraint $\delta(F_x - \overline{F}_x)$ forces the total flux $F_x - \overline{F}_x$ at x to vanish.

- The structure can be simplified by using linear combinations $k_{x,\nu} \in \mathbb{Z}$ and $l_{x,\nu} \in \mathbb{N}_0$ of the original variables $j_{x,\nu}$ and $\overline{j}_{x,\nu}$.

- Only the $k_{x,\nu}$ are subject to constraints.

Dual representation – III (final form)

- The original partition function is mapped exactly to a sum over configurations of the dual variables k and l:

$$Z \;=\; \sum_{\{k,l\}} \mathcal{W}(k,l)\, \mathcal{C}(k).$$

- Weight factor (real and positive):

$$\mathcal{W}(k,l) \;=\; \prod_{x,\nu} \frac{1}{(|\,k_{x,\nu}\,| + l_{x,\nu})!\; l_{x,\nu}!} \\ \times\; \prod_{x} e^{\,-\mu\, k_{x,4}}\; \mathcal{I}\Big(\sum_{\nu} [|\,k_{x,\nu}\,| + |\,k_{x-\widehat{\nu},\nu}\,| + 2(l_{x,\nu} + l_{x-\widehat{\nu},\nu})]\Big)$$

- Constraint (only for k-variables):

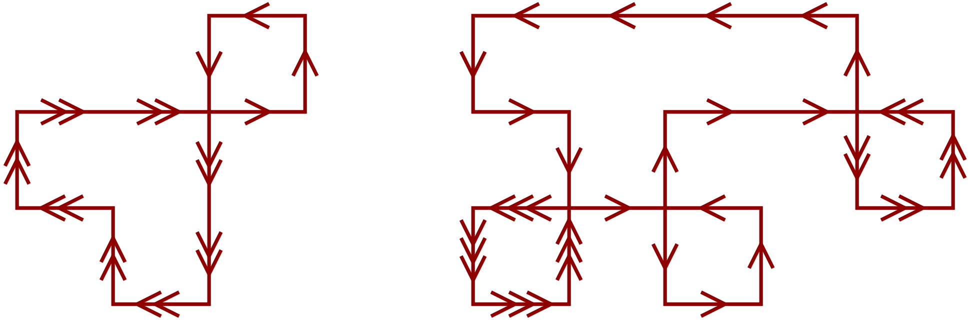
$$\mathcal{C}(k) \;=\; \prod_{x} \delta\Big(\sum_{\nu} [\, k_{x,\nu} - k_{x-\widehat{\nu},\nu}\,]\Big)$$

Admissible configurations are loops:

- Constraint from the integration over the U(1) phases:

$$\forall \; x \; : \qquad f_x \; = \; \sum_{\nu} \left[\, k_{x,\nu} - k_{x-\widehat{\nu},\nu} \, \right] \; = \; 0$$

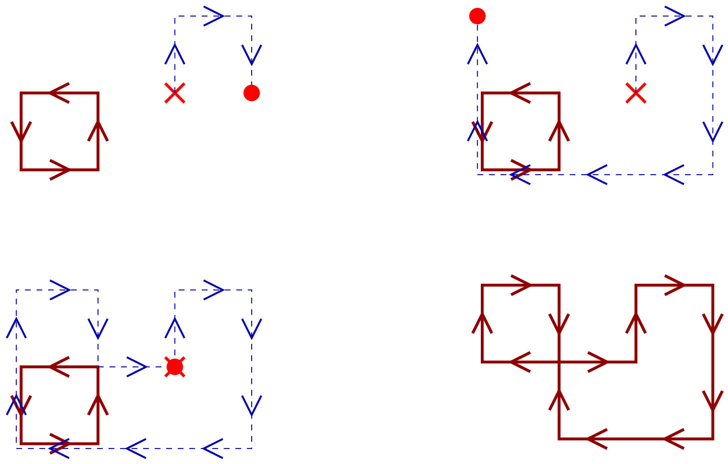
- Admissible configurations of dynamical variables are loops of flux:



- Chemical potential gives different weight to forward and backward temporal flux.

Worm algorithm

- A worm locally violates the constraint and propagates the defect until the worm closes and the constraint is healed.



- Every step is accepted with the Metropolis probability computed from

$$\mathcal{W} = \prod_{x,\nu} \frac{1}{(|k_{x,\nu}|+l_{x,\nu})!\, l_{x,\nu}!} \prod_{x} e^{-\mu\, k_{x,4}}\, \mathcal{I}\Big(\sum_{\nu} [|k_{x,\nu}| + |k_{x-\widehat{\nu},\nu}| + 2(l_{x,\nu}+l_{x-\widehat{\nu},\nu})]\Big)$$

Bulk observables

- Bulk observables are obtained as derivatives of the free energy with respect to the parameters.

- They have the form of averages and fluctuations of the dual variables.

- Observables related to the particle number:

$$n \;=\; \frac{T}{V}\frac{\partial \, \ln Z}{\partial \mu} \;=\; \frac{1}{N_s^3\, N_t}\frac{\partial \, \ln Z}{\partial \mu} \quad , \quad \chi_n \;=\; \frac{\partial \, n}{\partial \mu}$$

- Observables related to field expectation values:

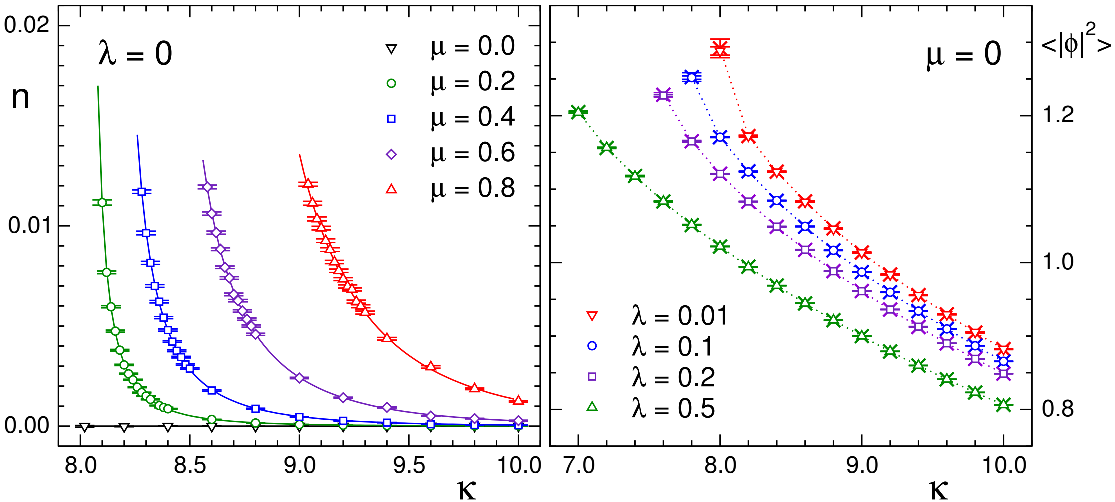
$$\langle | \, \phi \, |^2 \rangle \;=\; \frac{-T}{V}\frac{\partial \, \ln Z}{\partial \kappa} \;=\; \frac{-1}{N_s^3\, N_t}\frac{\partial \, \ln Z}{\partial \kappa} \quad , \quad \chi_\phi \;=\; \frac{-\partial \langle | \, \phi \, |^2 \rangle}{\partial \kappa}$$

- Dual forms:

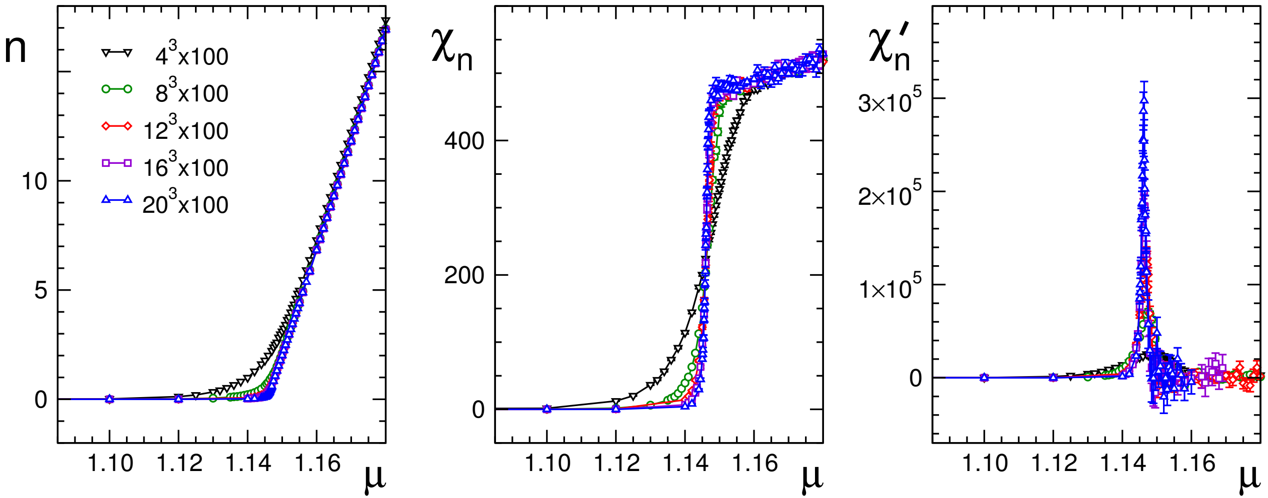
$$n \;=\; \frac{1}{N_s^3\, N_t}\left\langle \sum_x k_{x,4} \right\rangle \quad , \quad \langle | \, \phi \, |^2 \rangle \;=\; \frac{1}{N_s^3\, N_t}\left\langle \sum_x \frac{\mathcal{I}(f_x + 2)}{\mathcal{I}(f_x)} \right\rangle$$

Checks

Simulation with dual variables can be checked with high precision:



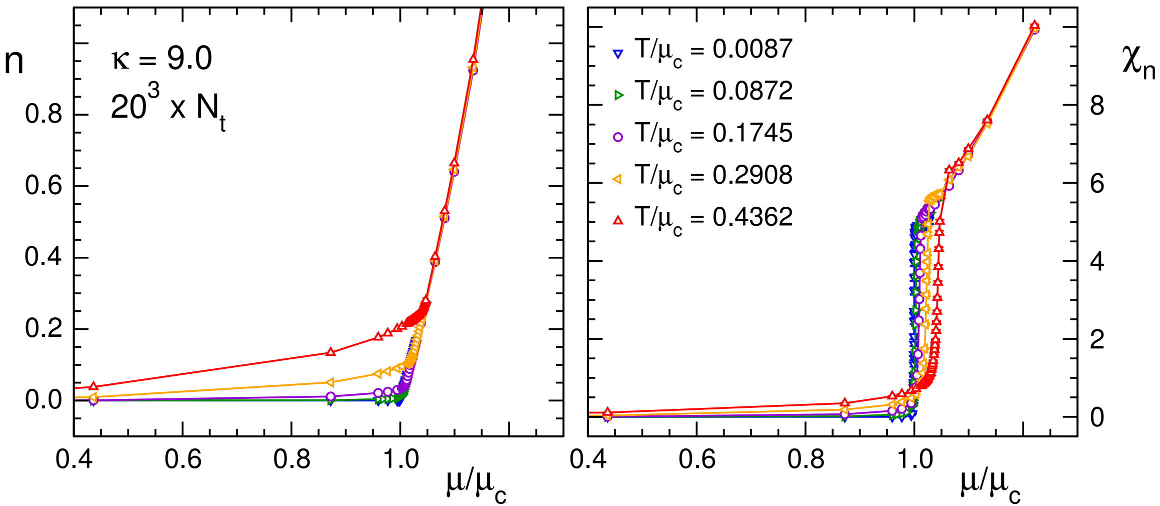
Thermodynamics at zero temperature



Second order transition at the end of the Silver Blaze region.

Cross checked with the complex Langevin study by G. Aarts.

Non-zero temperature

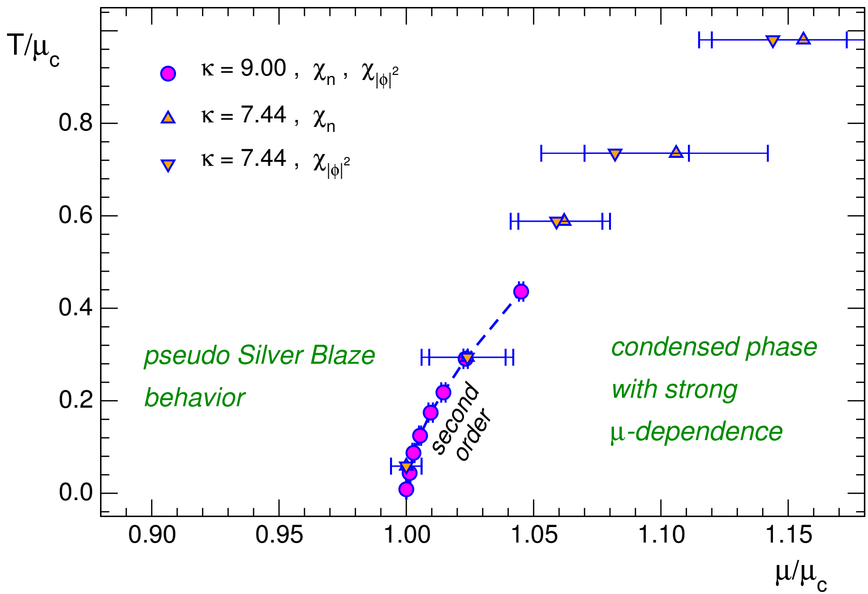


Observables depend on μ throughout.

Still pronounced transition behavior.

Phase diagram

Phase diagram in the $\mu - T$ plane:

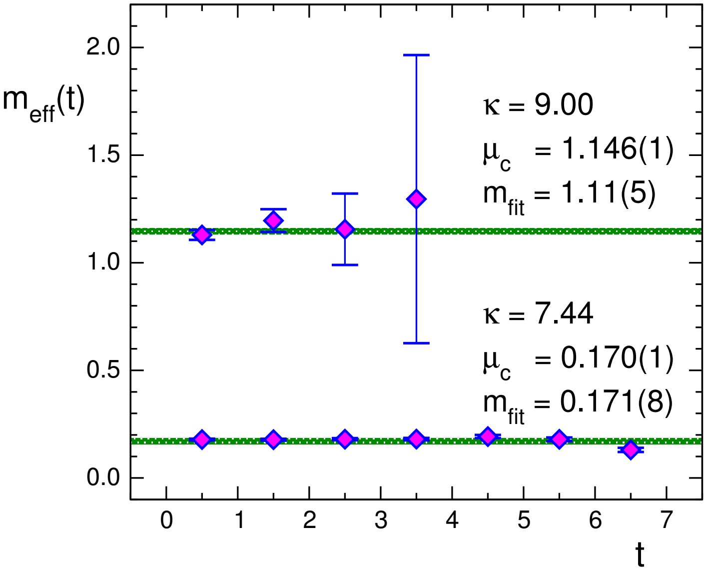


Silver Blaze transition persists for finite temperature.

Mass of lowest excitation

Important test: $\mu_c = m_1$???

Compare μ_c to effective masses from a conventional simulation:



Yes !!

Spectroscopy at finite density $\Rightarrow$ Dual spectroscopy

- Zero momentum propagator

$$C(t) \;=\; \sum_{\vec{x}} \langle\, \phi_{\vec{x},t}\, \phi^*_{\vec{0},0}\, \rangle \;\;\propto\;\; e^{-E_0 t}$$

$$\langle\, \phi_y\; \phi^*_z \,\rangle \;=\; \frac{1}{Z} \int D[\phi]\, e^{-S}\, \phi_y\, \phi^*_z \;=\; \frac{Z_{y,z}}{Z}$$

- Dual representation of the partition sum $Z_{y,z}$ with two insertions:

$$Z_{y,z} = \sum_{\{k,l\}} \prod_{x,\nu} \frac{1}{(|\,k_{x,\nu}\,| + l_{x,\nu})!\, l_{x,\nu}!} \prod_x \delta\Big(\sum_\nu \left[\, k_{x,\nu} - k_{x-\widehat{\nu},\nu}\,\right] - \delta_{x,y} + \delta_{y,z} \Big)$$
$$\times \prod_x e^{-\mu\, k_{x,4}}\, \mathcal{I}\Big(\sum_\nu [|\,k_{x,\nu}\,| + |\,k_{x-\widehat{\nu},\nu}\,| + 2(l_{x,\nu} + l_{x-\widehat{\nu},\nu})] + \delta_{x,y} + \delta_{y,z} \Big)$$

- Admissible configurations in $Z_{y,z}$:

  Closed loops of flux plus an open string of flux connecting y and z.

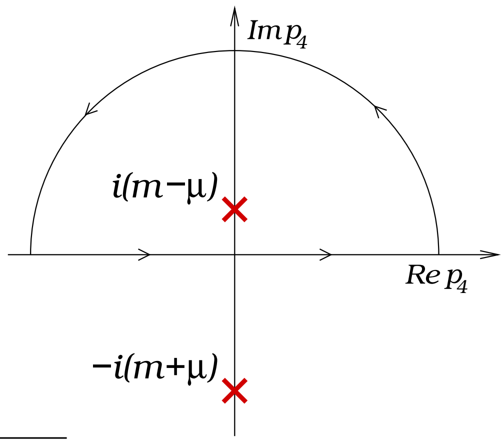
Worm strategy for correlators

- Since $Z_{y,z}$ consists of closed loop plus a single open string, every step of the worm corresponds to an admissible configuration for some $Z_{u,v}$.

- In our propagators we project to zero momentum, i.e., the spatial lattice indices are summed.

- To compute $C(t)$ one simply evaluates the temporal distance t of head and tail of the worm at every step and $C(t)$ is obtained as a histogram.

What do we expect? Analysis of the free case in the continuum.



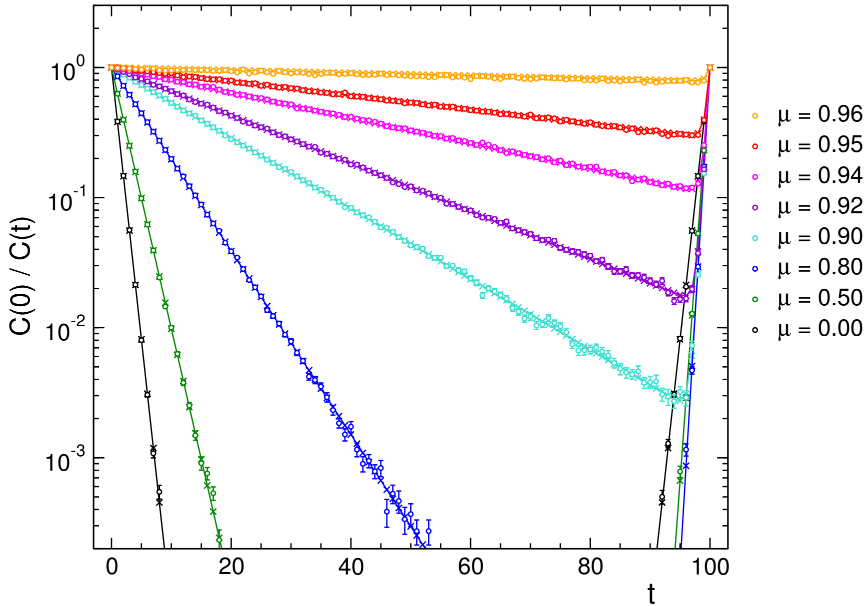
- Propagator in the continuum:

$$C(t) \; = \int \frac{dp_4}{2\pi} \, \frac{e^{ip_4 t}}{[p_4 - i(m-\mu)][p_4 + i(m+\mu)]}$$

- Asymmetry between forward and backward propagation:

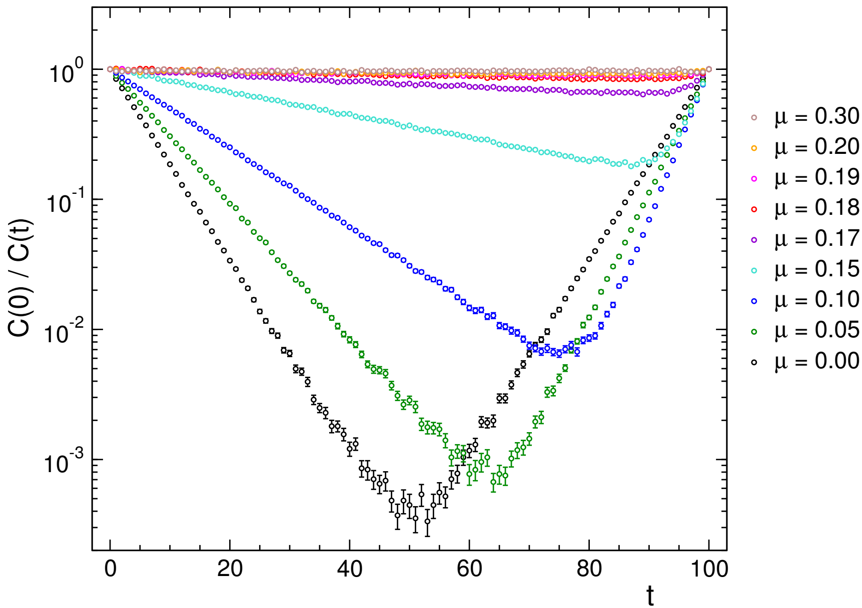
$$C(t) \; \propto \; \begin{cases} e^{-(m-\mu)t} & \text{for } t > 0 \\ e^{+(m+\mu)t} & \text{for } t < 0 \end{cases}$$

Test of free propagators against (lattice) Fourier transformation



Excellent agreement indicates that the finite density propagators computed from the dual representation are under control. $(16^3 \times 100,\ m = 1, \lambda = 0)$

Propagators at non zero coupling



Asymmetric propagation for $\mu < \mu_c \simeq 0.17$. Condensation ($=$ constant propagator) for μ above μ_c. $(16^3 \times 100,\, \kappa = 7.44, \lambda = 1)$

Scalar electrodynamics

.... adding U(1) gauge fields to the charged scalar ....

Surfaces for the gauge fields:

- Expansion of an individual plaquette term from the gauge action:

$$e^{\beta\, U_{x,\rho}\, U_{x+\hat{\rho},\sigma}\, U^\star_{x+\hat{\sigma},\rho}\, U^\star_{x,\sigma}} \;=\; \sum_{p_{x,\rho\sigma}} \frac{\beta^{\,p_{x,\rho\sigma}}}{(p_{x,\rho\sigma})!} \left[U_{x,\rho}\, U_{x+\hat{\rho},\sigma}\, U^\star_{x+\hat{\sigma},\rho}\, U^\star_{x,\sigma} \right]^{p_{x,\rho\sigma}}$$

- For gauge fields the expansion indices $p_{x,\rho\sigma}$ live on the plaquettes.

- The matter loops are dressed with gauge links.

- The new constraints at the links of the lattice force the combined flux from the matter variables $k_{x,\nu}$ and the plaquette variable $p_{x,\rho\sigma}$ to vanish.

- Admissible configurations of the plaquette variables $p_{x,\rho\sigma}$ have the interpretation of 2-D surfaces embedded in 4-D.

- The surfaces are either closed or bounded by matter flux.

Dual form of the partition function:

The original partition sum is mapped exactly to a sum over loop and surface configurations:

$$Z \;=\; \sum_{\{p,k,l\}} \mathcal{W}_G(p)\,\mathcal{W}_M(k,l)\,\mathcal{C}_L(p,k)\,\mathcal{C}_S(k)$$

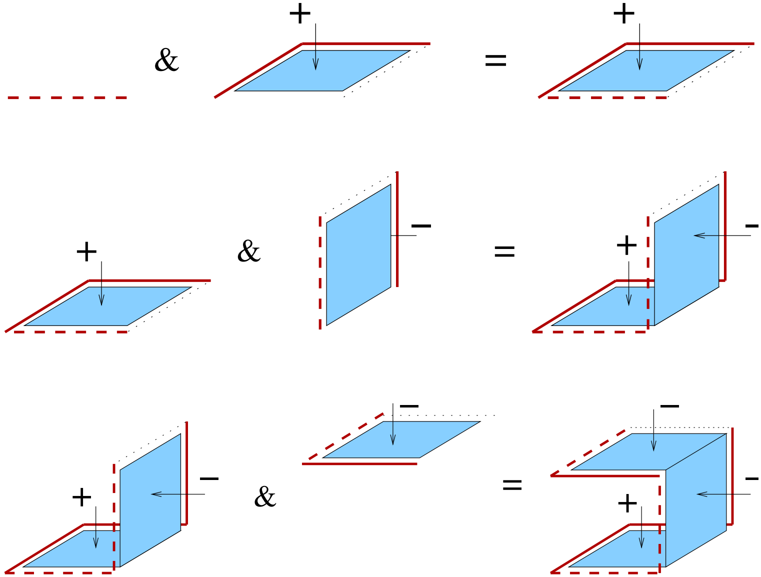
$\mathcal{W}_G(p)$: plaquette-based weight factor for gauge variables p
$\mathcal{W}_M(k,l)$: link-based weight factor for matter variables k,l
$\mathcal{C}_L(p,k)$: link-based constraint $\Rightarrow$ gauge surfaces
$\mathcal{C}_S(k)$: site-based constraint $\Rightarrow$ matter loops

$$\mathcal{C}_L[p,k] \;=\; \prod_x \prod_{\nu=1}^{4} \delta\left(\sum_{\rho:\nu<\rho} [p_{x,\nu\rho} - p_{x-\hat{\rho},\nu\rho}] - \sum_{\rho:\nu>\rho} [p_{x,\rho\nu} - p_{x-\hat{\rho},\rho\nu}] + k_{x,\nu} \right)$$

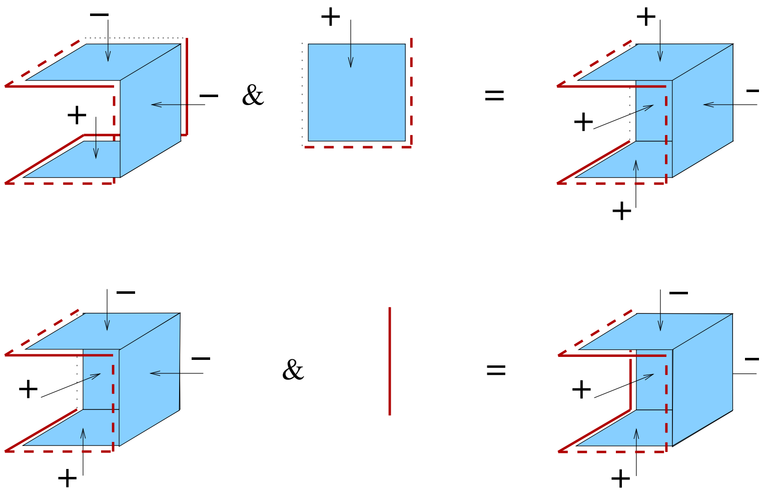
$$\mathcal{C}_S[k] \;=\; \prod_x \delta\left(\sum_{\nu=1}^{4} [k_{x-\hat{\nu},\nu} - k_{x,\nu}] \right)$$

Generalized worm algorithm for gauge Higgs systems:

Worm starts by inserting a unit of matter flux. Adding segments transports both the site and link defects across the lattice ....

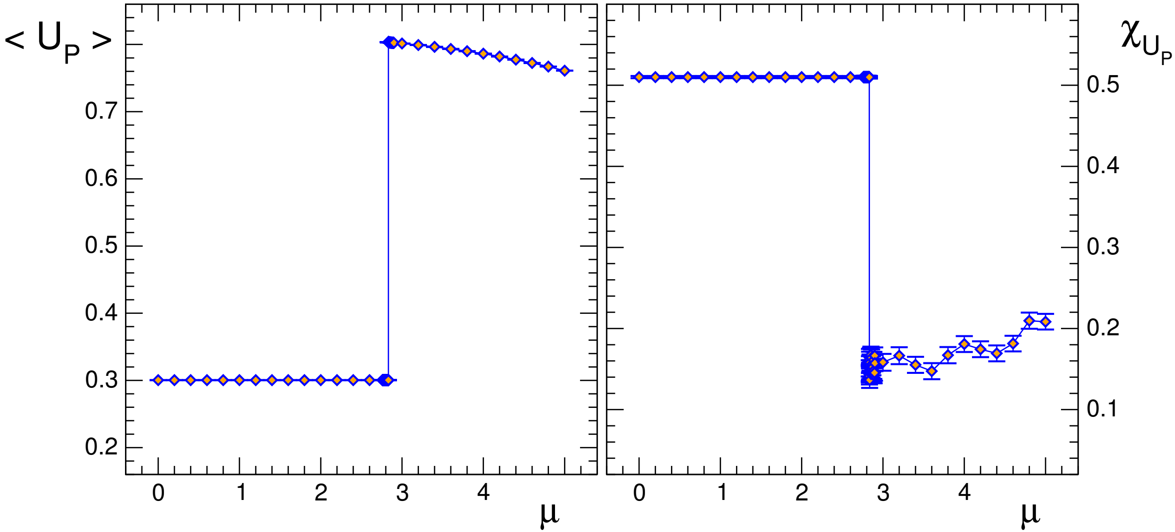


Generalized worm algorithm for gauge Higgs systems



.... until the worm terminates with the insertion of another line of flux.

Example of a Silver Blaze transition



Using two flavors of opposite charge one can couple a chemical potential.

$\Rightarrow$ Silver Blaze behavior

Summary:

- Considerable progress was made towards rewriting several systems in representations where the partition sum has only real and positive terms.

- Dual degrees of freedom are surfaces for gauge fields and loops for matter.

- Constraints for dual variables can be handled with worm-type algorithms.

- Interesting new algorithmic options when surfaces have boundaries.

- Spectroscopy is under control.

- Examples:
  - Relativistic Bose gas / charged scalar field.
  - Scalar electrodynamics.

- May serve as solved test cases for other approaches.