Necessity of indefinite metric 'Hilbert spaces' in covariantly gauged QED

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Introduction

Axiomatic formulations of quantum electrodynamics (QED) depart from standard QFTs in various regards. One of them is the presence of Krein spaces, also referred to as indefinite metric 'Hilbert spaces', in covariant gauges. While the necessity of such spaces is often claimed, it is difficult to find a satisfactory justification in the literature, especially beyond the Gupta-Bleuler gauge or in the presence of interaction. The aim of this thesis was to provide a systematic treatment of these matters in terms of two no-go theorems.

Axiomatic formulation of QED

A `theory of QED`[1] is given by a Wightman QFT including a hermitian antisymmetric 2-tensor field F and a hermitian vector field J such that Maxwell's equations,

$$\partial^\mu F_{\mu\nu} = J_\nu \quad \text{and} \quad \partial_{[\mu} F_{\nu\rho]} = 0 \tag{1}$$

hold. Free QED corresponds to $J_\nu = 0$.

Indefinite metric space as a state space

How can an indefinite metric space be a state space?[2] An `indefinite metric 'Hilbert space'` is a Hilbert space $\mathcal{H}$ together with an additional indefinite inner product $\langle , \rangle$. For such a space the subspaces $\mathcal{H}'$ and $\mathcal{H}''$ of states with non-negative and zero 'norm' give rise to an induced physical Hilbert space $\mathcal{H}_{\text{phys}} = \overline{\mathcal{H}'/\mathcal{H}''}$. Taking $\mathcal{H}$ as a state space we get an induced theory on $\mathcal{H}_{\text{phys}}$ (of only physical fields) where the usual probabilistic interpretation of states is recovered. A `gauge formulation of QED` is then given by any theory of a hermitian quantum field A_μ on a state space with (in general indefinite) metric $(\mathcal{H}, \langle , \rangle)$ which induces a theory of QED on $\mathcal{H}_{\text{phys}}$ by $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. If A transforms as a vector field, it is considered as `QED in a covariant gauge`.

Common Physics lore

States with negative scalar square are referred to as `ghost states`. These states occur generically in gauge theories, in particular in QED, as a consequence of the fact that the gauge fields propagate not only physical degrees of freedom. In order to resolve this issue gauge theories are usually constructed for a specific choice of gauge. There are two choices:

1. Modding out the non-physical degrees of freedom, usually breaking covariance and locality of the gauge field; e.g. Coulomb and axial gauge.
2. Insisting on locality and/or covariance, usually implying the existence of ghost states; e.g. Gupta-Bleuler gauge.

Triviality of free QED without gauge-fixing

A gauge formulation of free QED in which A satisfies $\Box A_\mu - \partial_\mu \partial A = 0$ and A is either covariant or local leads to a trivial two-point function of F.[1] Insisting on locality or covariance the equations of motion of free QED thus have to be modified or, in other words, a gauge has to be chosen. The modification has to vanish on matrix elements between physical states such that on the physical subspace Maxwell's equations are recovered.

Thesis' Results

A: Every covariant gauge formulation of free QED on a state space with a non-negative metric is trivial. Trivial means that the two point function of $F_{\mu\nu}$ vanishes:

$$\langle \Omega, F_{\mu\nu}(x) F_{\mu\nu}(y) \Omega \rangle = 0. \tag{2}$$

B: In every covariant gauge formulation of QED on a state space with a non-negative metric the Maxwell-tensor F cannot create massless states from the vacuum. With $\mathcal{H}^{(1)}$ being the space of massless states we have

$$\langle \mathcal{H}^{(1)}, F_{\mu\nu}(\cdot) \Omega \rangle = 0. \tag{3}$$

Methods

We study a Lorentz-covariant tempered distribution $W_{\mu\nu}$ transforming as a $1 \otimes 1$ tensor satisfying the differential equation

$$\Box W_{\mu\nu} - \lambda \partial_\mu \partial^\rho W_{\rho\nu} = 0, \quad \lambda \neq 1. \tag{4}$$

The most general such distribution is determined up to four constants $c_i, i = 1, .., 4$: $\hat{W}_{\mu\nu}(p) = \theta(p_0)\left(c_1\left(\eta_{\mu\nu}\delta(p^2) - \frac{\lambda}{1-\lambda} p_\mu p_\nu \delta'(p^2)\right) + c_2 p_\mu p_\nu \delta(p^2)\right) + c_3\left(\eta_{\mu\nu}\Box\delta(p) - \frac{4-\lambda}{24(1-\lambda)} p_\mu p_\nu \Box^2 \delta(p)\right) + c_4 \eta_{\mu\nu}\delta(p)$

This result applies to the two-point function of a vector field satisfying $\Box A_\mu - \lambda \partial_\mu \partial A = 0, \lambda \neq 1$, which is the most general covariant second-order PDE which is linear in the vector field A. Requiring definiteness of the metric yields $c_1 = c_3 = c_4 = 0$ as the corresponding terms are indefinite. Consequently, the two-point function is of the form $W_{\mu\nu} = \partial_\mu \partial_\nu K$ for some K. Accordingly, A is a pure gauge and the two-point function of F vanishes.

The second result of the thesis is an application to the two-point function of the gauge field of the interacting theory projected onto the subspace of massless states.

Summary and current research

In conclusion we have found that a satisfactory theory of (free or interacting) QED constructed in terms of a gauge field transforming covariantly as a vector field has to rely on a state space in which zero and negative 'norm' states are present. A complementary result from the literature draws the same implication for the case of a local gauge field in the presence of local and charged fields.[2] In total this implies a necessity to use indefinite metric spaces for local or covariant gauge formulations of QED.

Apart from well-known ways to evade the presence of negative 'norm' states like Coulomb and axial gauges, a quite recent approach describes QED in terms of a string-local potential $A_\mu(x, e) = \int_0^\infty ds F_{\mu\nu}(x + se) e^\nu$, e spacelike.[3] As this potential satisfies weakened conditions of locality and covariance it might be constructed on an ordinary state space. Its main advantage is that it possibly is sufficiently localized to apply adapted perturbative and renormalization methods.

REFERENCES

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[1] Note that this definition here without further specification of J is very broad. But for the results no further specification of J is necessary.
[2] Reminder: Physical states need to have a non-negative 'norm' for their probabilistic interpretation.