Density of States Techniques for Finite Density Lattice QCD

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Motivation

- On the lattice, finite density QCD suffers from the complex action problem: the Boltzmann factor e^{-S} becomes complex and cannot be used as a weight in Monte-Carlo simulations.
- The problem has no universal solution. To deal with finite density, we here investigate the density of states approach for QCD.

Density of states (DoS) [1]

DoS approach for a theory with bosonic fields Φ:

- Split action into real and imaginary parts:

$$S[\Phi] = S_R[\Phi] - iX[\Phi] \ .$$

- Introduce density of states:

$$\rho(x) = \int \mathcal{D}[\Phi] e^{-S_R[\Phi]} \delta(x - X[\Phi]) \ .$$

- Obtain observables via:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int dx \rho(x) e^{ix} \mathcal{O}(x) \ , \quad Z = \int dx \rho(x) e^{ix} \ .$$

$\Longrightarrow$ need to determine the density with high numerical precision.

- Usually, $\rho(x)$ falls off rapidly for large $|x|$ and is even $\Longrightarrow$ consider $\rho(x)$ on interval $[0, x_{\text{max}}]$.

Parametrization of the density

- Partition $[0, x_{\text{max}}]$ into N subintervals I_n of length Δ_n:

$$I_n = [x_n, x_{n+1}] \ , \quad x_n = \sum_{i=0}^{n-1} \Delta_i \ .$$

- Use exponential of piecewise linear function $L(x)$ as an ansatz for $\rho(x)$:

$$\rho(x) = e^{-L(x)} \ ,$$
$$L(x) = k_n x + d_n \ , \quad x \in I_n \ .$$

- Requiring continuity and normalizing $\rho(0) = 1$ fixes all d_n.

$\Longrightarrow$ task: find k_n. We employ the functional fit approach.

The functional fit approach (FFA) [2]

- Introduce restricted expectation values with a control parameter $\lambda \in \mathbb{R}$:

$$\langle\langle X \rangle\rangle_n(\lambda) = \frac{1}{Z_n(\lambda)} \int \mathcal{D}[\Phi] e^{-S_R[\Phi] + \lambda X[\Phi]} X[\Phi] \Theta_n(X[\Phi])$$
$$Z_n(\lambda) = \int \mathcal{D}[\Phi] e^{-S_R[\Phi] + \lambda X[\Phi]} \Theta_n(X[\Phi]) \ .$$

- Can be calculated via Monte-Carlo simulation (no sign problem).
- Support function $\Theta_n(x)$ restricts x to interval I_n:

$$\Theta_n(x) : \begin{cases} 1, & \text{if } x \in [x_n, x_{n+1}] \\ 0, & \text{if } x \notin [x_n, x_{n+1}] \end{cases} \ .$$

- $\langle\langle X \rangle\rangle_n(\lambda)$ can be computed explicitly using the parametrized density:

$$V_n(\lambda) = \frac{1}{\Delta_n}\Big(\langle\langle X \rangle\rangle_n(\lambda) - x_n \Big) - \frac{1}{2} = h\Big(\Delta_n \left(\lambda - k_n\right)\Big) \ ,$$
$$h(s) = \frac{1}{1 - e^{-s}} - \frac{1}{s} - \frac{1}{2} \ .$$

$\Longrightarrow$ determine k_n via a fit of the Monte-Carlo results to $h\Big(\Delta_n(\lambda - k_n)\Big)$.

Application to QCD

- The DoS formulation is intrinsically bosonic, but QCD has fermions $\Longrightarrow$ require bosonization.
- QCD partition sum:

$$Z = \int \mathcal{D}[U] e^{-S_G[U]} \det(D[U]) \ .$$

- U: gauge links, $S_G[U]$: gauge action, $D[U]$: Dirac operator.
- Use pseudofermion representation:

$$\det(D[U]) = \det(D^\dagger[U] D[U]) \frac{1}{\det(D^\dagger[U])} \propto \int \mathcal{D}[\phi] e^{-\phi^\dagger D^\dagger[U] \phi} .$$

- Prefactor is real and positive $\Longrightarrow$ can be treated with standard methods, e.g., multiboson techniques [3].
- Apply DoS FFA to $\phi^\dagger D^\dagger[U] \phi$.

Free theory in $1+1$ dimensions

- To gain first insights, set all gauge links $U = 1$ and restrict to $1+1$ dimensions $\Longrightarrow$ can compare to analytical result.
- We use Wilson fermions.

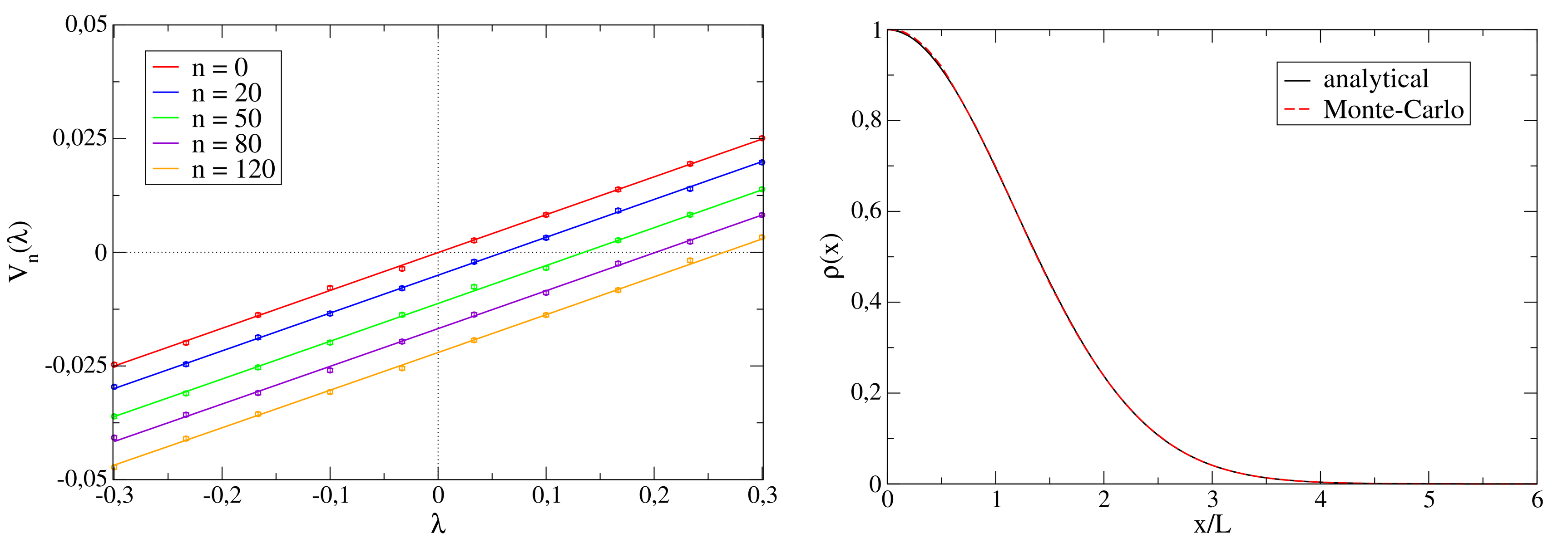

Fig.1: Left: Observable $V_n(\lambda)$ obtained by a simulation and fit of results. Right: Comparision between the DoS calculated via simulation and analytically.

Lattice size: $L \times L$ with $L = 16$, mass: $m = 0.1$, chemical potential: $\mu = 0.05$.

- Results already in good agreement.
- Calculate $\langle X \rangle$ as first simple observable:

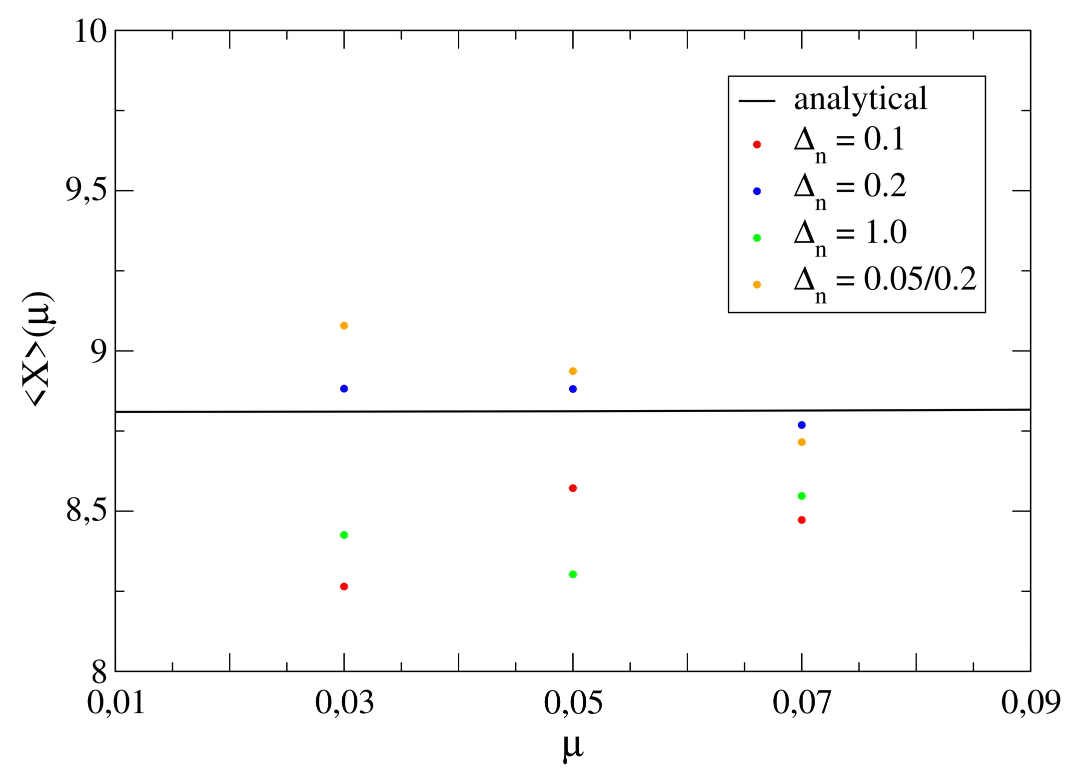

Fig.2: Expectation value of X for different interval lenghts Δ_n as a function of μ. 4×4 - lattice.

Summary and Outlook

- DoS FFA can be applied to a toy theory with the application to full QCD in mind. The results for $\rho(x)$ agree well with an analytic calculation.
- Room for improvement in accuracy to improve results for observables.
- First modern DoS implementation for theory with fermions.

References

[1] K. Langfeld, B. Lucini, and A. Rago. *The density of states in gauge theories.* Phys. Rev. Lett. **109**, 111601 (2012), arXiv: 1204.3243v1 [hep-lat]

[2] C. Gattringer, M. Giuliani, A. Lehmann and P. Törek. *Density of states techniques for lattice field theories using the functional fit approach (FFA).* PoS LATTICE2015 **195** (2015), arXiv: 1511.07176v1 [hep-lat]

[3] M. Cè, L. Giusti, and S. Schaefer. *Local factorization of the fermion determinant in lattice QCD.* Phys. Rev. D **95**, 034503 (2017), arXiv:1609.02419 [hep-lat]