Inhomogeneous Phases in Gross-Neveu Models

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with

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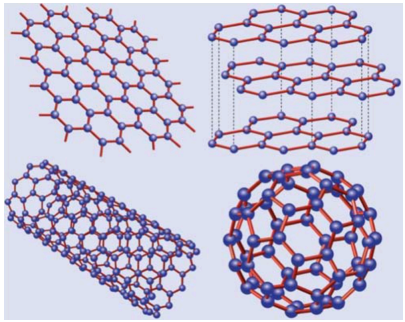
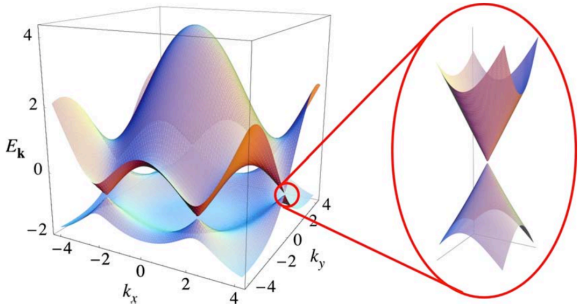
Outline

Introduction

Applications

Solid state physics

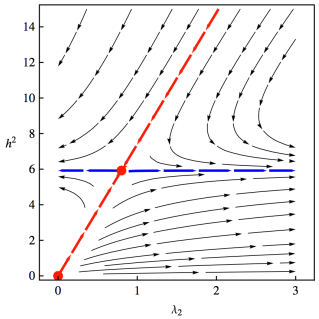


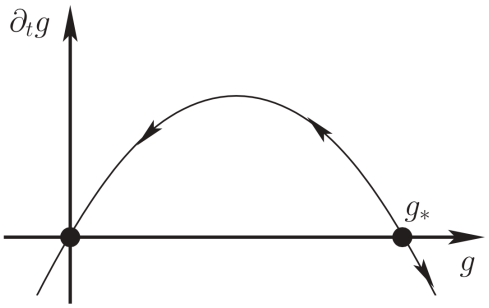


from [Castro Neto et al., 2009],

see also [Herbut et al., 2009]

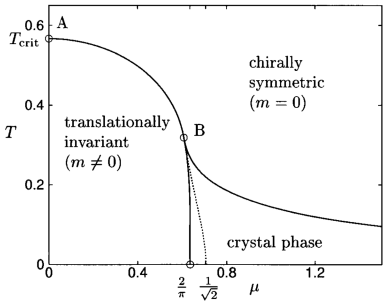
Asymp. free/save

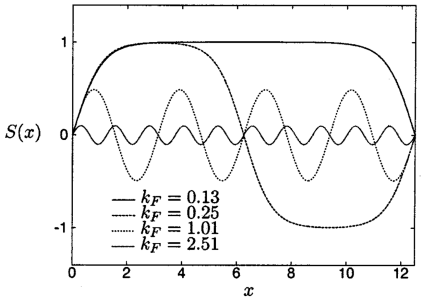




from [Braun et al., 2011]

QCD toy model





from [Thies et al., 2003]

Four-Fermion Theories

$$\mathcal{L} = \sum_{a=1}^{N_f} \overline{\psi}_a \left(\mathrm{i}\partial\!\!\!/ + \mathrm{i}\gamma^0 \mu \right) \psi_a - \frac{1}{4\lambda} \sum_{a,b=1}^{N_f} \left(\overline{\psi}_a M_1 \psi_a \right) \left(\overline{\psi}_b M_2 \psi_b \right)$$

$\psi, \overline{\psi}$ spinors,	a, b flavour index,
μ chemical potential,	N_f number of flavours
λ (inverse) coupling,	M_1, M_2 matrices

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Gross-Neveu (GN)	Thirring (Th)	Nambu-Jona-Lasinio (NJL)
$\left(\bar{\psi}\psi\right)^2$	$\left(\bar{\psi}\gamma_\mu\psi\right)\left(\bar{\psi}\gamma^\mu\psi\right)$	$\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\mathrm{i}\gamma_5\psi\right)^2$

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$$\mathcal{L} = \mathrm{i}\bar{\psi}\left(\partial\!\!\!/ + \gamma^0\mu - \sigma \right)\psi + N_\mathrm{f}\lambda\sigma^2, \qquad \langle\sigma\rangle = -\frac{N_\mathrm{f}}{2\lambda}\left\langle\bar{\psi}\psi\right\rangle$$

Symmetries

1. flavor symmetry $U(N_\mathrm{f})$:

$$\psi_a \mapsto \left(e^{\mathrm{i}\alpha}\right)_{ab} \psi_b, \qquad \bar{\psi}_a \mapsto \left(e^{-\mathrm{i}\alpha}\right)_{ab} \bar{\psi}_b$$

2. chiral symmetry $U(N_\mathrm{f})$:

$$\psi_a \mapsto \left(e^{\mathrm{i}\gamma_5\alpha}\right)_{ab} \psi_b, \qquad \bar{\psi}_a \mapsto \left(e^{\mathrm{i}\gamma_5\alpha}\right)_{ab} \bar{\psi}_b$$

3. (discrete) translational symmetry $\subset \mathbb{R}^D$

4. (discrete) rotational symmetry $\subset O(D)$

Fermion Representations

1+1D	1+2D	
unique irr. rep.	2 inequiv. irr. rep.	
2-comp. spinors	2-c. irred. Spinors	4-c. red. Spinors
chirality: ✓	×	✓
$\mathrm{U_c}(N_\mathrm{f}) \times \mathrm{U_f}(N_\mathrm{f})$	$\mathrm{U_f}(N_\mathrm{f}^\mathrm{irr} = 2N_\mathrm{f})$ (only f)	$\mathrm{U}(2N_\mathrm{f})$ (c & f)
Sign problem: ☺	☹	☺

1+1 Dimensional Models

Knowledge on Large-N

quantum $\int \mathscr{D}\sigma \; e^{-N_\mathrm{f} S_\mathrm{eff}} \xrightarrow{N_\mathrm{f}\to\infty} e^{-N_\mathrm{f} \min(S_\mathrm{eff})}$ semi-classical

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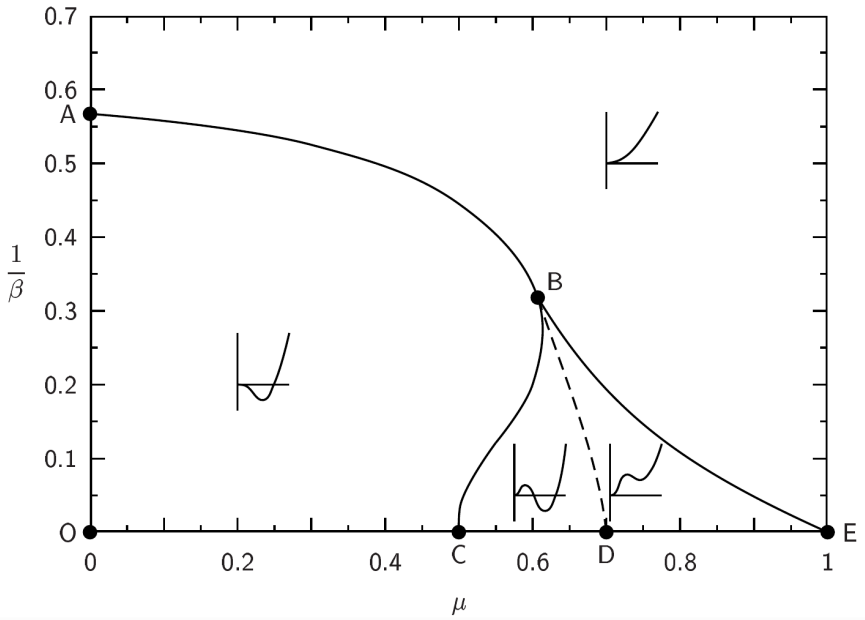
Completely solved:

- homogeneous [Wolff 1985]



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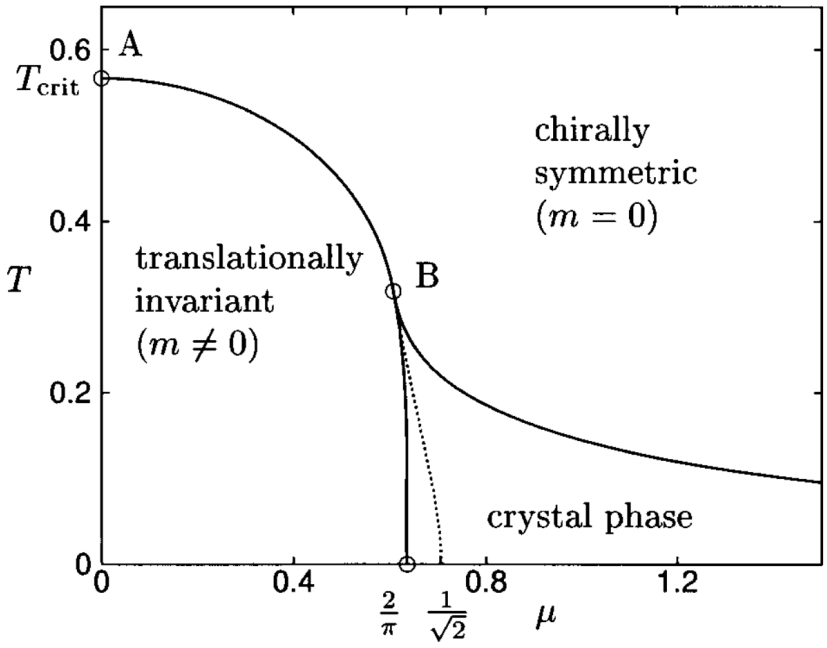

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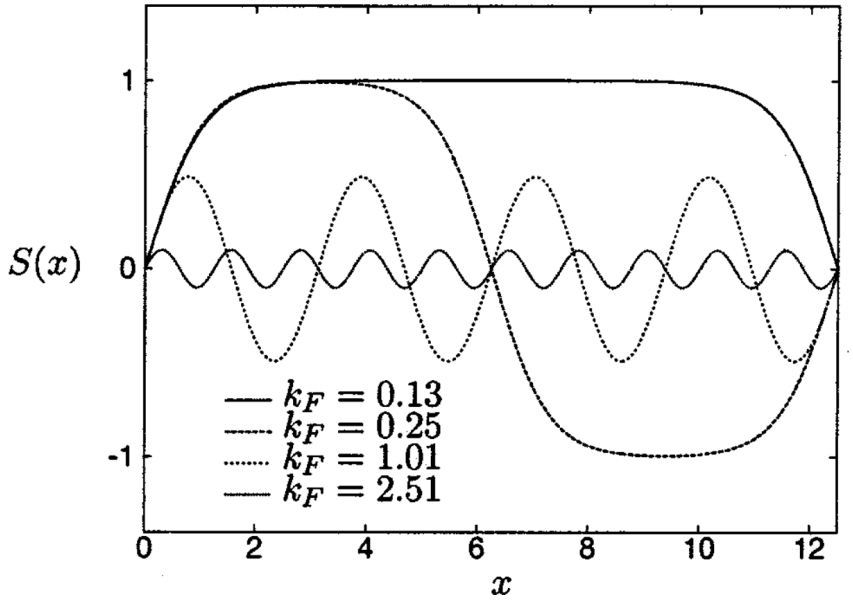

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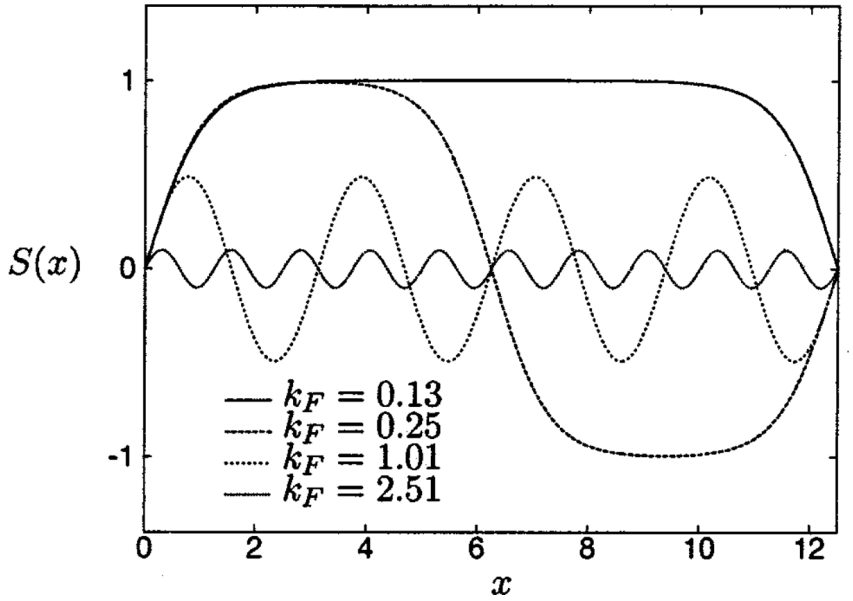
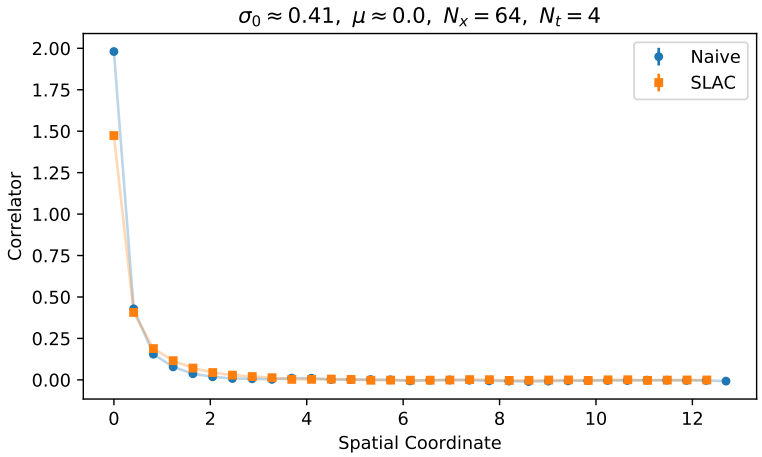
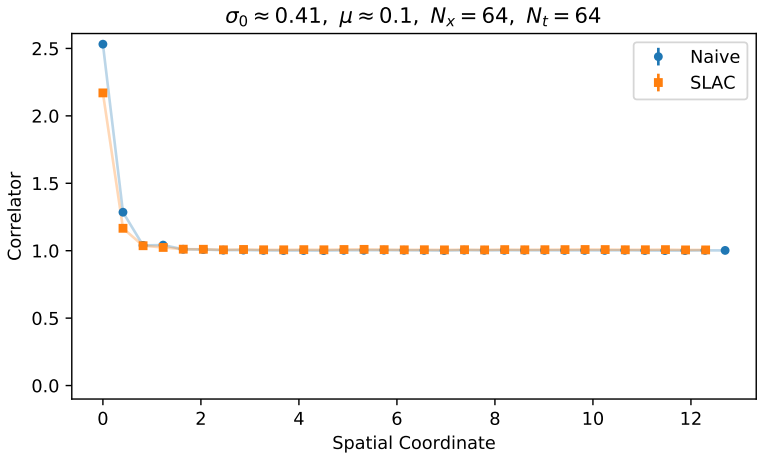


Do we find a similar phase at finite flavor number?

Examples: $C(x) = \langle \sigma_{tx} \sigma_{t0} \rangle$





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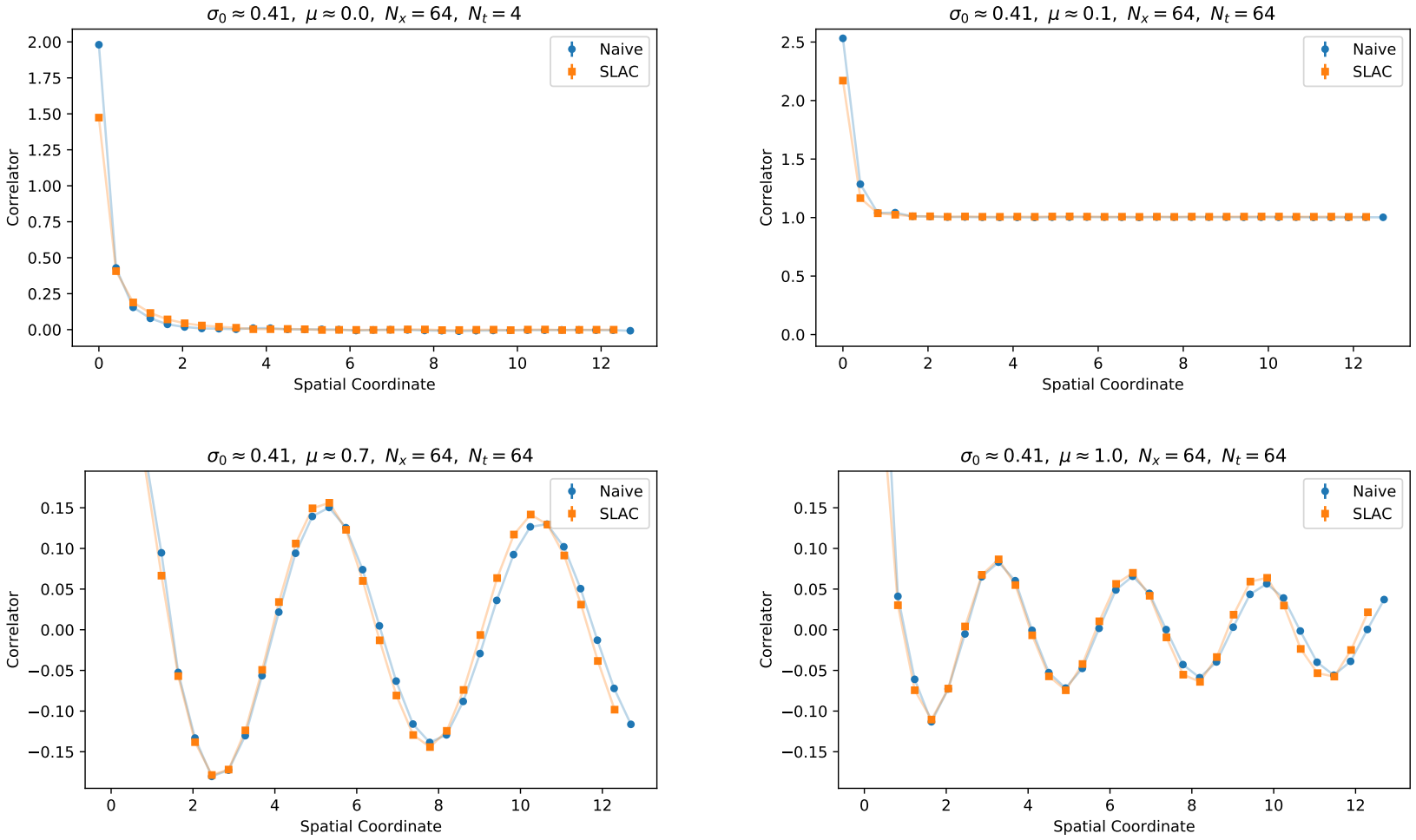
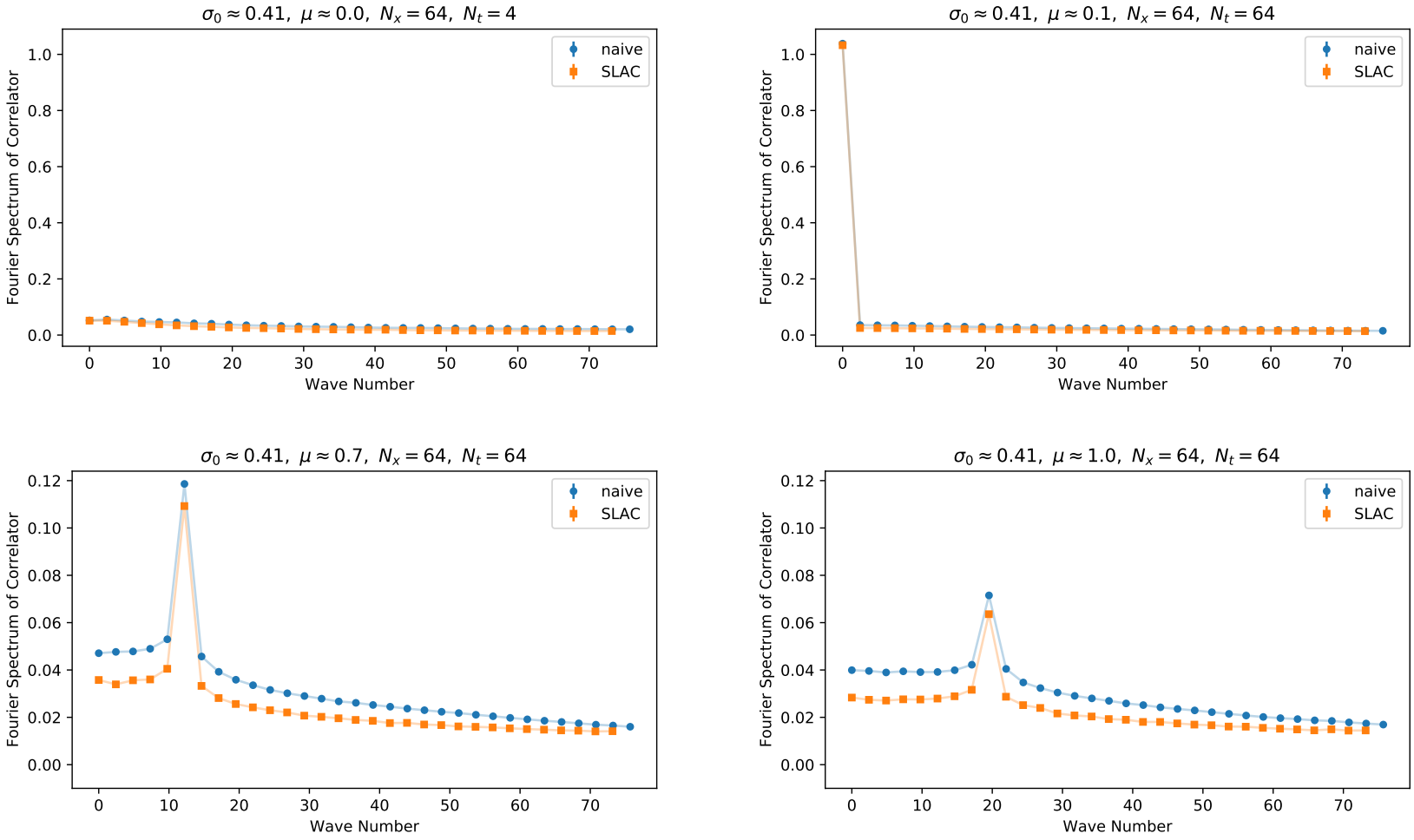
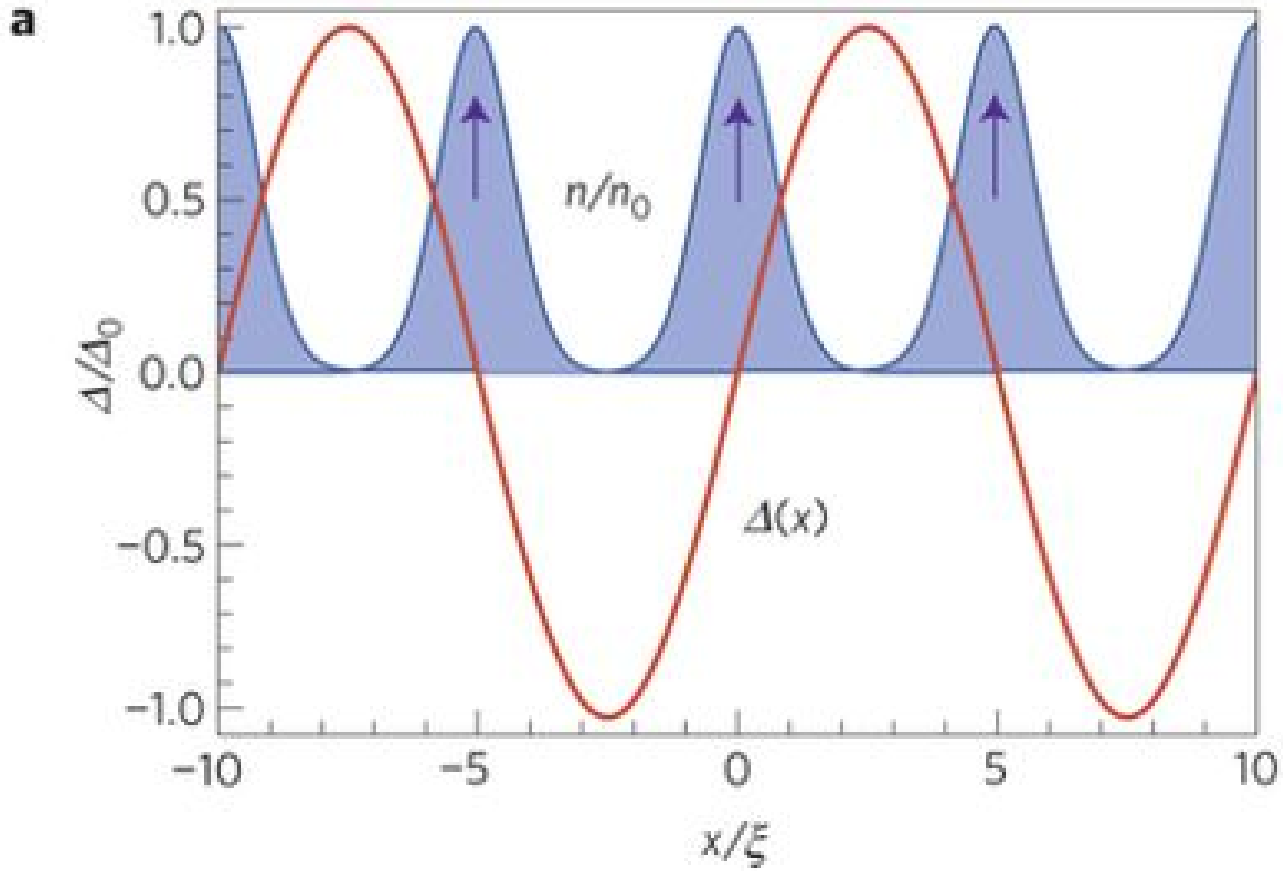


Examples: Fourier[C](k)

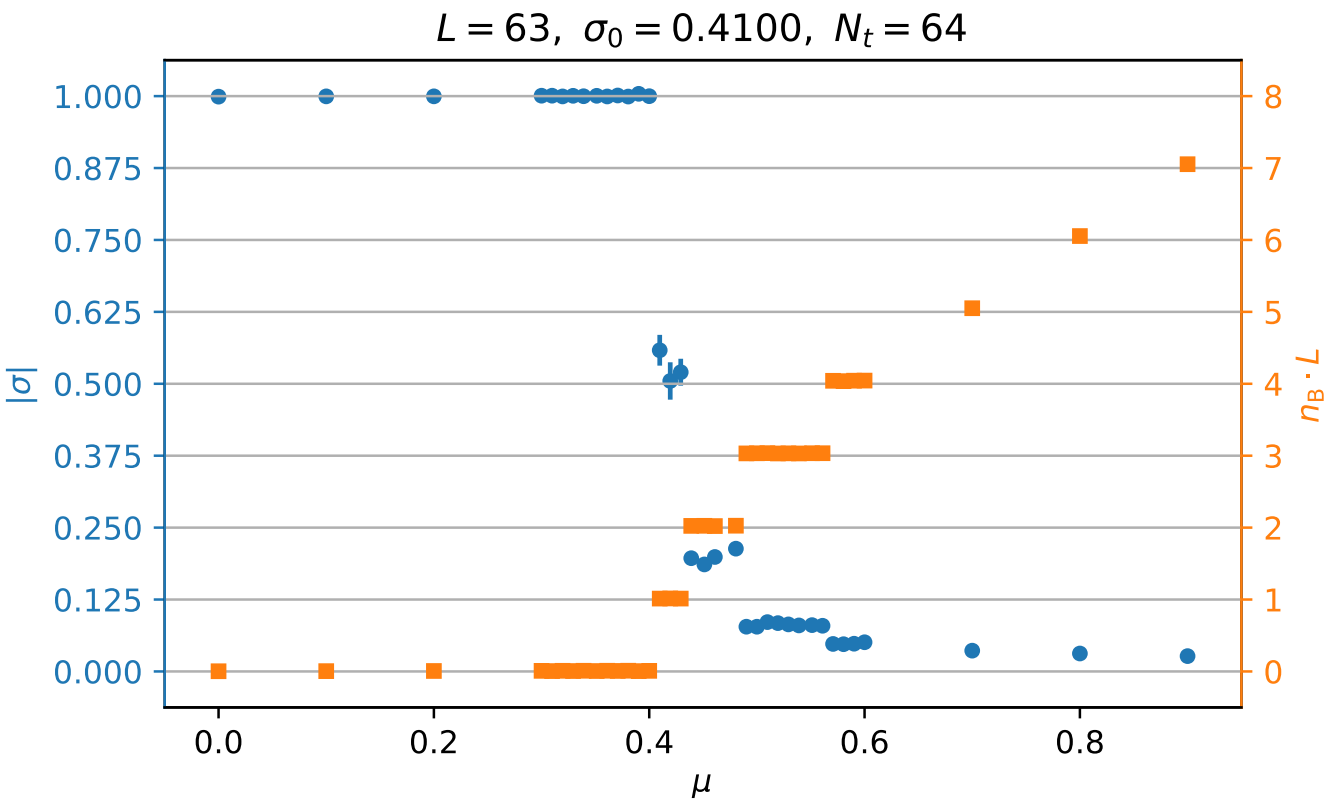


Baryon Crystal

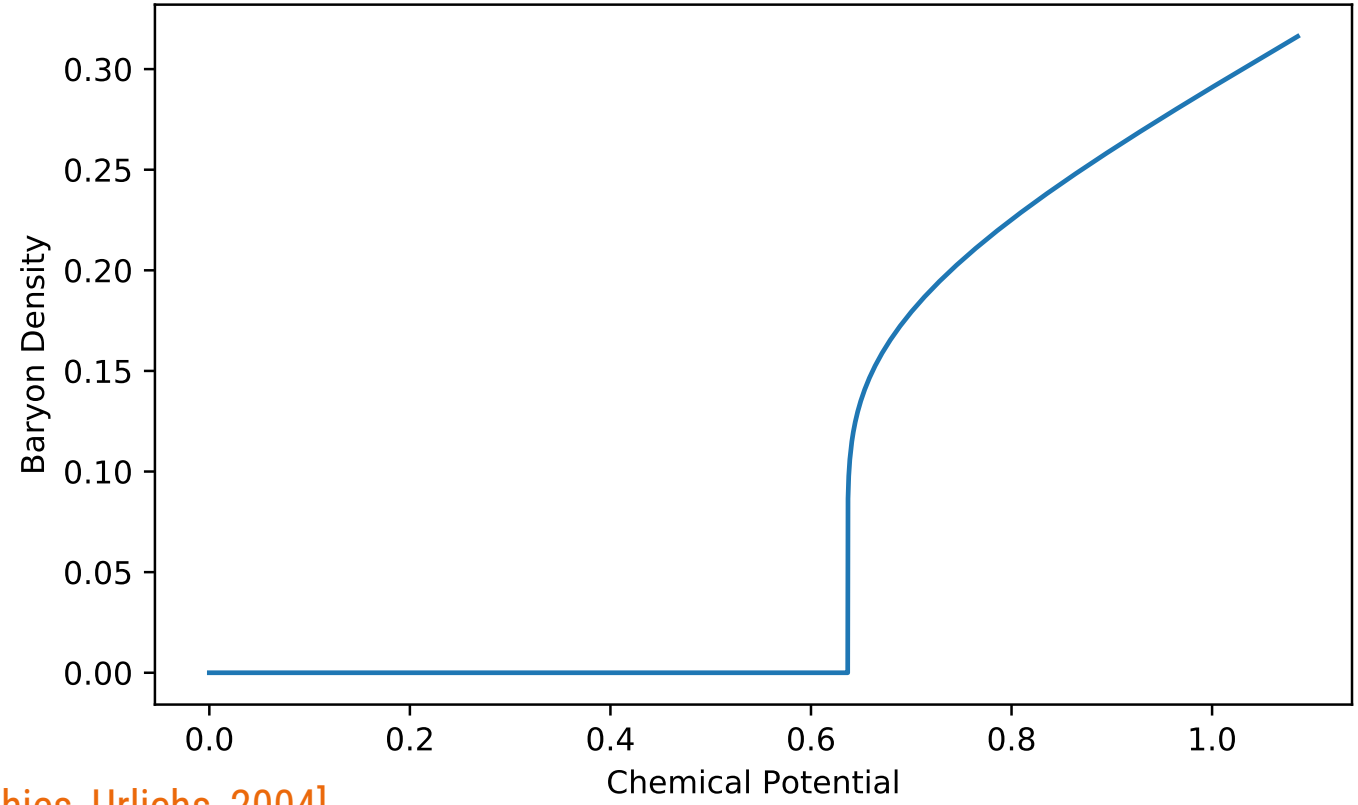


[Mayaffre et al., 2014]

Baryon Density ($T = 0$, $N_f = 8$)

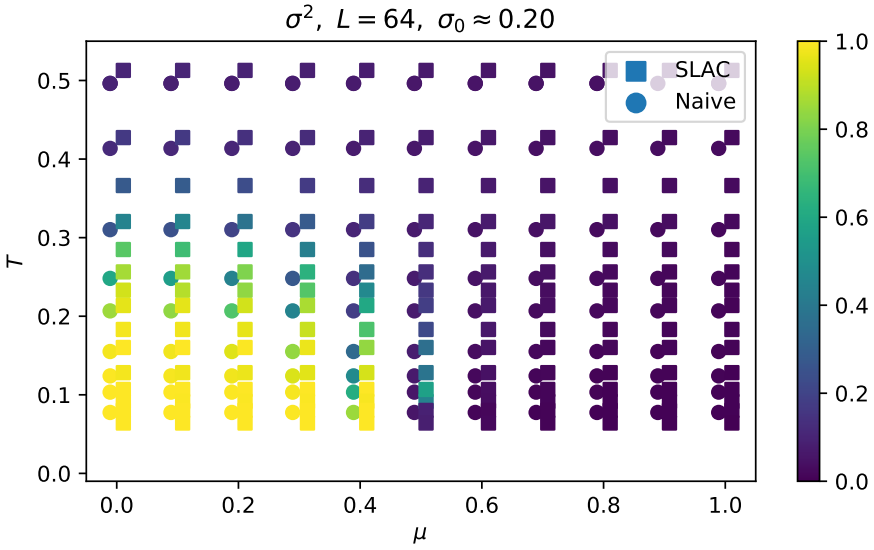


Baryon Density ($T = 0, N_f = \infty$)



[Schnetz, Thies, Urlichs, 2004]

The Phase Diagram ($N_f = 8$)



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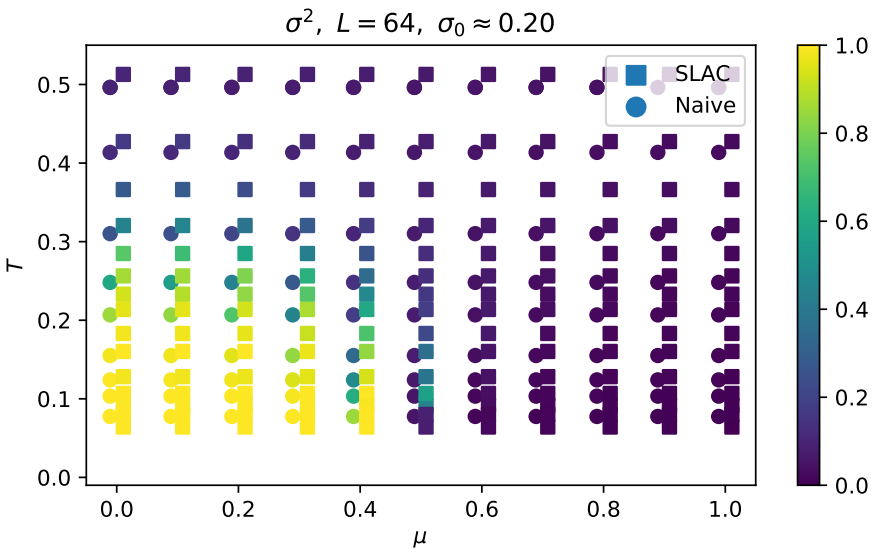
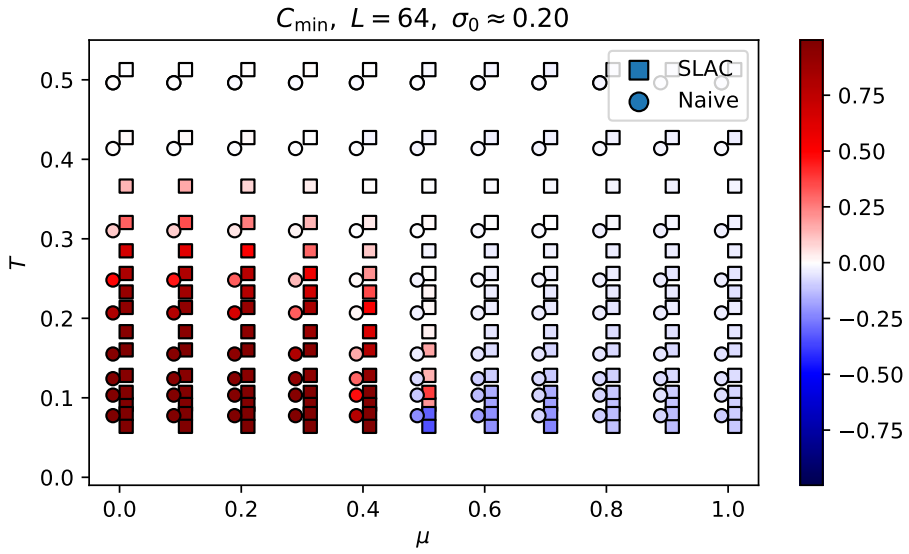
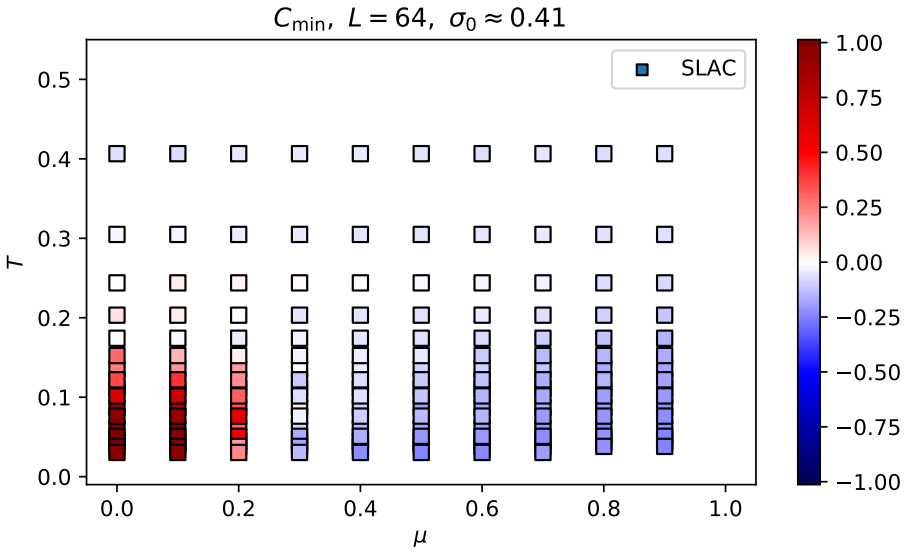




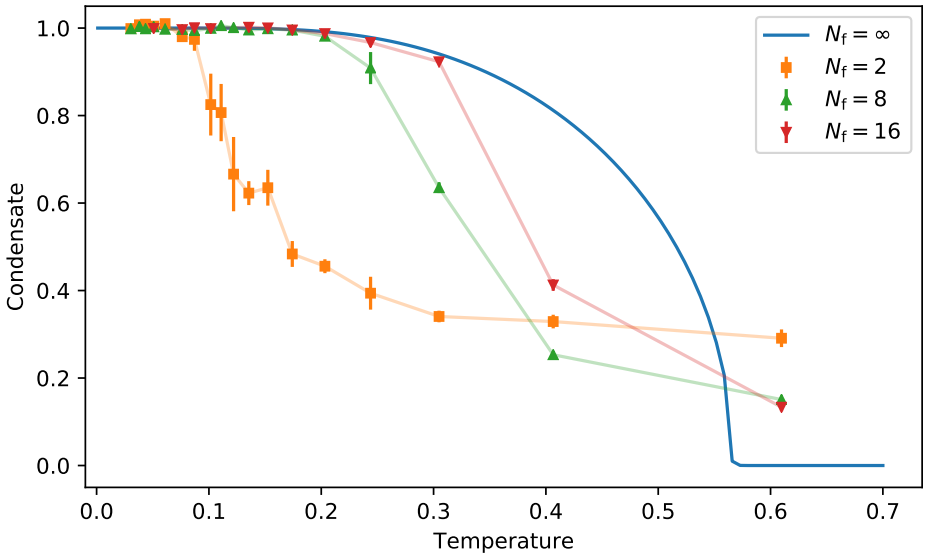
$$C_{\min} = \min_x C(x) \begin{cases} > 0 & \text{homogeneously broken} \\ \approx 0 & \text{unbroken} \\ < 0 & \text{inhomogeneously broken} \end{cases}$$

N_f Dependency (preliminary)

$N_f = 2$

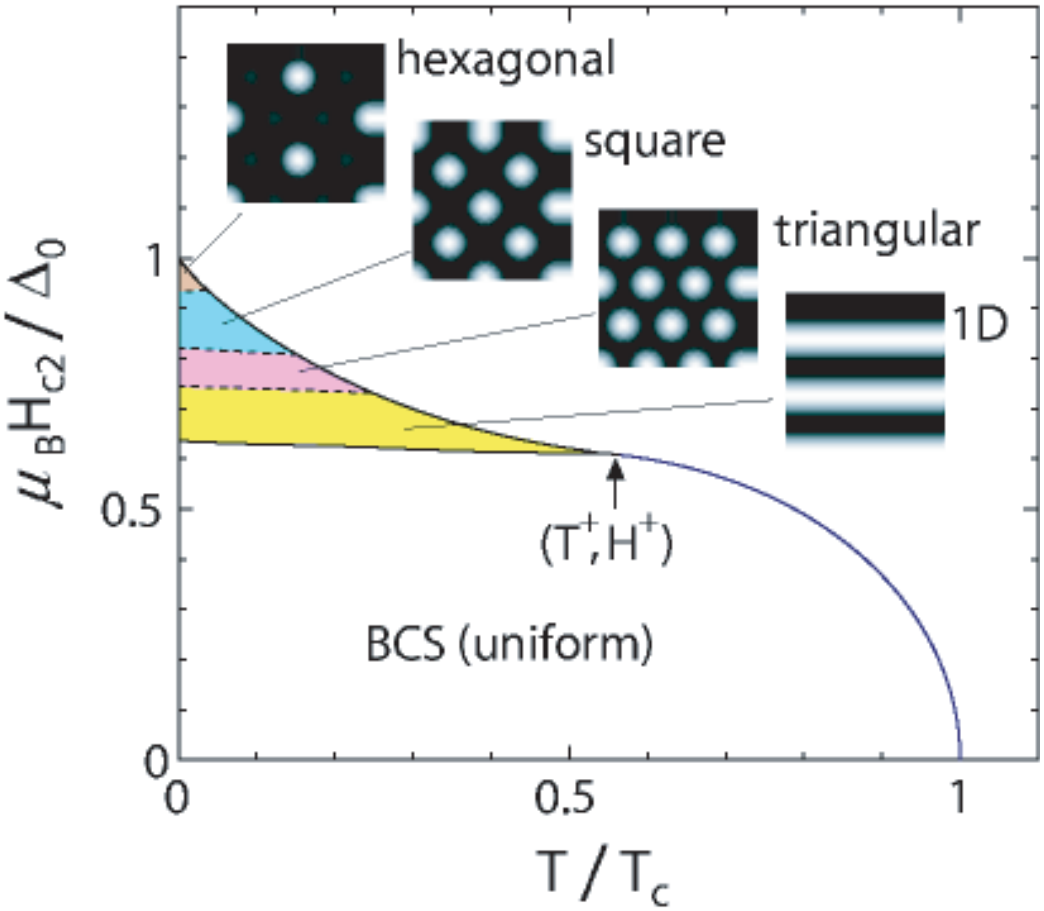


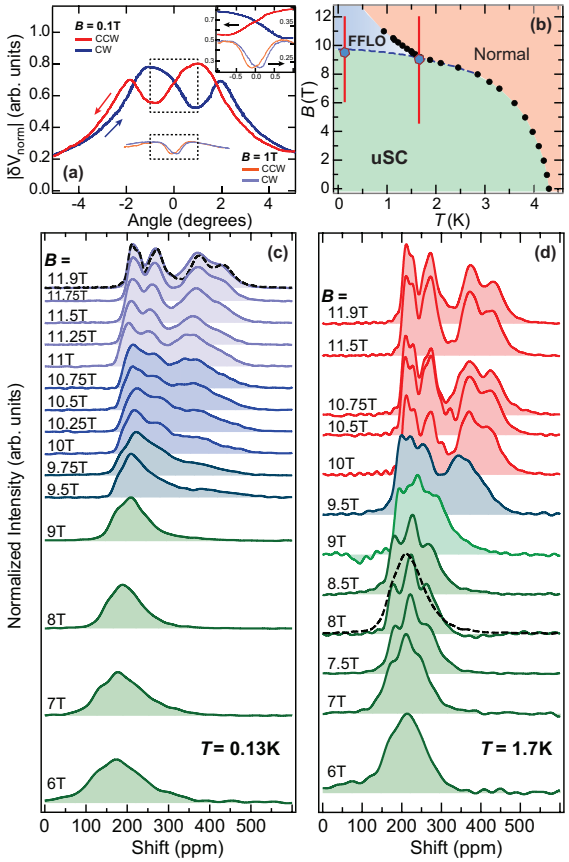
$\mu = 0$



1+2 Dimensional Models (preliminary)

Mean-Field Results I (Solid State)

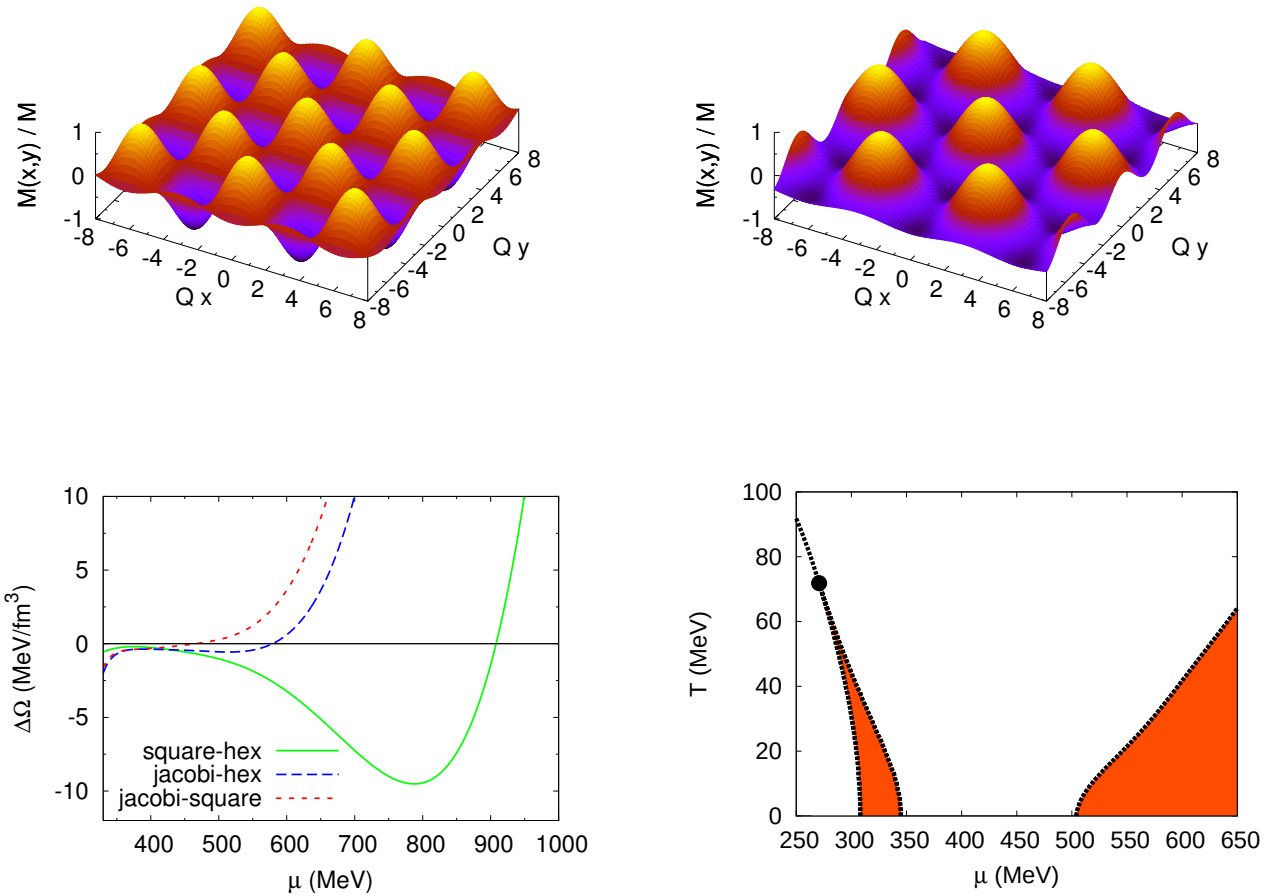




[Matsuda, Shimahara, 2007]

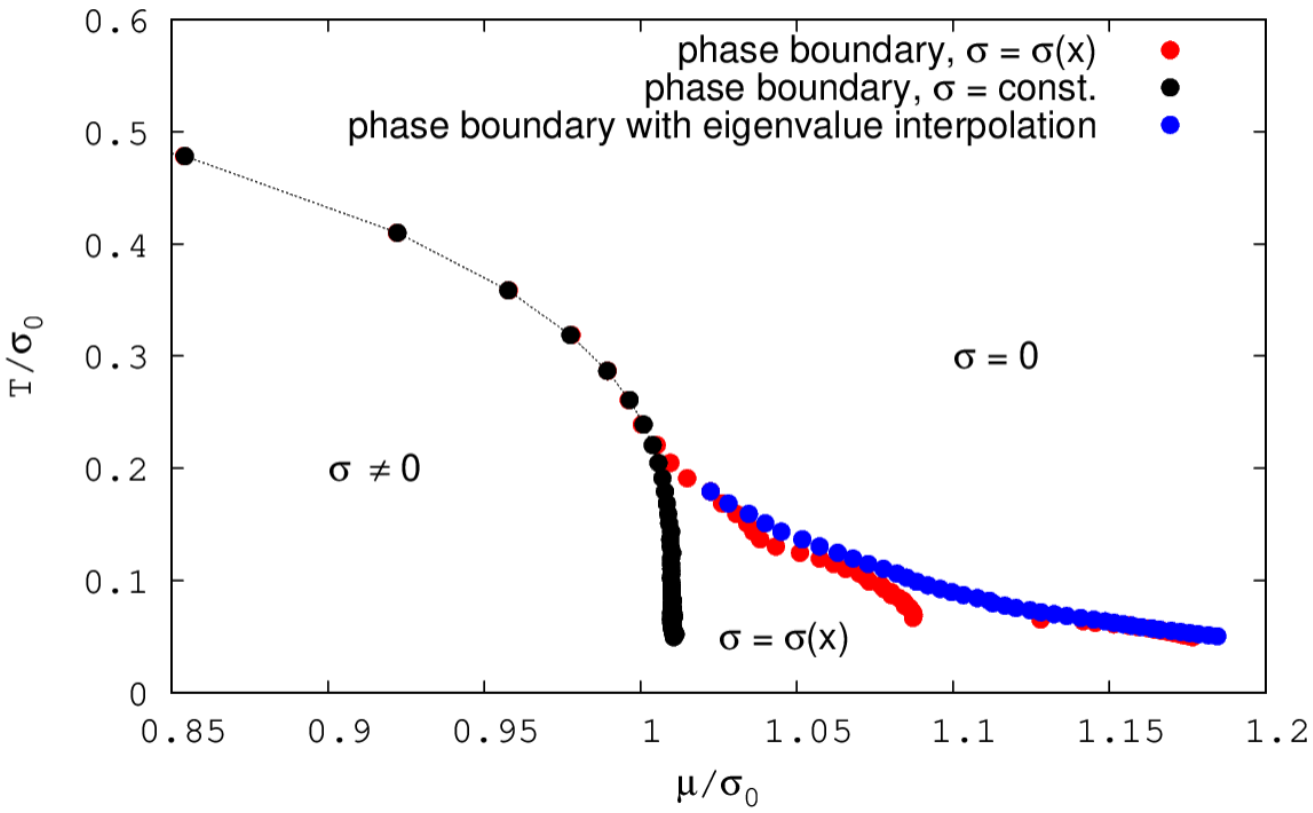
[Koutroulakis et al., 2016]

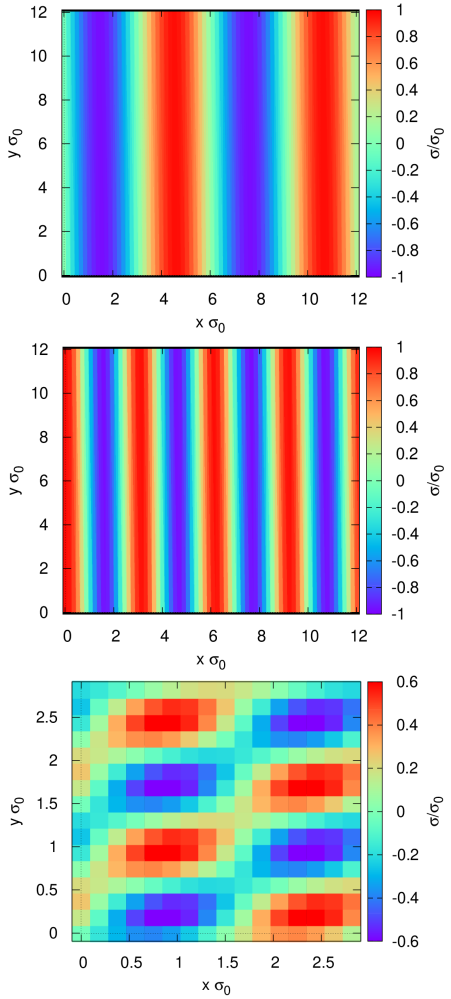
Mean-Field Results II (NJL)



[Buballa, Carignano, 2011/14]

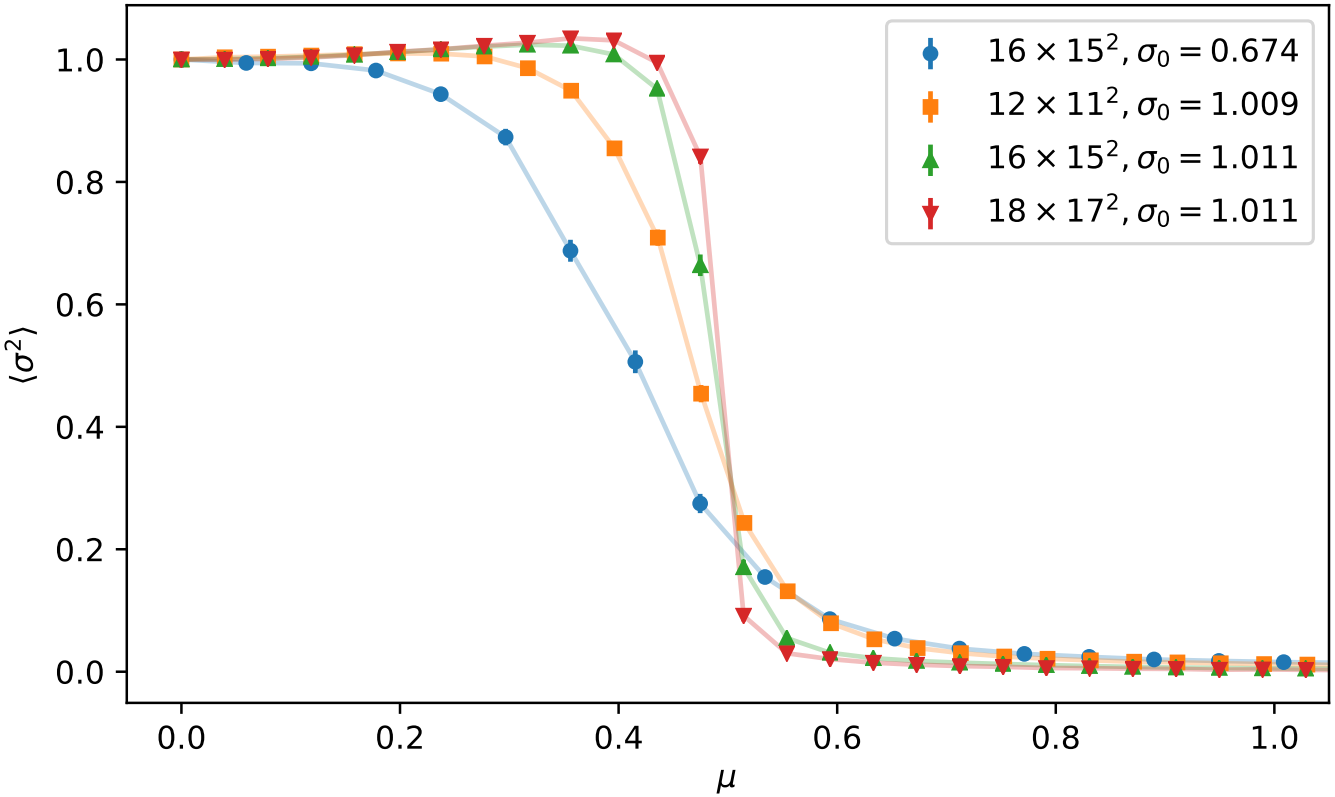
Mean-Field Results III (GN)



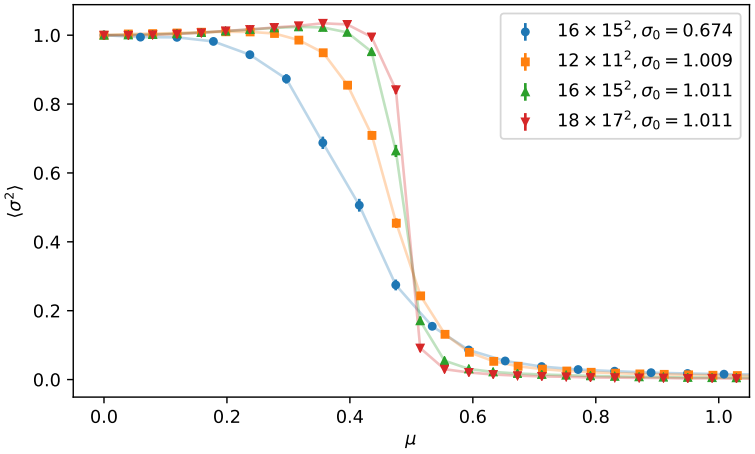


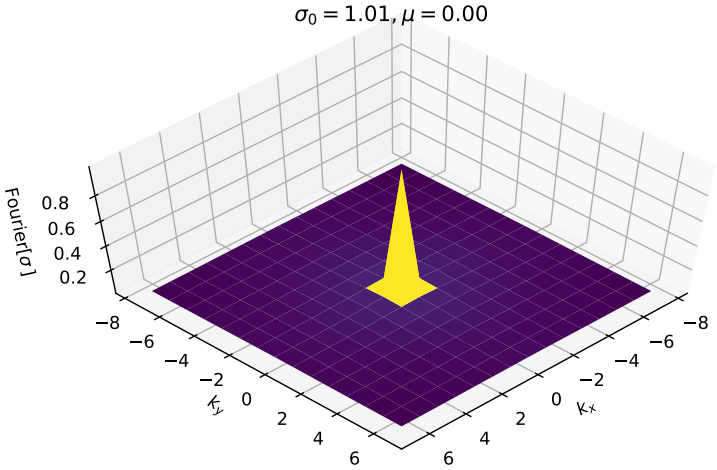
[Winstel, Stoll, Wagner, 2019]

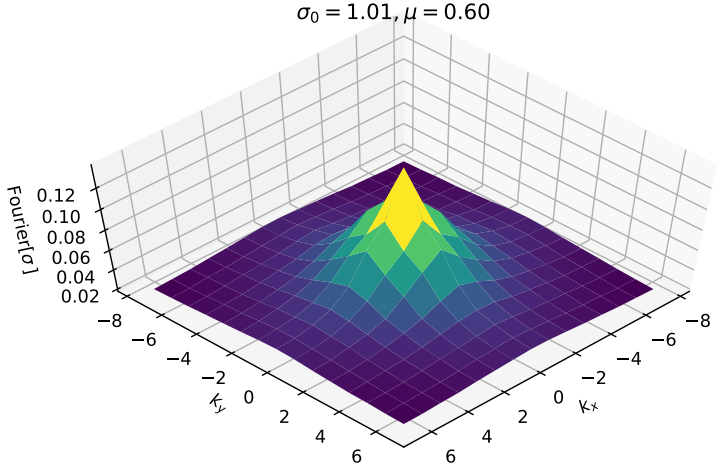
Prel. Results: Homogen. Condensate

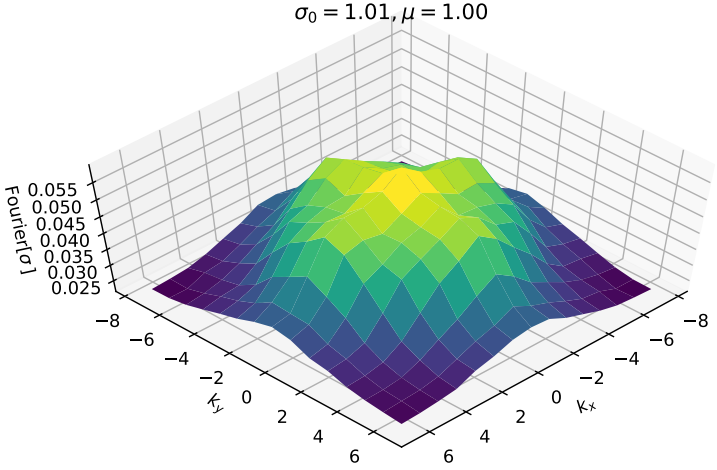


Prel. Results: μ Dependence

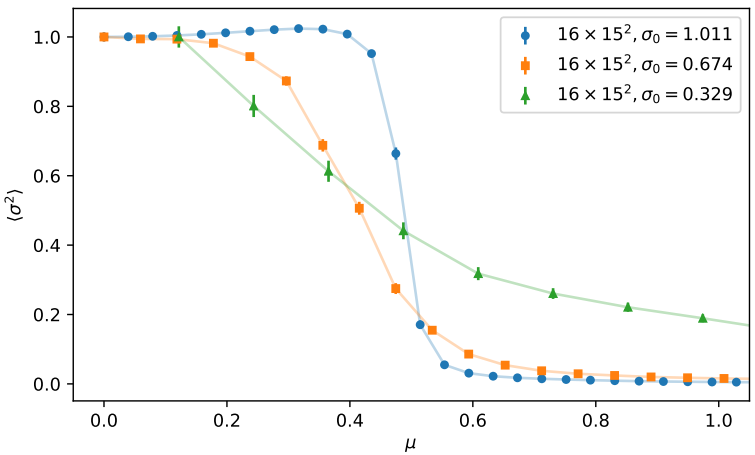


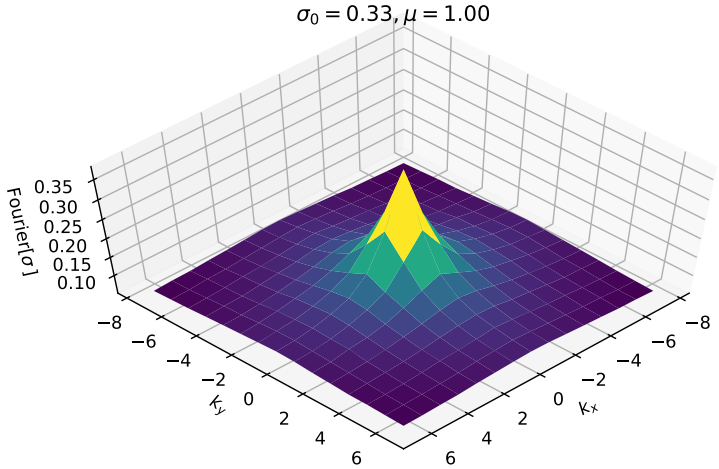


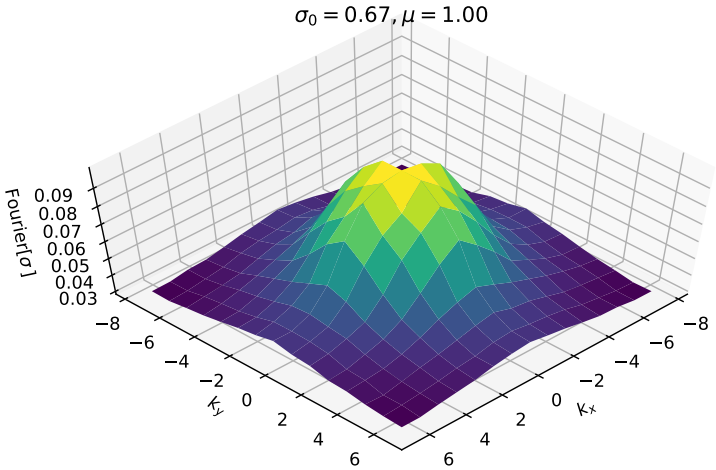


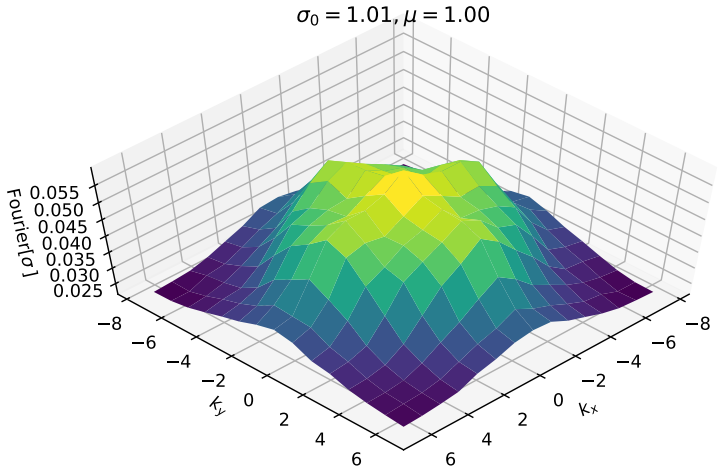


Prel. Results: Momentum Resolution









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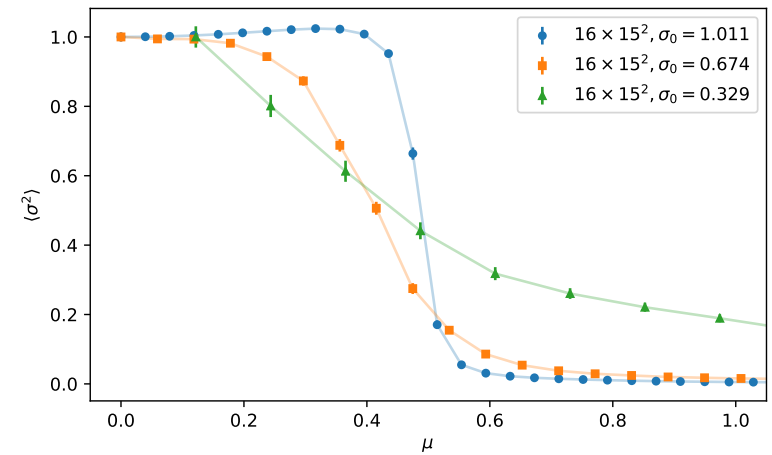

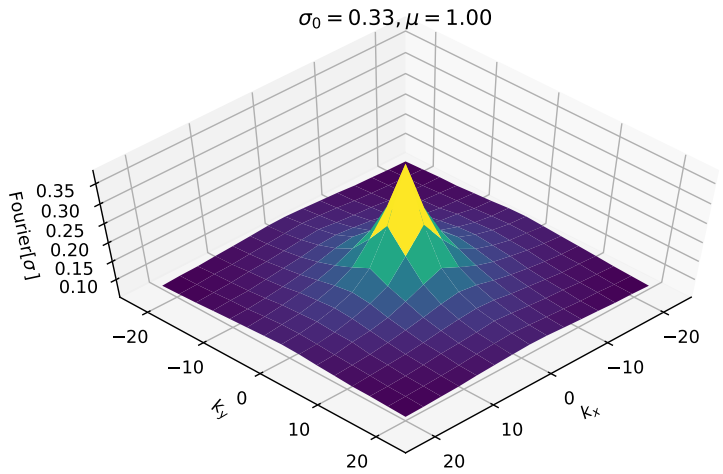

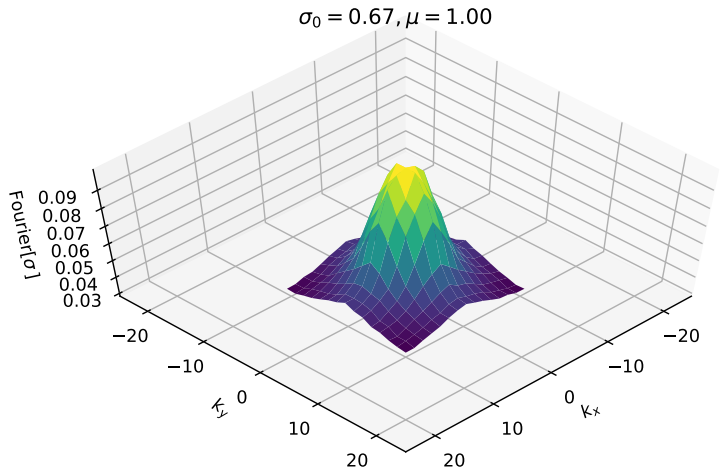

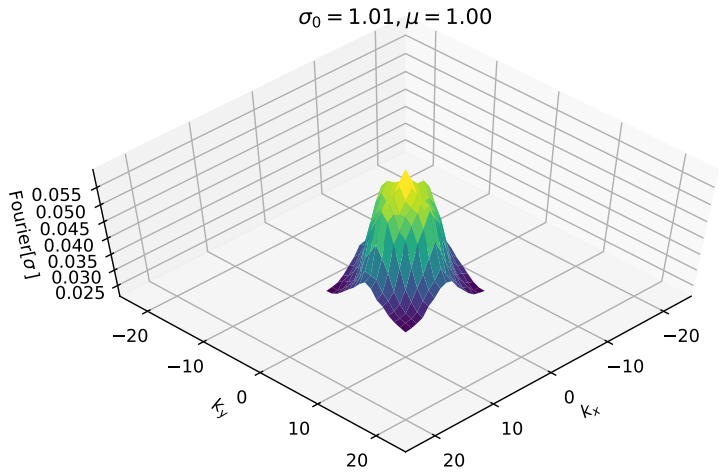

Conclusions

Conclusions and Outlook

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Inhomogeneous Phases in GN Models.

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- **1+2D:**
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  - Phase Diagram

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- **1+2D:**
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  - Phase Diagram
- **External Fields (magnetic, gravitational,…)**

Appendix

Goldstone Theorem

Goldstone, Salam, Weinberg 1962:

Spontaneous breaking of a continuous, uniform, global symmetry implies the existence of a massless mode in the spectrum, i.e.

$$\lim_{k \to 0} E_k = 0.$$

Idea: This mode will be

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$$= \sum_n \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-\mathrm{i}E_{n,k}t} \varphi_\Omega(k) \langle 0 | j^0(0) | n_k \rangle \langle n_k | \phi | 0 \rangle - \dots$$

Goldstone Theorem

Goldstone, Salam, Weinberg 1962:

Spontaneous breaking of a continuous, uniform, global symmetry implies the existence of a massless mode in the spectrum, i.e.

$$\lim_{k\to 0} E_k = 0.$$

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Finally, $\Omega \to \infty$ implies time-independence of LHS and $\varphi_\Omega(k) \to \delta(k^2)$.

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Coleman/Mermin-Wagner: Stat. Phys.: [Mermin, Wagner 1966]; QFT: [Coleman 1973]

There are no Goldstone phenomena in two dimensions.

Proof: $\delta(k^2)$ is no well-defined distribution in 2D. ($\sim$ Cauchy-Schwarz)

Goldstone Counting

Lorentz Invariance:

In a Lorentz invariant system, there is a Goldstone mode for every broken symmetry generator with dispersion relation $E_k \sim k$.

Without Lorentz invariance:

- always at least 1.
- various theorems [Nielsen, Chadha, 1976], [Schäfer et al., 2001], [Watanabe, Brauner, 2011]
- dispersion relations can be

$$E_k \sim k^n, \qquad n \in \mathbb{N}$$

$\overline{\text{2D}}$-No-Go Theorems

Coleman 1973: (QFT)

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$\overline{\text{2D}}$-No-Go Theorems

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Mermin, Wagner 1966: (Stat. Phys.)

At any nonzero temperature, a one- or two-dimensional isotropic spin-S Heisenberg model with finite-range exchange interaction can be neither ferro- nor antiferromagnetic.

Proof (2D): $|s_z| < \frac{const.}{\sqrt{T|\ln h|}}$ via Bogoliubov's ineq. ($\sim$ Cauchy-Schwarz)

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General: [Nielsen, Chadha, 1976; Schäfer et al., 2001; Watanabe, Brauner, 2011]

In general, there is at least one Goldstone mode with some dispersion relation of

$$\text{Type I: } E_k \sim k^{2n-1} \quad \text{or} \quad \text{Type II: } E_k \sim k^{2n}, \qquad n \in \mathbb{N},$$

$$N_{\mathrm{I}} + 2N_{\mathrm{II}} \geq \#\text{ broken generators}$$

Dispersion Rel. as a Cure

$$\mathrm{i}\Delta_F(k) = \lim_{\varepsilon \to 0} \frac{1}{(k^0)^2 - \left(E_{\vec{k}} - \mathrm{i}\varepsilon\right)^2}$$

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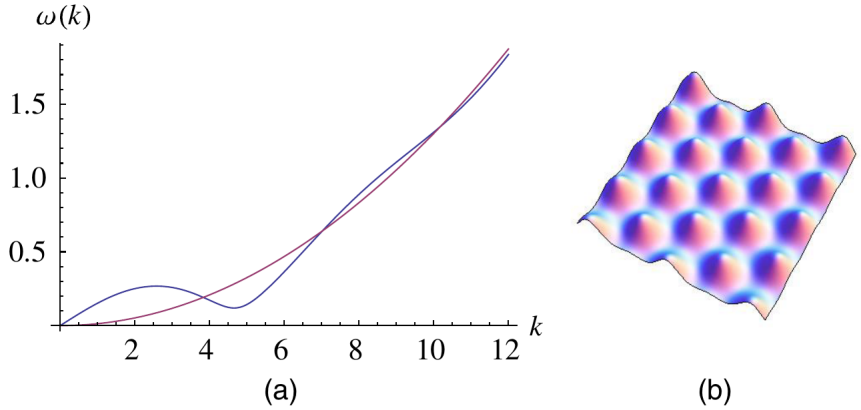


[Watanabe, Brauner, 2012] in a supersolid model

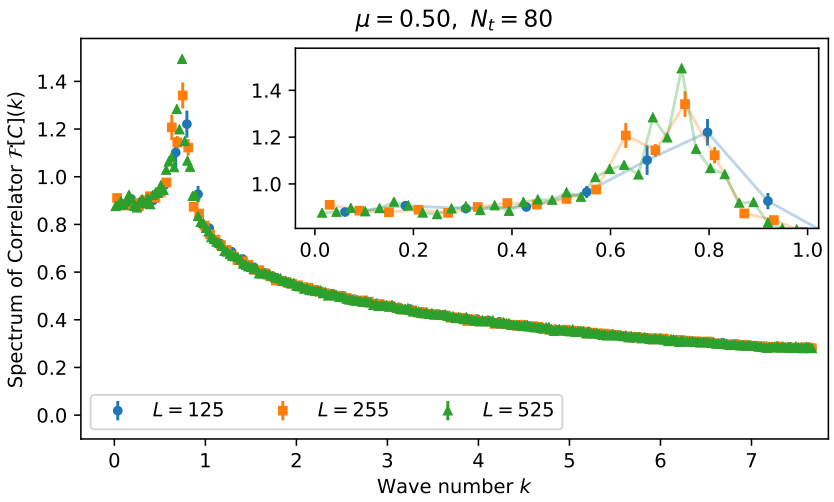
Large-N as a Cure

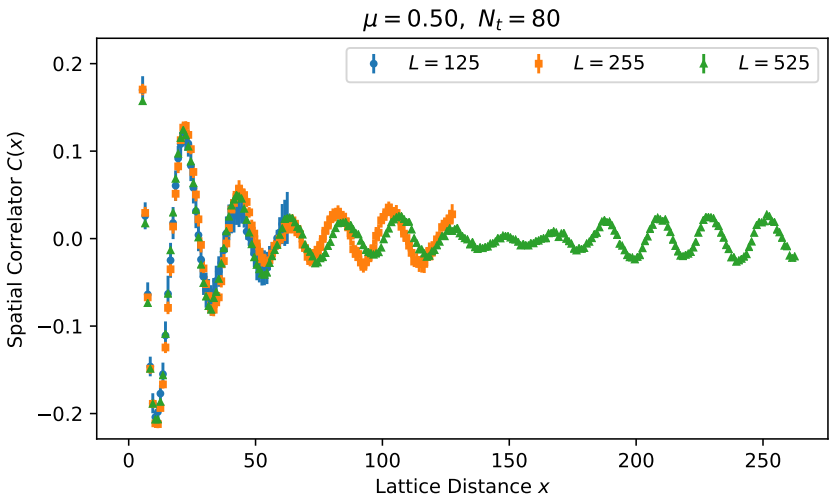
In an exactly soluble 2D model [Witten, 1978] showed that

- a continuous (chiral) symmetry remains unbroken
- physical fermions are chirally neutral and can acquire a mass
- correlations fall off as $\sim |x|^{-1/N_f}$
- there is no IR-divergence because

$$G(x, y) \sim 0 \cdot (\text{div. Term})$$

Long Range Behavior I ($N_f = 2$)





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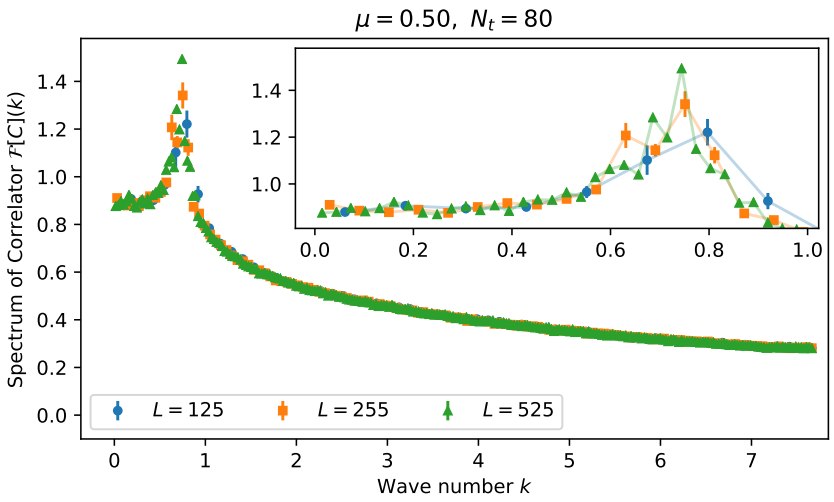

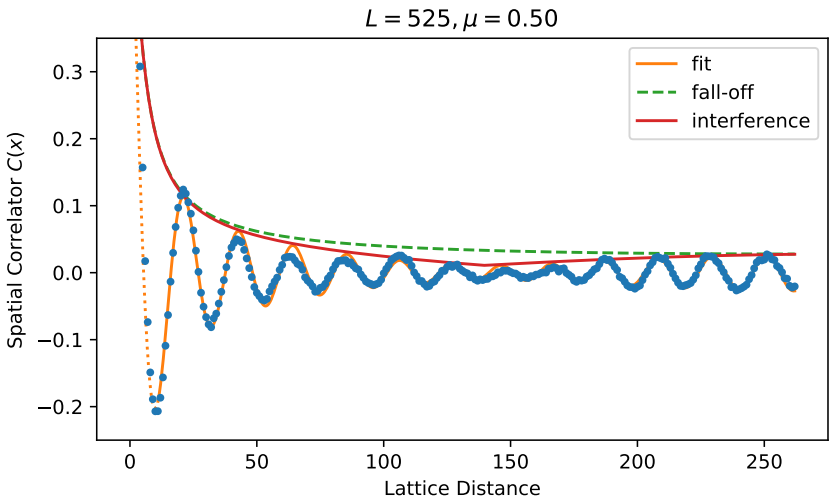

$$\text{fit}(x) \sim \frac{1}{|x|^{1/2}} \left(\cos(k_1 x) + a \cdot \cos(k_2 x)\right)$$

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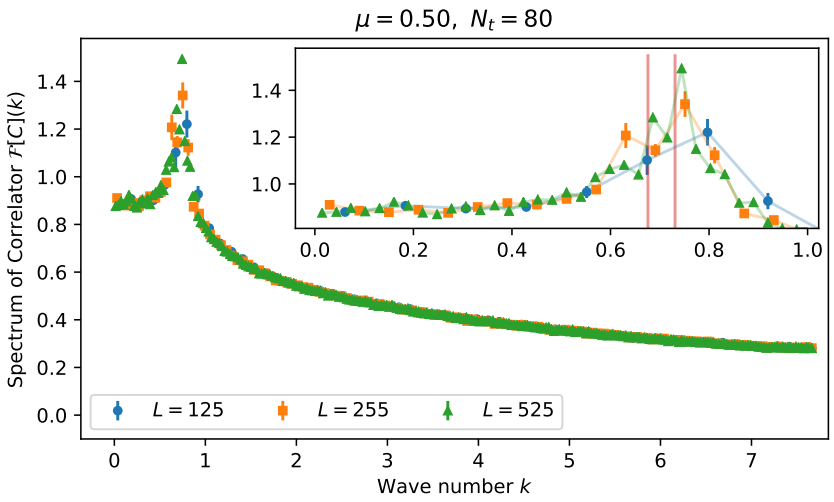
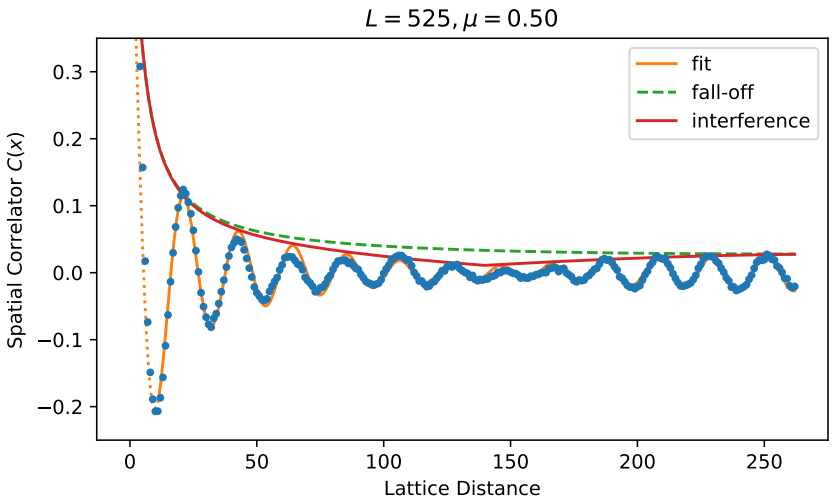
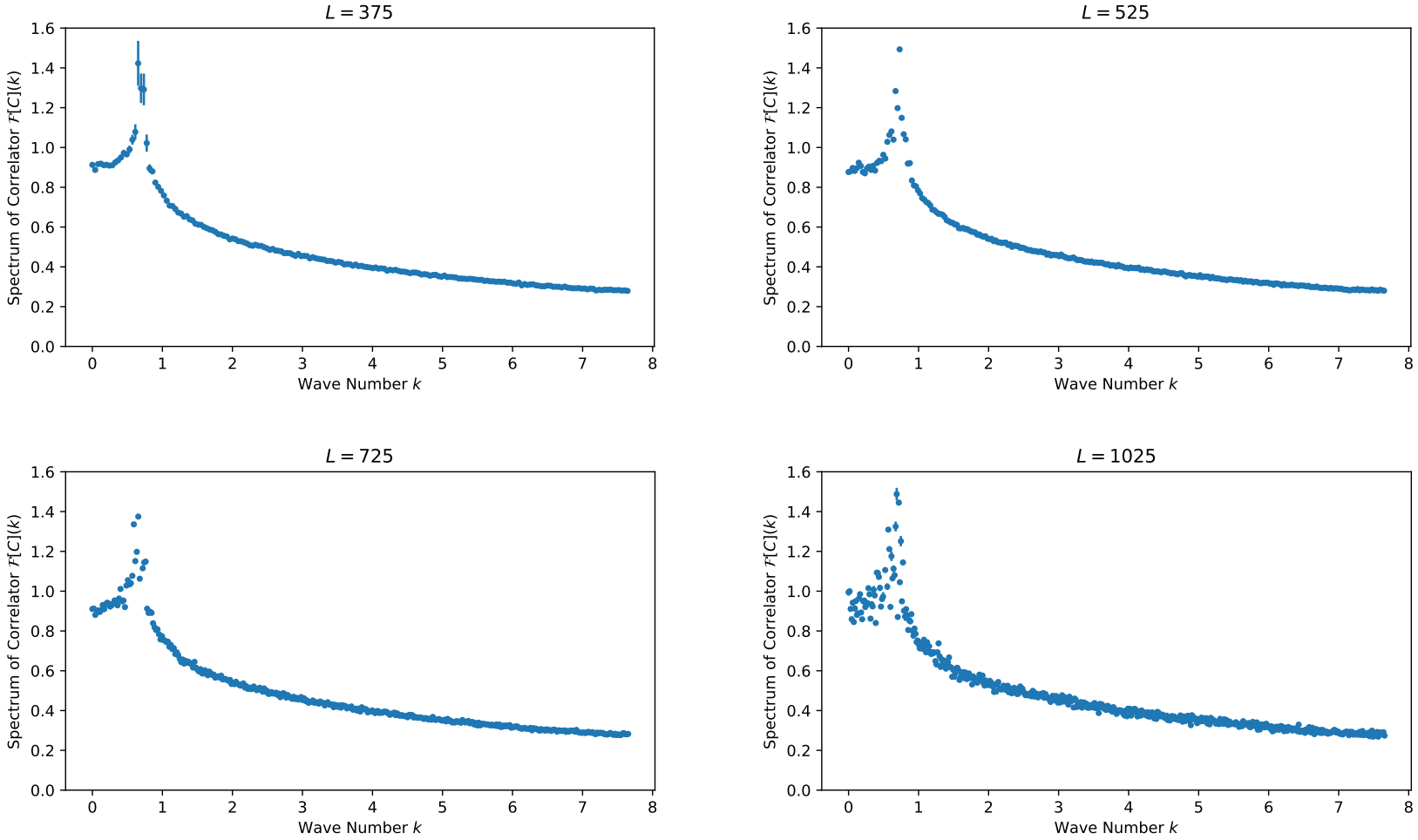




$$\text{fit}(x) \sim \frac{1}{|x|^{1/2}} \left(\cos(k_1 x) + a \cdot \cos(k_2 x)\right)$$

$$k_1 = 0.6763(4) \qquad k_2 = 0.7311(1)$$

Long Range Behavior II ($N_f = 2$)



Fermions on the Lattice

Naive	SLAC
$(\partial_\nu \psi)_x = \frac{1}{2}\left(\psi_{x+e_\nu} - \psi_{x-e_\nu}\right)$	$\mathcal{F}\left[\partial_\nu \psi\right]_p = \mathrm{i} p_\nu \mathcal{F}\left[\psi\right]_p$
local	non-local
exactly chiral	exactly chiral
2^2 doublers	no doublers
$\frac{1}{2}\left(e^{\mu}\psi_{x+e_0} - e^{-\mu}\psi_{x-e_0}\right)$	$\left(\mathrm{i} p_0 + \mu\right)\mathcal{F}\left[\psi\right]_p$

The Observables I

Homogeneous:

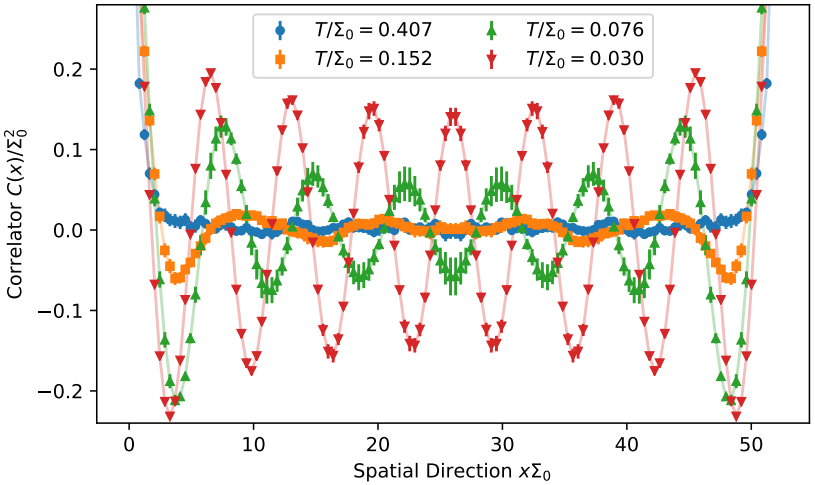
$$|\Sigma| = \frac{1}{V}\sum_{t,x}\langle|\sigma_{tx}|\rangle\,, \qquad \chi = \frac{1}{V}\sum_{t,x}\left[\left\langle\sigma_{tx}^2\right\rangle - \langle\sigma_{tx}\rangle^2\right]$$

Inhomogeneous:

$$C(x) = \frac{1}{N_t}\sum_{t}\langle\sigma_{tx}\,\sigma_{t0}\rangle\,, \qquad \tilde{C}(k) = \mathcal{F}[C](k)$$

Temperature Dependency ($N_f = 8$)

Correlator $C(x)$



Fourier Transformed $\mathcal{F}[C](k)$