Quantum Distillations, Semi-classics and mixed anomalies

Mithat Ünsal

NCSU

Statement of the problem

- My aim it to generalize the connections between the Hilbert space of QFT, the path integral formulation and thermodynamics.

- The motivations comes from the following observation. Thermal partition functions or state sums (with appropriate Boltzmann weights) of a QFT usually exhibit a phase transition as a function of the inverse temperature or rapid cross-overs.

- Hilbert space is huge. (e.g. even 500 spin 1/2 particle, its dimension is 2^500), and density of states grow rather rapidly. We do not really care about the details of all states.

- Asymptotic freedom: phenomena at scale of T becomes weakly coupled, and partly calculable. However, at such T, partition function is extremely contaminated. (Every state contribute.)

- Given a general QFT, can we construct a state sum which remains analytic while staying in thermodynamic limit?

Statement of the problem-II

$$\mathcal{Z}(\beta) = \mathrm{tr}\left[e^{-\beta H}\right]$$

- Thermal partition functions or state sums (with appropriate Boltzmann weights) of a QFT usually exhibit a phase transition (or rapid crossover) as a function of the inverse temperature.

- You can for example show that in order to get the well-known Stefan-Boltzmann law of blackbody radiation, the density of states must grow

$$\mathcal{F} = -V_3 \times \frac{\pi^2}{45} N^2 T^4 \qquad \rho_{SB}(E) \sim e^{E^{3/4} N^{1/2} {V_3}^{1/4}}$$

- So, for such a phenomena, we cannot be ignorant of high energy states or the growth of density of state.

One version of background story

- In 2012, w/Gerald Dunne, we introduced the idea of resurgence and trans-series in QFT. As an example, we studied CP(N) model by using an SU(N) symmetry-twisted boundary conditions on $S^1 \times \mathbb{R}^1$
- Quite remarkably, almost all interesting non-perturbative properties of the compactified theory $S^1 \times \mathbb{R}^1$ matches to the expected properties of infinite volume limit. E.g.
- Mass gap of the order of strong scale
- Renormalons
- Multi-branched theta angle dependence
- These results are in sharp distinction with elegant work of Affleck (82) which study the same theory with thermal compactification.

Background story

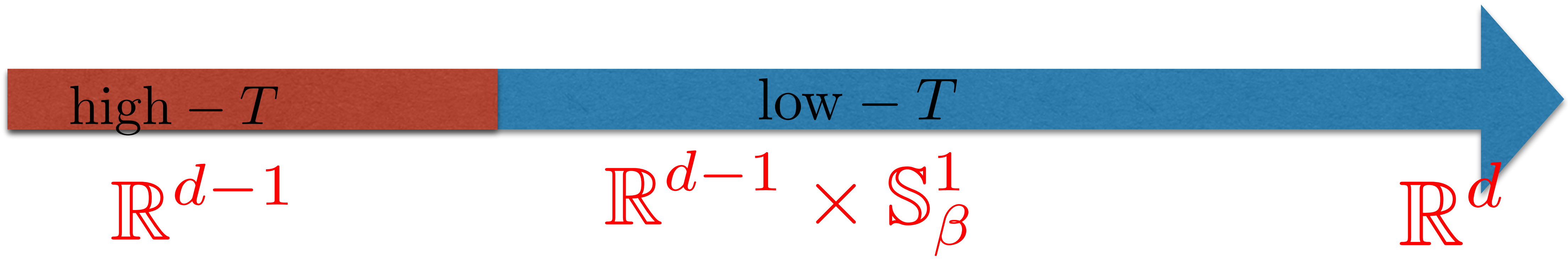
- One puzzle is why a theory compactified on a tiny circle (smaller than the strong length scale) knows so much about full QFT. (We called this idea adiabatic continuity. Physical observables are smooth functions of radius even in thermodynamic limit.)

- This lead us to think more carefully on the Hilbert space implications of the global symmetry twisted boundary conditions.

- The story that takes place in CP(N-1) is first understood by Sulejmanpasic(2016). He interpreted t.b.c. in Hilbert space. I will explain and generalize his argument.

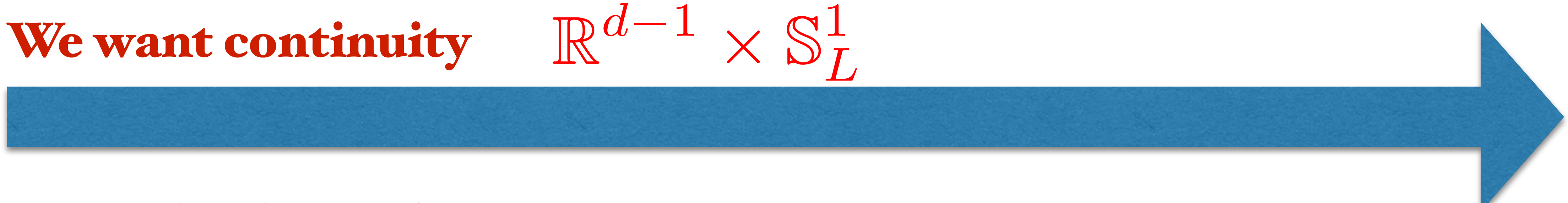
- What is recently understood is that the t.b.c. that Gerald and I used was **unique** in the sense that a mixed 't Hooft anomaly that exists on infinite volume persists upon compactification if and only if one uses the specific t.b.c.

Idea of adiabatic continuity



Thermal: Rapid crossover at finite-N, phase transition at large-N $Z(\beta) = \mathrm{tr} e^{-\beta H}$

We want continuity $\mathbb{R}^{d-1} \times \mathbb{S}^1_L$



New idea from physics: Prevent phase transition by using circle compactification or judicious matter choice or (twisted/non-thermal) boundary conditions.

Supersymmetric theories: Continuity and analyticity (Witten,80).

$$Z(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

Non-supersymmetric theories, including QCD-like theories:
The possibility of is realized in 2007 (M.Ü., Yaffe07) and also see Ogilvie, Myers.).
Semi-classical version of the beautiful large-N reduction idea (Eguchi, Kawai 82)!

Idea of adiabatic continuity in theories with global symmetries

- First, as a toy example, I will describe the ideas in a simple example.
- Broad-brush the solution of a non-trivial QFT on small $S^1 \times \mathbb{R}^1$
- Explain what is going on in the Hilbert space. (quantum distillations)
- Explain uniqueness of the choice of the t.b.c. by using mixed anomaly

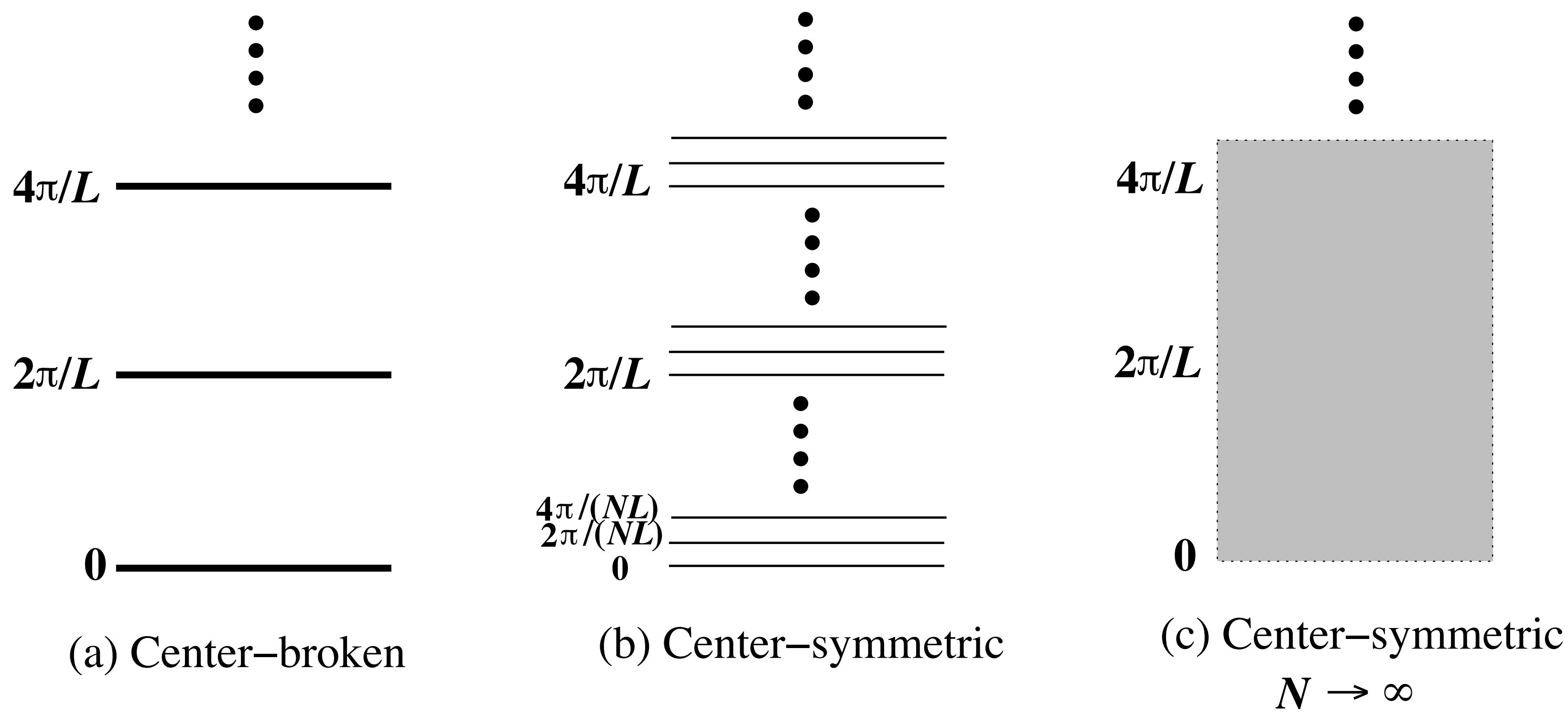
Quick review of CP(N-1) with tbc.

Twisted boundary conditions= Turning on a background SU(N) field

$$n_i(x_1, x_2 + L) = \Omega_{ij} n_j(x_1, x_2)$$

$$\Omega = \mathrm{Diag}\left(1, \omega, \omega^2, \ldots \omega^{N-1}\right), \qquad \omega = e^{i2\pi/N}$$

The dependence of perturbative spectrum to the flavor -holonomy background

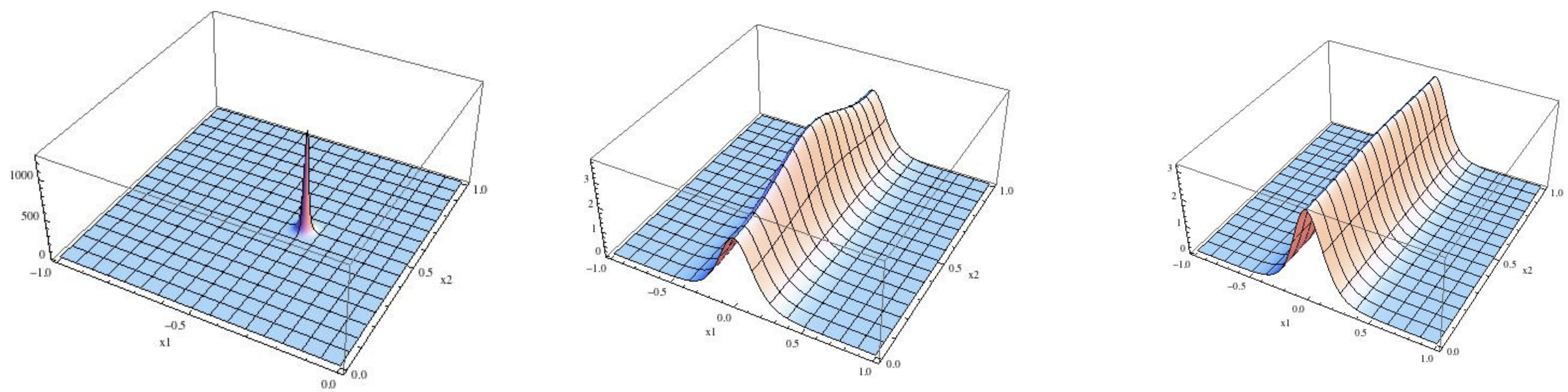


Same as gauge theory on R3 x S1: Spectrum become dense in the L=fixed, and N-large
$\Longrightarrow$ Imprint of the large-N volume independence (large-N or Eguchi-Kawai reduction).

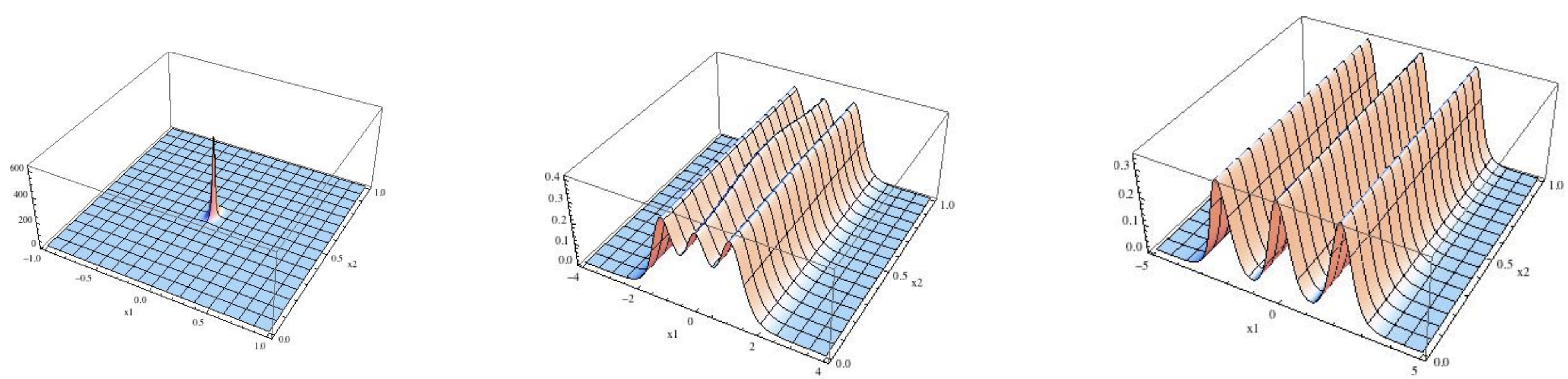
Here, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

In thermal box, and high T, associated with trivial holonomy, the fractionalization does not occur (Affleck, 80s). Plot is for CP(2)



In spatial box, and small-L, associated with non-trivial holonomy, the fractionalization does occur. Large-2d BPST instanton in CP(2) fractionates into 3-types. (Dunne,MÜ, 2012)



Gauge theory counter-part on R_3 x S_1 :

Monopole-instantons or 3d-instanton and twisted instanton.

(caloron constituents) : van Baal, Kraan, (97/98), Lee-Yi (97)

Topological configurations, 1-defects, formally

Kink-instantons: (1d-instanton and twisted instantons) Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The twisted-instanton is present only because the theory is locally 2d! Also derived in Bruckmann et.al.(07, 09)

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$

$$\mathcal{I}_{2d} \sim e^{-\frac{4\pi}{g^2}} = \left(\frac{\Lambda}{\mu}\right)^{\beta_0} \sim \prod_{j=0}^{\mathfrak{r}} [\mathcal{K}_j]^{k_j^\vee}, \qquad \beta_0 = h^\vee = \sum_{i=0}^{\mathfrak{r}} k_i^\vee, \qquad \mathfrak{r} = \mathrm{rank}[\mathfrak{su}(N)]$$

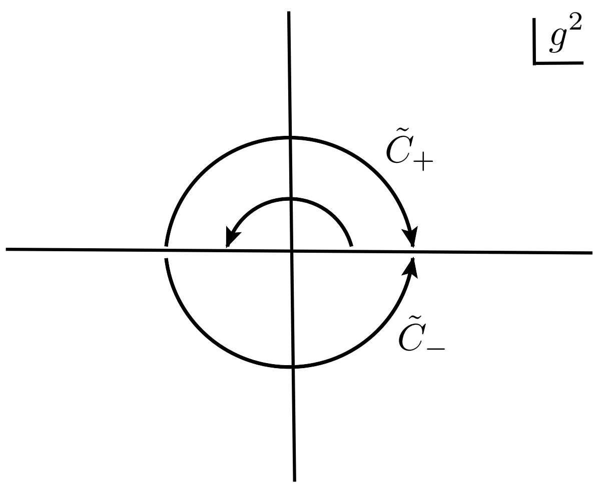
$k_i^\vee$ Dual Kac labels, all 1 for su(N) algebra, multiplicity of kink-instanton

Sum of Dual Kac labels= Dual Coxeter number= Beta function (This is quite non-trivial in general sigma models.)

Fractionalization formula of the instanton

Neutral bion and non-perturbative ambiguity in semi-classical expansion

We can unambiguously calculate the second order in semi-classical contributions by using Picard-Lefschetz theory to quasi-zero mode integral. Since we are on a Stokes line, the amplitude is 2-fold ambiguous.



$$
\begin{aligned}
[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \mathrm{Re}\,[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\
&= \left(\log\left(\frac{\lambda}{8\pi}\right) - \gamma\right)\frac{16}{\lambda}e^{-2S_0} \pm i\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}
\end{aligned}
$$

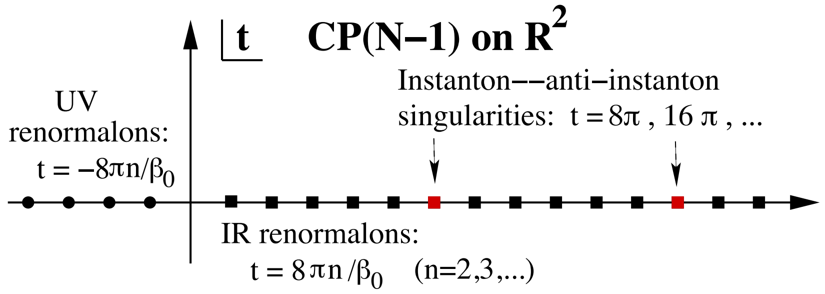
As it stands, this looks terrible. Is semi-classical expansion at second order void of meaning? This is a general statement valid for many QFTs admitting semi-classical approximation, including the Polyakov model (77)! And it has not been addressed in literature until recently.

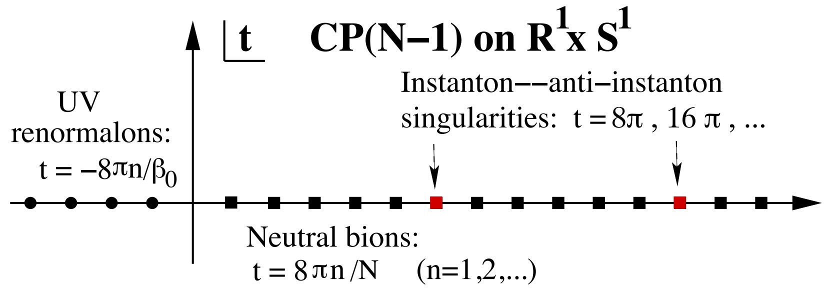
In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. The truth is far more subtler!

Semi-classical renormalons as neutral bions

The ambiguities which cancel are at exp[-$2S_I/N$] order.
Exactly in the IR-renormalon territory ['t Hooft(77), David(81)].

Claim: Neutral bions and neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion.





More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD (QFT)?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that, we have a chance, at least, in the semi-classical regime of QFT.

This description was the missing link between 't Hooft renormalons (late 70s) and van Baal's fractional-instantons/caloron constituents (late 90s).

Why things work the way they do? Hilbert space distillation idea

Sulejmanpasic (2016), Dunne, Tanizaki, MU (2018)

$n_i(x_1, x_2 + L) = \Omega_{ij} n_j(x_1, x_2)$ In path integral formalism, becomes:

$$Z_\Omega^{\mathrm{CP(N-1)}} = \mathrm{tr}[e^{-\beta H} \prod_k^{N_f} e^{i\frac{2\pi k}{N_f} Q_k}]$$

in operator formalism,

where Q_k is the number operator for n_k quanta.

Hilbert space distallation in simple QM

If you do distallation for N-dimensional SHO ($N = 4$ in the picture), this leads to cancellation of all states in the Hilbert space except for level number $\lambda = Nk, k \in \mathbb{N}$.

$$\vdots$$

$$\lambda = 2 \quad \boxed{\underline{q^1}\ \ \underline{q^2}\ \ \underline{q^3}\ \ \underline{q^4}\ \ \underline{q^1}\ \ \underline{q^2}\ \ \underline{q^3}\ \ \underline{q^4}\ \ \underline{q^2}\ \ \underline{q^4}} \quad 0$$

$$\lambda = 1 \quad \boxed{\underline{q^1}\ \ \underline{q^2}\ \ \underline{q^3}\ \ \underline{q^4}} \quad 0$$

$$\lambda = 0 \quad \underline{q^4}$$

$$\begin{aligned} \mathcal{Z}(\beta) &= e^{-\beta N/2}\left(1 + N\,e^{-\beta} + \cdots + \begin{pmatrix} 2N-2 \\ N-1 \end{pmatrix} e^{-(N-1)\beta} + \begin{pmatrix} 2N-1 \\ N \end{pmatrix} e^{-N\beta} + \ldots\right) \\ \mathcal{Z}_\Omega(L) &= e^{-LN/2}\left(1 + 0 \times e^{-L} + \cdots + \quad 0 \times e^{-(N-1)L} \quad + 1 \times e^{-NL} + \ldots\right). \end{aligned}$$

Same Hilbert space, dramatic cancellation in the graded sum.

Large-N limit: **Only ground state contribute**, couter-part of super-symmetric Witten index in a boring bosonic QM.

Hilbert space distallation in QFT: CP(N-1)

- The global symmetry of CP(N-1) model is actually PSU(N) but not SU(N). Hilbert space constitute reps of PSU(N). There is no gauge invariant fundamental rep of SU(N) in the physical spectrum. There are only meson-like excitations. E.g. CP(1)

$$(nn^\dagger)^k_j(x) \in \mathrm{Adj}_N, \qquad n(x)(e^{i\int_x^y a})(n^\dagger)(y) : \text{singlet}$$

$$\mathcal{Z}(\beta) \supset +(N^2-1) \times e^{-\beta E_{\mathrm{adj}}} + 1 \times e^{-\beta E_{\mathrm{singlet}}} + \ldots \underbrace{\rightarrow}_{N\rightarrow\infty} N^2 \times e^{-\beta E_{\mathrm{adj}}}$$

$$\mathcal{Z}_\Omega(L) \supset (-1) \times e^{-\beta E_{\mathrm{adj}}} \qquad + 1 \times e^{-\beta E_{\mathrm{singlet}}} + \ldots \underbrace{\rightarrow}_{N\rightarrow\infty} 0 \qquad \text{HMMM!}$$

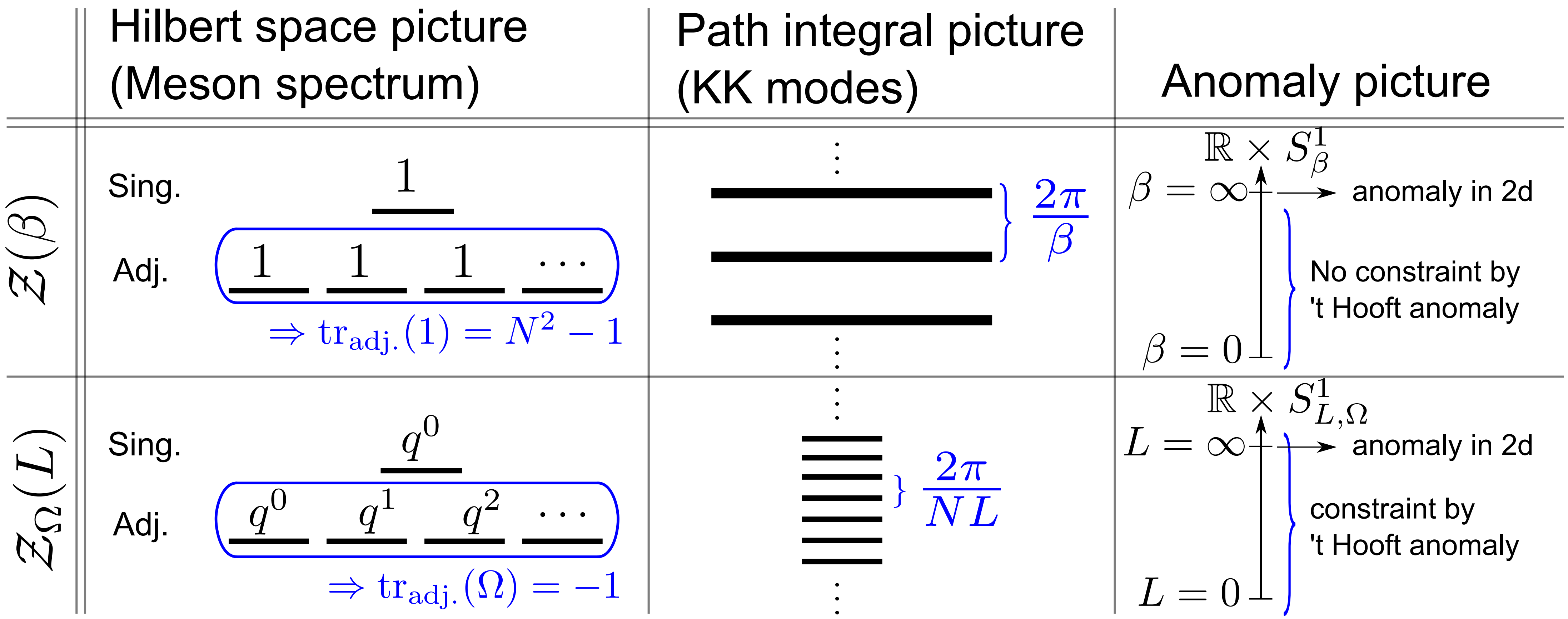
Large-N limit: **Only ground state contribute**, couter-part of super-symmetric Witten index in a non-trivial bosonic QFT!

Mixed Anomaly in CP(N-1)

- There is a mixed 't Hooft anomaly between $PSU(N)$ and C at $\theta = \pi$. If we gauge $PSU(N)$, topological charge happens to be quantized in units of $1/N$ and theta angle becomes periodic in units of $2\pi N$. As a result, C ceases to be a symmetry at $\theta = \pi$ implying mixed anomaly.

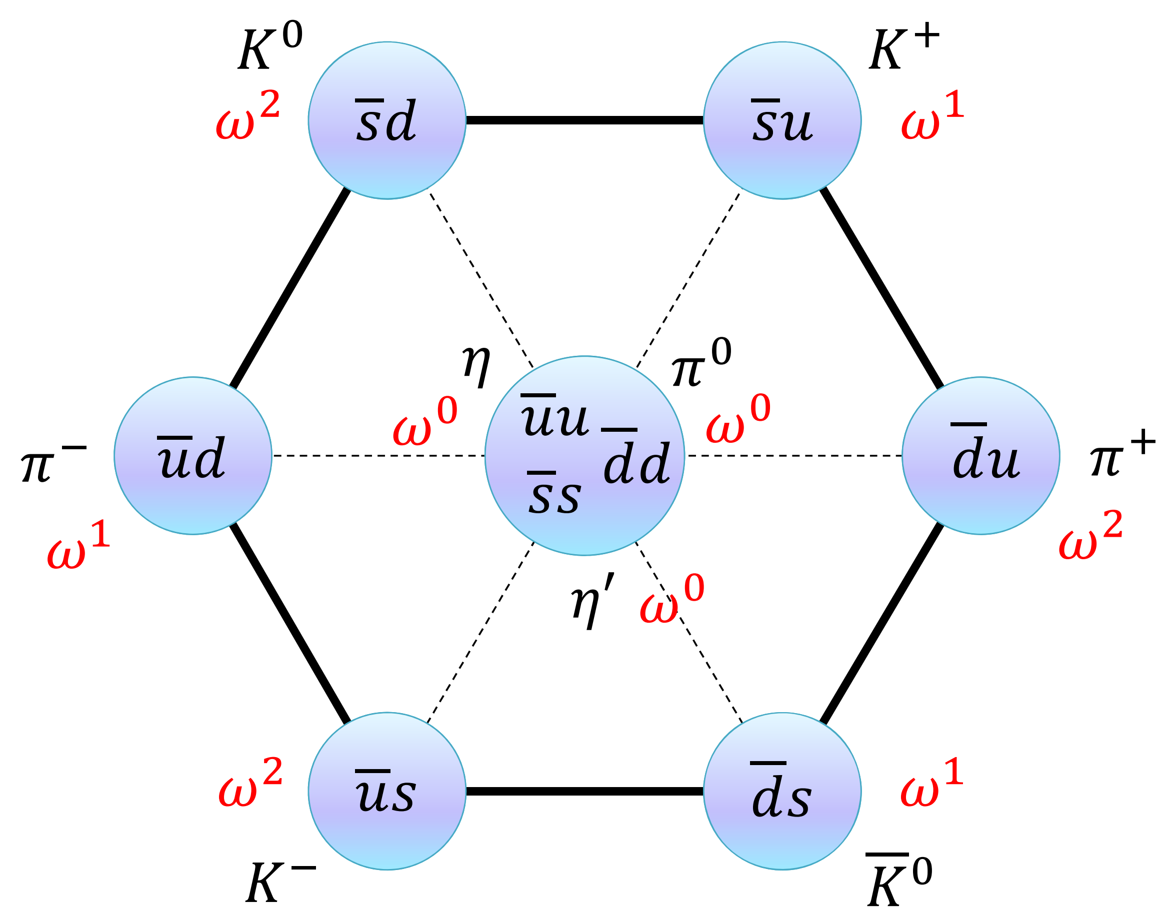
- Mixed anomaly persists on $\mathbb{R}^1 \times S^1$ if and only if the tbc is $\mathbb{Z}_N$ symmetric.

Hilbert space distillation,
Path integral KK-modes,
Persistent mixed anomaly



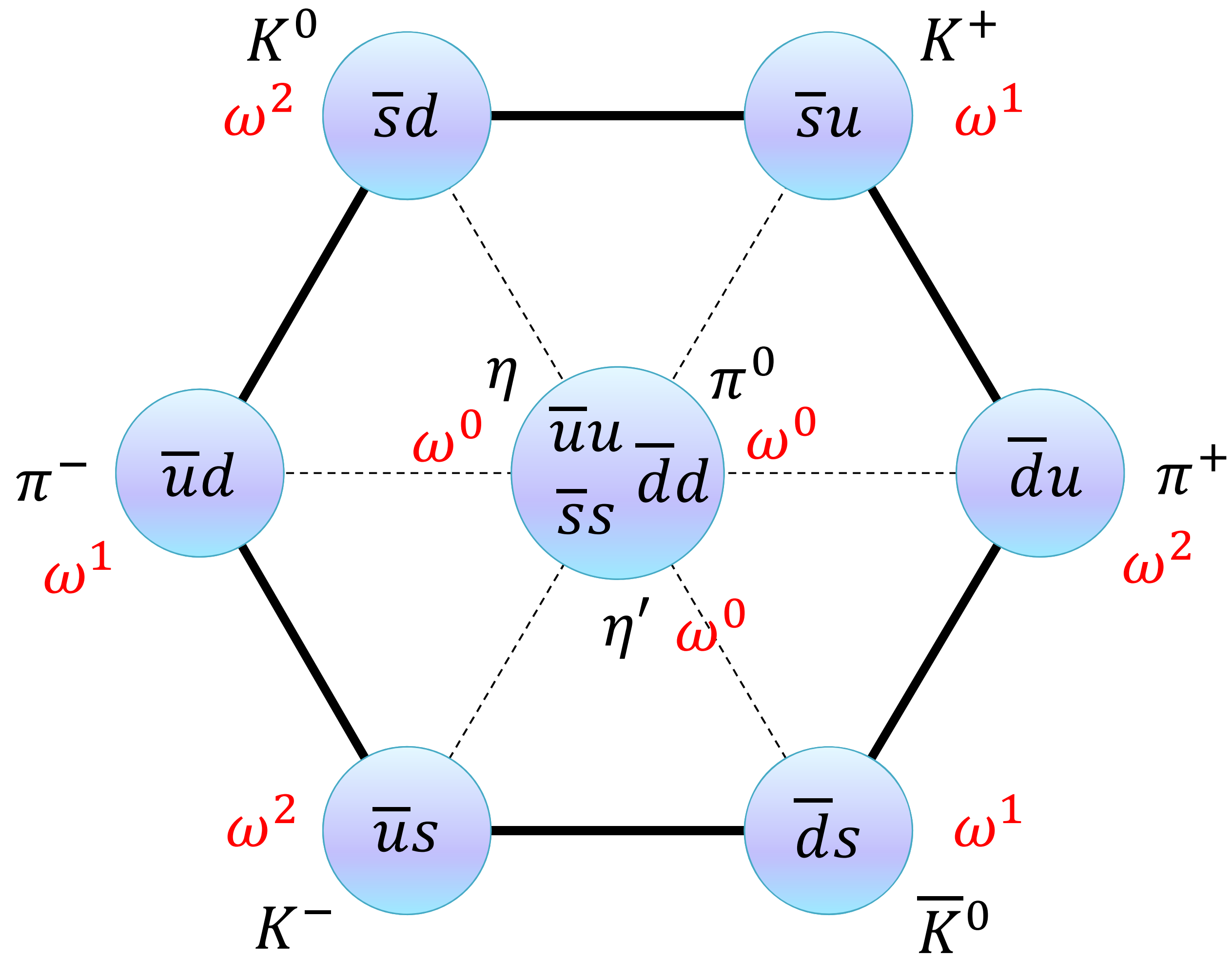
Can we do this in QCD?

• Assume $m_u = m_d = m_s \geq 0$ limit and as an example, consider scalar meson sector.



Use $$Z_\Omega^{\mathrm{QCD}} = \mathrm{tr}[e^{-\beta H^{\mathrm{QCD}}} \prod_k^{N_f} e^{i\frac{2\pi k}{N_f} Q_k}]$$ What happens?

Can we do this in QCD?



Morally: $$(N_f^2-1)e^{-\beta E_\pi}+1e^{-\beta E_{\eta'}}\underbrace{\rightarrow}_{\Omega}(-1)e^{-\beta E_\pi}+1e^{-\beta E_{\eta'}}$$

Less than 1 particle contributing to graded P. F. instead of N_f^2

QCD(F) with grading

$$Z_{\Omega}^{\rm QCD} = {\rm tr}[e^{-\beta H^{\rm QCD}} \prod_{k}^{N_f} e^{i \frac{2\pi k}{N_f} Q_k}] \approx {\rm tr}[e^{-\beta H^{\rm YM}}] = Z^{\rm YM}$$

To get more physical intuition, calculate free-energy for Nf=Nc theory. First, remember the standard Stefan-Boltzmann result.

$$\mathcal{F}^{\rm QCD} \sim -\frac{\pi^2}{45} N_c^2 V_3 T^4 - \frac{7}{8}\frac{\pi^2}{45} N_f N_c V_3 T^4$$

$$\mathcal{F}_{\Omega}^{\rm QCD} \sim -\frac{\pi^2}{45} N_c^2 V_3 T^4 - \frac{7}{8}\frac{\pi^2}{45} \frac{1}{N_f^3} N_c V_3 T^4 \approx \mathcal{F}^{\rm YM}$$

In the twisted background, it is as if there are no quarks in the microscopic theory!

QCD(F/adj): A 4d theory with adiabatic continuity?

$$\mathcal{L} = \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \overline{\psi}_a \gamma_\mu D_\mu \psi^a + 2 \operatorname{tr} \overline{\lambda} \overline{\sigma}_\mu D_\mu \lambda$$

Global symmetry (acting faithfully on Hilbert space)

$$\mathbf{G} = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_c,N_f)}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}.$$

Extra U(1) axial compared to QCD(F). Turn on a mass deformation for adjoint fermion to remove it.

Graded partition function

$$\mathcal{Z}(\beta, \epsilon_a) = \mathrm{tr}\left[e^{-\beta H}(-1)^F \prod_{a=1}^{N_f} e^{i\epsilon_a Q_a}\right]$$

In path integral formulation:

$$\mathcal{Z}(\beta, \epsilon_a) = \int_{\substack{A(\beta) = +A(0) \\ \lambda(\beta) = +\lambda(0) \\ \psi(\beta) = +\psi(0)\overline{\Omega}_F e^{i\pi}}} DA_\mu D\psi D\lambda \;\; e^{-S[A_\mu, \psi, \lambda]}$$

Color-Flavor center symmetry in QCD(F/adj) and QCD(F)

QCD with fundamental quark does not have a 1-form center-symmetry due to existence of quarks. The compactified theory does not have a 0-form center-symmetry either. Both are explicitly broken.

However, compactified theory with TBC has an exact 0-form CFC-symmetry under which Polyakov loop is an exact order parameter. (de the absence of 1-form center.)

Order parameters and color-flavor center symmetry in QCD
A. Cherman, S. Sen, M. Unsal, M. L. Wagman, L. G. Yaffe

$$\psi(x_4 + \beta) = -\psi(x_4)\overline{\Omega}_F^0$$

$$\Omega_F^0 = \mathrm{diag}(1, \omega, \cdots, \omega^{N_f-1}), \qquad \omega = e^{2\pi i/N_f}$$

Under a gauge rotation aperiodic up to an element of the center, the aperiodicity of the fermion field can be undone by a cyclic flavor rotatio

CFC-symmetry: $\mathrm{tr}\,\Omega(\mathbf{x}) \mapsto \omega\,\mathrm{tr}\,\Omega(\mathbf{x}), \qquad \psi_a \mapsto \psi_{a+1}$

One-loop gauge-holonomy potential: Without twists

$$V_{1-\mathrm{loop,thermal}}(\Omega) = \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\left[(-1+(-1)^n)\frac{1}{n^4}|\,\mathrm{tr}\,(\Omega^n)|^2 + N_f\frac{(-1)^n}{n^4}\left(\,\mathrm{tr}\,(\Omega^n) + \mathrm{c.c.}\right)\right]$$

With twist

$$V_{1-\mathrm{loop},\Omega_\mathrm{F}} = V^{\mathrm{gauge}}_{1-\mathrm{loop}} + V^{\lambda}_{1-\mathrm{loop}} + V^{\psi}_{1-\mathrm{loop},\Omega_\mathrm{F}}\,,$$

$$V^{\mathrm{gauge}}_{1-\mathrm{loop}} + V^{\lambda}_{1-\mathrm{loop}} = (-1+1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}|\,\mathrm{tr}\,(\Omega^n)|^2 = 0\,,$$

$$V^{\psi}_{1-\mathrm{loop},\Omega_\mathrm{F}} = \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left[\mathrm{tr}\,(\Omega^n)\,\mathrm{tr}\,(\overline{\Omega}_F^n) + \mathrm{c.c.}\right].$$

For generic twist, one-loop potential not invariant under the center. But for special twist, it is invariant under CFC as promised.

One-loop gauge-holonomy potential with special twist

$$V_{\text{1-loop},\Omega_F^0} = \frac{2}{\pi^2 N_f^3 \beta^4} \sum_{k=1}^{\infty} \frac{(-1)^{N_f k}}{k^4} \left[\operatorname{tr}\left(\Omega^{N_f k}\right) + \text{c.c.} \right]$$

The minimization of this potential maps to a problem in additive number theory. Studied by Erdos et.al. (1961).

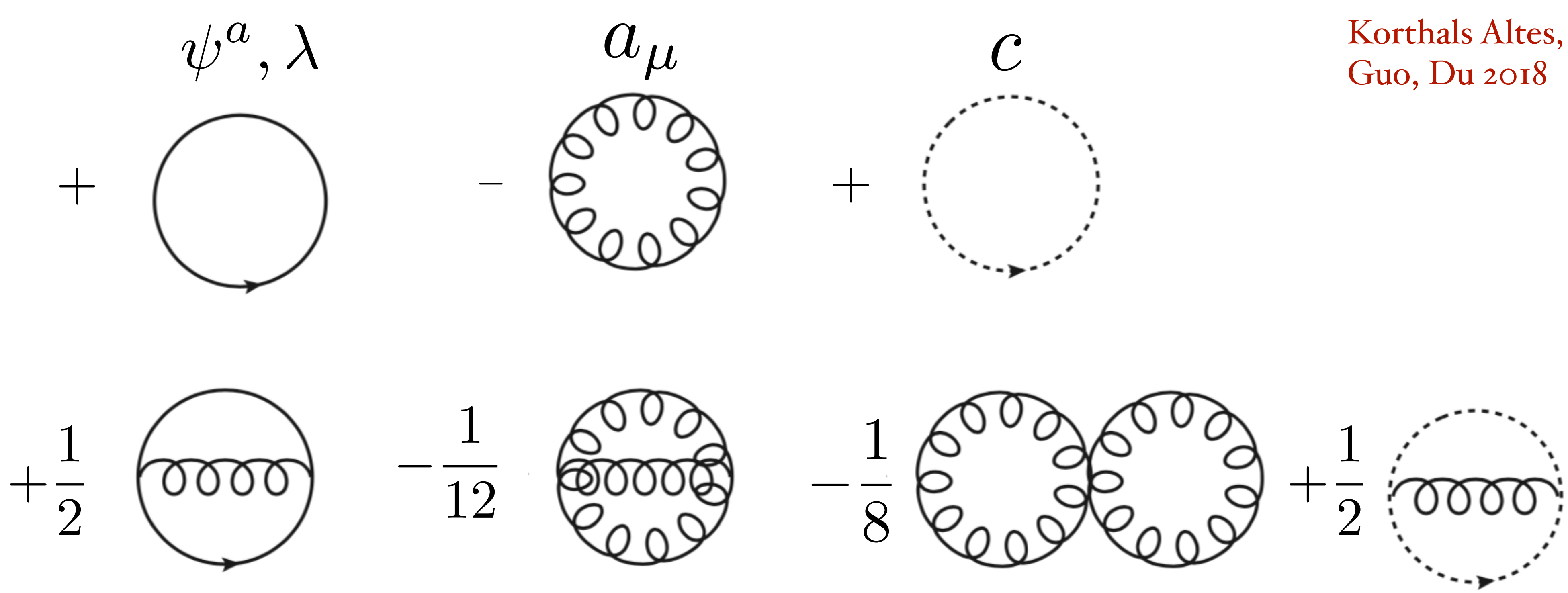
$$\mathfrak{N}_{\min}(N) \approx \frac{(2N-1)!}{(N!)^2} \underbrace{\longrightarrow}_{\text{large}-N} \frac{2^{2N-1} N^{-3/2}}{\sqrt{\pi}} .$$

At N=infinity limit, there is a moduli space at one-loop level. No potential generated as in susy theories. Quantum distillation at work?

N	2	3	4	5	6	7	8	9	10	11	12
Minima Exact	1	4	9	26	76	246	809	2704	9226	32066	112716
Approx.	1.5	3.33	8.75	25.2	77	245.14	804.38	2701.11	9237.8	32065.1	112673.16
Maxima Exact	2	6	10	50	80	490	810	5400	9252	64130	112720

We need two-loop potential to decide: With no twist,

Korthals Altes, Pisarski, Sinkovics 99
Guo, Du 2018



These two refs look like they do not agree. But due to many non-trivial Bernoulli polynomial identities, they in fact do. There are also consistency checks.

With Takuya Kanazawa, we generalized these to incorporated grading/twisting.

One-loop gauge-holonomy potential with special twist

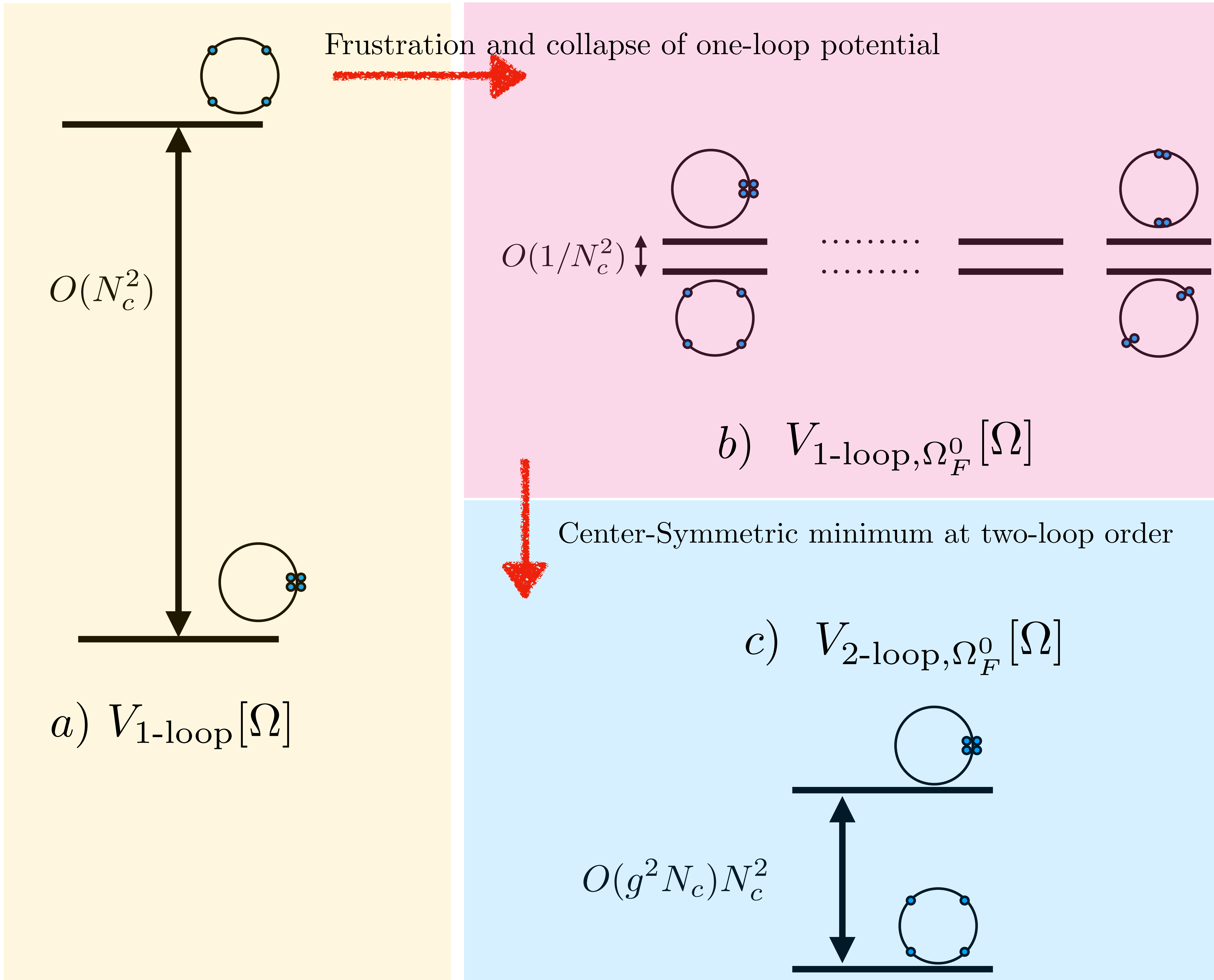
$$V_{1\text{-loop},\Omega_F^0} = \frac{2}{\pi^2 N_f^3 \beta^4} \sum_{k=1}^{\infty} \frac{(-1)^{N_f k}}{k^4} \left[\mathrm{tr}\left(\Omega^{N_f k}\right) + \mathrm{c.c.} \right]$$

Two-loop gauge-holonomy potential with special twist

Kanazawa, MU 2018

$$V^{\psi}_{2\text{-loop},\Omega_F^0} \to \frac{g^2}{\beta^4}\frac{3}{\pi^4}\left\{ -\frac{N_c^2-1}{8N_c N_f^3}\sum_{n=1}^{\infty}\frac{(-1)^{N_f k}}{k^4}[\mathrm{Tr}(\Omega^{N_f k}) + \mathrm{c.c.}] + \frac{N_f}{24}\sum_{n=1}^{\infty}\frac{\left|\mathrm{Tr}(\Omega^n)\right|^2}{n^4} \right\}$$

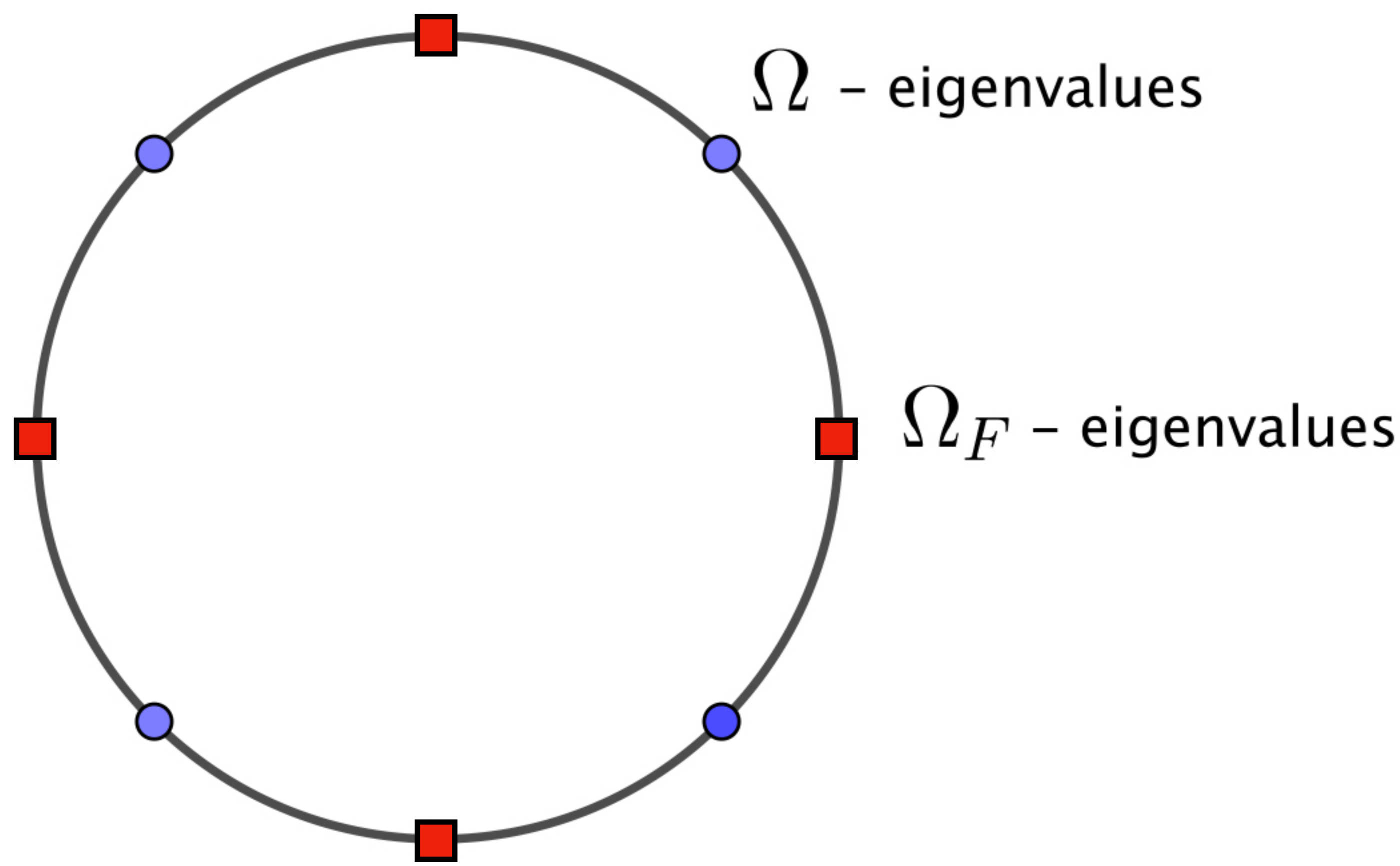
It is not the fact that the formulation has CFC symmetry, rather CFC is stabilized at small-circle that is extremely important. This was almost impossible, how did this happen? (with pictures.)



Frustration and collapse of one-loop potential
$O(N_c^2)$
$a)\ V_{1\text{-loop}}[\Omega]$
$O(1/N_c^2)$
$b)\ V_{1\text{-loop},\Omega_F^0}[\Omega]$
Center-Symmetric minimum at two-loop order
$c)\ V_{2\text{-loop},\Omega_F^0}[\Omega]$
$O(g^2 N_c)N_c^2$

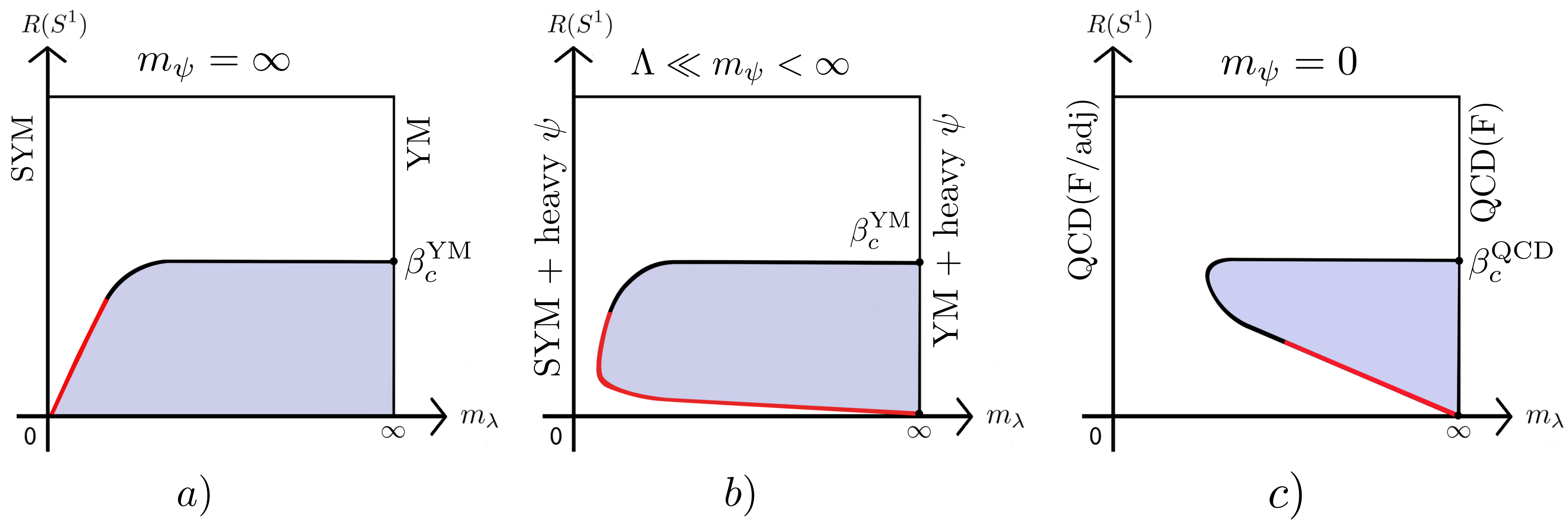
Center-symmetric flavor symmetric background (a choice) implies that the gauge holonomy is minimized (dynamically) at the center-symmetric gauge holonomy background.

$N_c = N_f = 4$



This implies we can do semi-classical weak coupling analysis of the non-perturbative dynamics, e.g. we can prove chiral symmetry breaking in this regime by semi-classical methods. We will show this.

How robust is this upon making adjoint fermion massive?



Transitions shown in red are analytically calculable.

SYM: Poppitz, Schaefer, MU, 2012

SYM Lattice: Bergner, Piemonte, MU 2018

Dynamics and monopole-operators

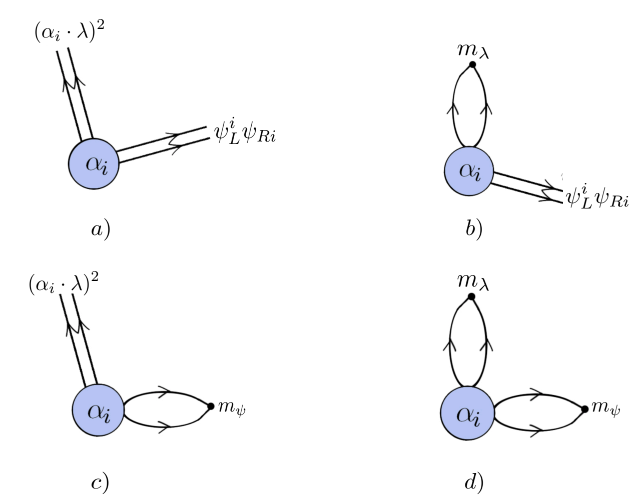
Due to adjoint Higgsing induced by gauge holonomy, dynamics abelianize dynamically at small-circle.

$$SU(N_c) \to U(1)^{N_c - 1}\,.$$

N-types of monopoles Yi, Lee, Kraan, van Baal, 97-99

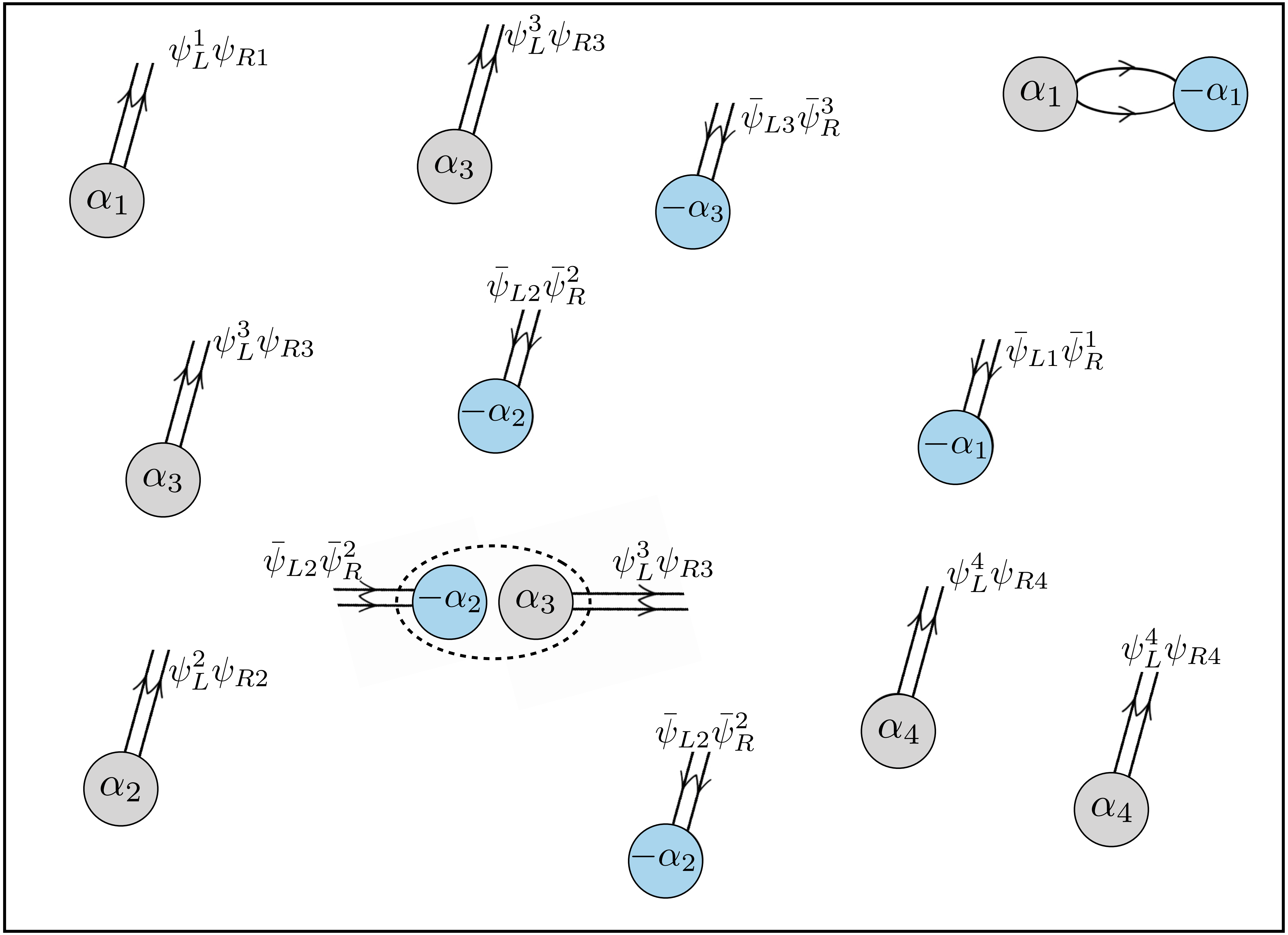
Zero modes of monopole operators:

$$\mathcal{M}_i = \begin{cases} e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi^i_L)(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi = 0 \\ e^{-S_i} f_\lambda\, e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi^i_L), & m_\lambda > 0,\ m_\psi = 0 \\ e^{-S_i} f_\psi e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi > 0 \\ e^{-S_i} f_{\lambda\psi} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}, & m_\lambda > 0,\ m_\psi > 0 \end{cases}$$



Zero modes are dictated by Nye-Singer index thm.
We will focus on QCD(F) with massive adjoint. (b)

Euclidean vacuum: Dilute gas of monopole-instantons



Can gauge fluctuations acquire chiral charge?

In the absence of monopoles, the dual photon has a topological shift symmetry which protests its gaplessness.

$$[U(1)_J]^{N_c-1} : \sigma \to \sigma + \varepsilon, \qquad \mathcal{J}_\mu = \partial_\mu \sigma$$

In Polyakov model and deformed YM, this symmetry is broken by monop
operators explicitly and this is how gauge fluctuations acquire a mass gap.

Polyakov 77,
Yaffe-MU 2008

The story in the presence of fermion zero modes is very interesting, and the existence of monopoles is by no means synonymous with mass gap.

Since the chiral symmetry is a genuine (non-anomalous) microscopic symmetry of the theory, topological configurations must respect it. But inspecting the monopole operator, we find ourselves in a puzzle.

$$\mathcal{M}_i = e^{-S_i}\; e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i)$$

Monopole amplitude indeed violate topological shift symmetry, but it also violates the chiral symmetry. This is impossible.
Let us inspect more carefully.

Magnetic charge violation in some background of monopoles:

$$\begin{aligned}
\Delta \mathbf{Q}_m &= \mathbf{Q}_m(t=\infty) - \mathbf{Q}_m(t=-\infty) = \int d^2x F_{12}\Big|_{t=-\infty}^{t=+\infty} \\
&= \int_{S^2_\infty} F_{12} \\
&= \frac{4\pi}{g}\sum_{i=1}^{N_c} n_i \alpha_i \\
&= \frac{4\pi}{g}\left(n_1 - n_{N_c}, n_2 - n_1, n_3 - n_2, \ldots, n_{N_c-1} - n_{N_c}\right)
\end{aligned}$$

Chiral charge violation in the same background of monopoles:

$$\begin{aligned}
\Delta \mathbf{Q}^5 &= \mathbf{Q}^5(t=\infty) - \mathbf{Q}^5(t=-\infty) \\
&= \sum_{A=1}^{N_f} n_A \alpha_A \\
&= 2\left(n_1 - n_{N_f}, n_2 - n_1, n_3 - n_2, \ldots, n_{N_f-1} - n_{N_f}\right)
\end{aligned}$$

$$\tilde{\mathbf{Q}} = \frac{g}{4\pi}\mathbf{Q}_m - \mathbf{Q}^5, \qquad \text{such that} \qquad \Delta\tilde{\mathbf{Q}} = 0$$

The non-invariance of the monopole-operator under chiral symmetry is absorbed by the topological shift symmetry.

$$\mathcal{M}_i = e^{-S_i}\ e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i)$$

$$[U(1)^{N_c-1}]_{AJ} = \mathrm{Diag}\left([U(1)^{N_c-1}]_A \times [U(1)^{N_c-1}]_J\right)$$

Gauge fluctuations acquire a chiral charge!
Quite surprising, but this is what semi-classics tells us.

Similar phenomena on R3 by Affleck, Harvey and Witten, 82

Chiral symmetry breaks by the choice of a point on the sigma-field manifold.

$$\langle \mathrm{VAC}|e^{-\alpha_i\cdot z}|\mathrm{VAC}\rangle = e^{-\frac{4\pi}{g^2}(v_{i+1}-v_i)}\langle e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi+i\alpha_i\cdot\sigma}\rangle = e^{-S_0}e^{i\delta_i},$$

$$\mathrm{Diag}[e^{i\delta_1},\ldots,e^{i\delta_{N_f}}]\in \mathbf{T}^{N_f-1}$$

Chiral field $\Sigma(x) = \begin{bmatrix} e^{i\alpha_1\cdot\sigma} & 0 & & \\ 0 & e^{i\alpha_2\cdot\sigma} & & \\ & & \ddots & \\ & & & e^{i\alpha_{N_f}\cdot\sigma} \end{bmatrix}$

Chiral Lagrangian: $S = \int_{\mathbb{R}^3\times S^1} \frac{f_\pi^2}{4}\,\mathrm{tr}\,|\partial_\mu\Sigma|^2$

Mixed Anomaly in QCD(F)

There is a mixed anomaly between QCD(F) and QCD(adj/F) between

$$SU(N_f)_V/\mathbb{Z}_N \qquad\qquad (\mathbb{Z}_{2N_F})_A$$

It is important that there is a quotient by center of gauge group in the faithful symmetry of the theory. Turning on SU(Nf) background is not enough. You have to gauge the center. On R4, we have

$$\begin{aligned}
\mathcal{Z}(h(A,B)) &= \exp\left[-i\frac{(4N_f N_c)}{2\mathrm{gcd}(N_f,N_c)}\frac{1}{4\pi}\int B\wedge B\right]\mathcal{Z}((A,B)) \\
&= \exp\left[-i\frac{2\mathrm{lcm}(N_f,N_c)}{4\pi}\int B\wedge B\right]\mathcal{Z}((A,B)) \\
&= \exp\left[-i2\pi\frac{2\mathrm{lcm}(N_f,N_c)}{(\mathrm{gcd}(N_f,N_c))^2}\right]\mathcal{Z}((A,B))
\end{aligned}$$

Mixed Anomaly with compactification is subtle!

Does anomaly survive compactification? Does the compactly theory achieves the same mixed anomalies?

For original 't Hooft, 0-form global symmetries, no!

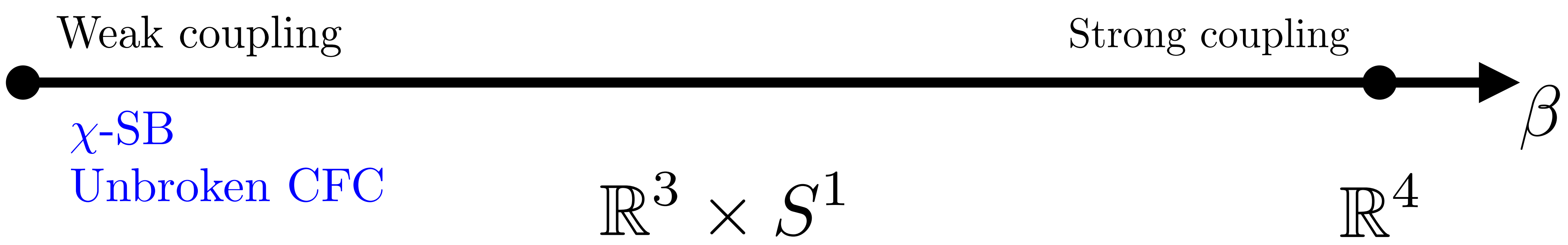
For symmetries involving 1-form symmetries in some way, the answer is yes. Seiberg, Kapustin, Komargodski, Gaiotto 2017

In our case, where there is no center to begin with, the answer is subtle. But can be made to work with an iff condition, our TBC!

Misumi, Sakai, Tanizaki 2017
Cherman, MU 2017

Mixed Anomaly and persistent order

Adiabatic continuity?



Persistent mixed anomaly:

$$\mathcal{Z}_{\Omega_F^0}(h(A_K, B^{(2)}, B^{(1)})) = \exp\left[-i\frac{2\mathrm{lcm}(N_f, N_c)}{2\pi}\int B^{(2)} \wedge B^{(1)}\right] \mathcal{Z}_{\Omega_F^0}(A_K, B^{(2)}, B^{(1)})$$

With thermal b.c., mixed anomaly does not survive. With the TBC as above, it does. This means, even without (A, B, B) background fields above, the graded partition function satisfy persistent order. Impossible to have a trivial phase.

Does this mean we can solve the QCD(F) at strong coupling? Far away, so close!

We solved QCD(F) + 1 heavy adjoint fermion at weak coupling on small $R_3 \times S_1$. There is persistent order, mixed anomaly. Does this imply the solution (a ground state continuously connected to what we found) at strong coupling at large $R_3 \times S_1$ or R_4?

The answer is no. The constraint allows three types of possibilities. So anomaly permitted phase transitions are allowed!

In other words, the solution at strong coupling is far away, so close!