Dualities and inhomogeneous phases in dense quark matter with chiral and isospin imbalances in the framework of effective models

Tamaz Khunjua [1] my colleagues:
Konstantin Klimenko [2]
Roman Zhokhov [3]
Dietmar Ebert (HU Berlin)
Vladimir Zhukovsky (MSU)

[1] University of Georgia, Georgia, Tbilisi

[2] Institute for High Energy Physics, NRC "Kurchatov Institute", Protvino, Moscow Region, Russia

[3] Russian Academy of Sciences, Institute of Terrestrial Magnetism, Ionosphere, and Radio Wave Propagation, 142092 Troitsk, Moscow Region, Russia

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Discrete symmetry in (1+1)-dim NJL model

Discrete symmetry in (1+1)-dim NJL model

A district duality property between condensates is not very new concept. But, before recently, such a duality correspondence was a well-known feature of only some (1+1)-dimensional 4F theories:

- In 1977 Ojima and Fukuda mentioned that as a result of Pauli–Gürsey symmetry the chiral phase in (1+1)–dimensional 4F model could be interpreted as a difermion superconducting phase. [Prog. Theor. Phys. **57**, 1720 (1977)]
- In 2003 Thies has shown that in addition to the duality between condensates there is also duality between fermion number-μ and chiral charge-μ_5 chemical potentials. [Phys. Rev. D **68**, 047703 (2003)]

Pauli–Gürsey transformation in (1+1)-dim

$$PG: \qquad \psi_k(x) \longrightarrow \tfrac{1}{2}(1-\gamma^5)\psi_k(x) + \tfrac{1}{2}(1+\gamma^5)C\bar{\psi}_k^T(x)$$

$$\mathcal{L} = -\frac{g^2}{2}(\overline{\psi}\psi)^2 = +\frac{g^2}{4}\overline{\psi}\psi^C\overline{\psi^C}\psi$$

Lagrangian of the model

In 2014 [1] we investigated the model that includes both condensates and chemical potentials with the following Lagrangian:

$$\mathcal{L} = \bar{\psi}_k \left[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5 \gamma^0 \gamma^5 \right] \psi_k + \frac{G_1}{N}(4F)_{ch} + \frac{G_2}{N}(4F)_{sc}, \quad \text{where}$$

$$(4F)_{ch} = \left(\bar{\psi}_k \psi_k\right)^2 + \left(\bar{\psi}_k i\gamma^5 \psi_k\right)^2, \quad (4F)_{sc} = \left(\psi_k^T C \psi_k\right)\left(\bar{\psi}_j C \bar{\psi}_j^T\right).$$

Definitions

- ψ_k $(k = 1, ..., N)$ – fundamental multiplet of the $O(N)$
- ψ_k – four-component (reducible) Dirac spinor
- γ^ν $(\nu = 0, 1, 2)$ and γ^5 – gamma-matrices
- $C \equiv \gamma^2$ – charge conjugation matrix

D. Ebert, T. G. Khunjua, K. G. Klimenko, V. Ch. Zhukovsky
Phys. Rev., D90:045021, 2014.

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Notations

- μ – fermion number chemical potential
- μ_5 – chiral (axial) chemical potential
- G_1, G_2 – coupling constants

D. Ebert, T. G. Khunjua, K. G. Klimenko, V. Ch. Zhukovsky
Phys. Rev., D90:045021, 2014.

Semi-bosonized version of the Lagrangian

Let us introduce the semi-bosonized version of the Lagrangian that contains only quadratic powers of fermionic fields as well as auxiliary bosonic fields $\sigma(x)$, $\pi(x)$, $\Delta(x)$ and $\Delta^*(x)$:

$$\widetilde{\mathcal{L}} = \bar{\psi}_k \Big[\gamma^\nu i\partial_\nu + \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \sigma - i\gamma^5\pi\Big]\psi_k -$$
$$- \frac{N(\sigma^2+\pi^2)}{4G_1} - \frac{N\Delta^*\Delta}{4G_2} - \frac{\Delta^*}{2}[\psi_k^T C\psi_k] - \frac{\Delta}{2}[\bar{\psi}_k C\bar{\psi}_k^T], \quad \text{where}$$

Bosonic fields

$$\sigma = -2\frac{G_1}{N}(\bar{\psi}_k\psi_k), \quad \pi = -2\frac{G_1}{N}(\bar{\psi}_k i\gamma^5\psi_k);$$
$$\Delta = -2\frac{G_2}{N}(\psi_k^T C\psi_k), \quad \Delta^* = -2\frac{G_2}{N}(\bar{\psi}_k C\bar{\psi}_k^T);$$

- σ and π – are real fields
- Δ and Δ^* – are Hermitian conjugated complex fields

Duality correspondence and Pauli–Gürsey transformation

Before studying the thermodynamics of the model, we want first of all to consider its duality property.

Pauli–Gürsey transformation of the fields

$$PG: \quad \psi_k(x) \longrightarrow \tfrac{1}{2}(1-\gamma^5)\psi_k(x) + \tfrac{1}{2}(1+\gamma^5)C\bar{\psi}_k^T(x)$$

$$\sigma(x) \rightleftarrows \tfrac{\Delta(x)+\Delta^*(x)}{2}; \quad \pi(x) \rightleftarrows \tfrac{\Delta(x)-\Delta^*(x)}{2i}$$

Taking into account that all spinor fields anticommute with each other, it is easy to see that under the action of the transformations each element (auxiliary Lagrangian) $\mathcal{L}(G_1, G_2; \mu, \mu_5)$ is transformed into another element according to the following rule:

$$\mathcal{L}(G_1, G_2; \mu, \mu_5) \longrightarrow \mathcal{L}(G_2, G_1; -\mu_5, -\mu)$$

Owing to the relation there is a connection between properties of the model when free model parameters G_1, G_2, μ and μ_5 vary in different regions. Due to this reason, we call the relation the duality property of the model.

Effective action

The effective action $\mathcal{S}_{\rm eff}(\sigma,\pi,\Delta,\Delta^*)$ of the considered model is expressed by means of the path integral over fermion fields:

$$\exp(i\mathcal{S}_{\rm eff}(\sigma,\pi,\Delta,\Delta^*)) = \int \prod_{l=1}^{N} [d\bar{\psi}_l][d\psi_l] \exp\Big(i\int \widetilde{\mathcal{L}}\, d^3x\Big),$$

where

$$\mathcal{S}_{\rm eff}(\sigma,\pi,\Delta,\Delta^*) = -\int d^3x \left[\frac{N}{4G_1}(\sigma^2+\pi^2) + \frac{N}{4G_2}\Delta\Delta^*\right] + \widetilde{\mathcal{S}}_{\rm eff}, \quad \text{and}$$

$$e^{(i\widetilde{\mathcal{S}}_{\rm eff})} = \int [d\bar{\psi}_l][d\psi_l] e^{\left\{i\int\left[\bar{\psi}(\gamma^\nu i\partial_\nu+\mu\gamma^0+\mu_5\gamma^0\gamma^5-\sigma-i\gamma^5\pi)\psi-\frac{\Delta^*}{2}(\psi^T C\psi)-\frac{\Delta}{2}(\bar{\psi}C\bar{\psi}^T)\right]d^2x\right\}}$$

Henceforth we omit the index k from quark fields.

Effective action

The effective action $\mathcal{S}_{\text{eff}}(\sigma, \pi, \Delta, \Delta^*)$ of the considered model is expressed by means of the path integral over fermion fields:

$$\exp(i\mathcal{S}_{\text{eff}}(\sigma, \pi, \Delta, \Delta^*)) = \int \prod_{l=1}^{N} [d\bar{\psi}_l][d\psi_l] \exp\Big(i \int \widetilde{\mathcal{L}}\, d^3x\Big),$$

The ground state expectation values $\langle\sigma\rangle$, $\langle\Delta\rangle$, etc. of the composite bosonic fields are determined by the saddle point equations:

$$\frac{\delta\mathcal{S}_{\text{eff}}}{\delta\sigma} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\pi} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\Delta} = 0, \quad \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\Delta^*} = 0.$$

Inhomogeneous anzatz (CDW)

In vacuum, i.e. in the state corresponding to an empty space with zero particle density and zero values of the chemical potentials μ and μ_5, the above mentioned quantities $\langle\sigma(x)\rangle$, etc. do not depend on space coordinates. However, in a dense medium, when $\mu \neq 0$ and/or $\mu_5 \neq 0$, the ground state expectation values of bosonic fields might have a nontrivial dependence on the spatial coordinate x. In particular we will use the following ansatz:

$$\langle\sigma(x)\rangle = M\cos(2bx), \quad \langle\pi(x)\rangle = M\sin(2bx),$$
$$\langle\Delta(x)\rangle = \Delta\exp(2ib'x), \quad \langle\Delta^*(x)\rangle = \Delta\exp(-2ib'x),$$

where M, b, b' and Δ are real constant quantities. In fact, they are coordinates of the global minimum point of the thermodynamic potential (TDP) $\Omega(M, b, b', \Delta)$.

Thermodynamical potential (TDP)

In the leading order of the large N-expansion it is defined by the following expression:

$$\int d^2x\Omega(M,b,b',\Delta) = -\frac{1}{N}\mathcal{S}_{\text{eff}}\{\sigma(x),\pi(x),\Delta(x),\Delta^*(x)\}\Big|_{\sigma(x)=\langle\sigma(x)\rangle,\pi(x)=\langle\pi_a(x)\rangle,\ldots}$$

which gives

$$\int d^2x\Omega(M,b,b',\Delta) \;=\; \int d^2x\left(\frac{M^2}{4G_1}+\frac{\Delta^2}{4G_2}\right)$$
$$+\frac{i}{N}\ln\left(\int\prod_{l=1}^{N}[d\bar{\psi}_l][d\psi_l]\exp\left(i\int d^2x\left[\bar{\psi}_k\mathcal{D}\psi_k\;\left(-\frac{\Delta e^{-2ib'x}}{2}(\psi_k^T\epsilon\psi_k)-\frac{\Delta e^{2ib'x}}{2}(\bar{\psi}_k\epsilon\bar{\psi}_k^T)\right]\right)\right)$$

where $\mathcal{D}=\gamma^\rho i\partial_\rho+\mu\gamma^0+\mu_5\gamma^1-Me^{2i\gamma^5bx}$.

Thermodynamical potential (TDP)

In order to simplify the problem of spatial dependance we perform Weinberg (or chiral) transformation of spinor fields, $q_k = \exp[i(\gamma^5 b - b')x]\psi_k$ and $\bar{q}_k = \bar{\psi}_k \exp[i(\gamma^5 b + b')x]$. Since Weinberg transformation of fermion fields does not change the path integral measure in, we see that the system is reduced to a uniform form, i.e. we obtain the following expression for the thermodynamic potential:

$$\int d^2x \Omega(M, b, b', \Delta) = \int d^2x \left(\frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} \right)$$
$$+ \frac{i}{N} \ln \left(\int \prod_{l=1}^{N} [d\bar{q}_l][dq_l] \exp \left(i \int d^2x \left[\bar{q}_k D q_k - \frac{\Delta}{2}(q_k^T \epsilon q_k) - \frac{\Delta}{2}(\bar{q}_k \epsilon \bar{q}_k^T) \right] \right) \right),$$

where

$$D = \gamma^\nu i\partial_\nu + (\mu - b)\gamma^0 - M + \gamma^1(\mu_5 - b').$$

Thermodynamical potential (TDP)

After path integration over the fermionic fields, the TDP has the following expression:

$$\Omega(M,b,b',\Delta) \equiv \Omega^{un}(M,b,b',\Delta) \quad = \quad \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} + \frac{i}{2}\int \frac{d^2p}{(2\pi)^2} \ln\Big[\lambda_1(p)\lambda_2(p)\Big],$$

where $\lambda_{1,2}(p)$:

$$\begin{aligned} \lambda_{1,2}(p) \quad &= \quad p_0^2 - \tilde{\mu}^2 - p_1^2 + \tilde{\mu}_5^2 + M^2 - \Delta^2 \\ &\pm \quad 2\sqrt{M^2p_0^2 - M^2p_1^2 + \tilde{\mu}^2p_1^2 - 2p_0\tilde{\mu}_5\tilde{\mu}p_1 + p_0^2\tilde{\mu}_5^2}, \end{aligned}$$

where $\tilde{\mu} = \mu - b$ and $\tilde{\mu}_5 = \mu_5 - b'$.

Vacuum case ($\mu = \mu_5 = 0$)

We have in vacuum case the following expression for the renormalized effective potential:

$$V_0^{reg}(M,\Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \int_0^\Lambda \frac{dp_1}{2\pi}\left(\sqrt{p_1^2 + (M+\Delta)^2} + \sqrt{p_1^2 + (M-\Delta)^2}\right).$$

Since this expression diverges at $\Lambda \to \infty$, it is necessary to renormalize it, assuming that $G_1 \equiv G_1(\Lambda)$ and $G_2 \equiv G_2(\Lambda)$ have an appropriate Λ-dependencies:

$$\frac{1}{4G_1} \equiv \frac{1}{4G_1(\Lambda)} = \frac{1}{2\pi}\ln\frac{2\Lambda}{M_1}, \quad \frac{1}{4G_2} \equiv \frac{1}{4G_2(\Lambda)} = \frac{1}{2\pi}\ln\frac{2\Lambda}{M_2},$$

where M_1 and M_2 are some finite and cutoff independent parameters with dimensionality of mass. In the limit $\Lambda \to \infty$ a finite and renormalization invariant expression:

$$4\pi V_0(M,\Delta) = M^2 \ln\frac{|M^2-\Delta^2|}{M_1^2} + \Delta^2 \ln\frac{|M^2-\Delta^2|}{M_2^2} + 2M\Delta \ln\left|\frac{M+\Delta}{M-\Delta}\right| - \Delta^2 - M^2$$

General case ($\mu \neq 0; \mu_5 \neq 0$)

$$[\lambda_1(p)\lambda_2(p)] \equiv p_0^4 + \alpha p_0^2 + \beta p_0 + \gamma = (p_0^2 + r p_0 + q)(p_0^2 - r p_0 + s)$$

where r, q and s are some real valued quantities:

$$q = \frac{1}{2}\left(\alpha + R - \frac{\beta}{\sqrt{R}}\right), \quad s = \frac{1}{2}\left(\alpha + R + \frac{\beta}{\sqrt{R}}\right), \quad r = \sqrt{R},$$

and R is an arbitrary positive real solution of the equation

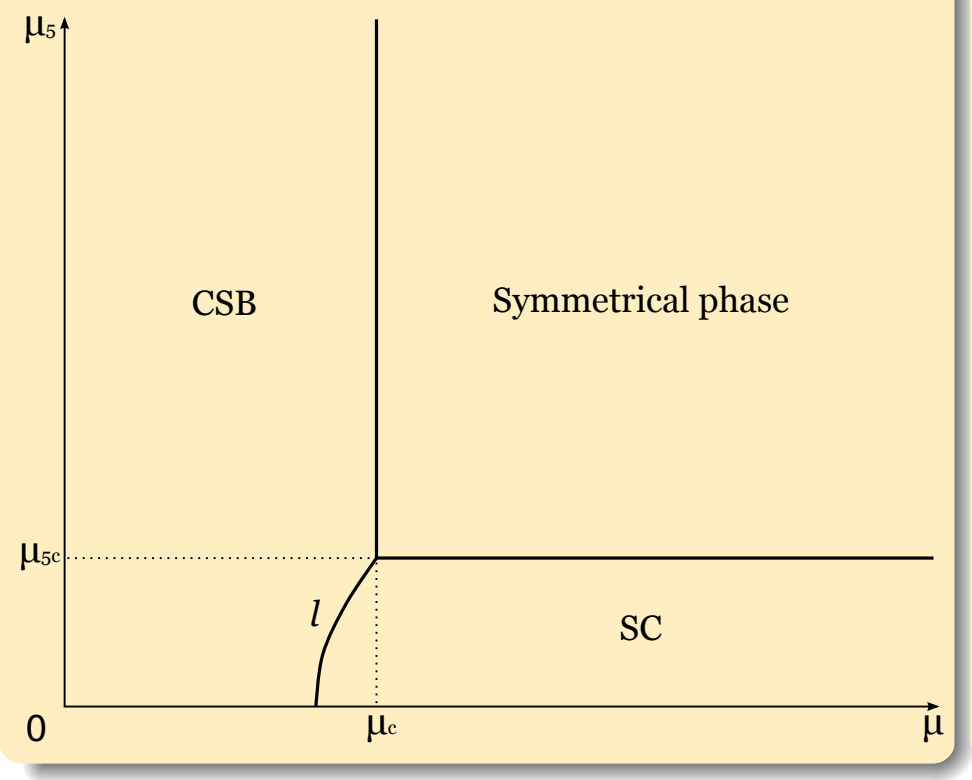
$$X^3 + AX = BX^2 + C$$

with respect to a variable X, and

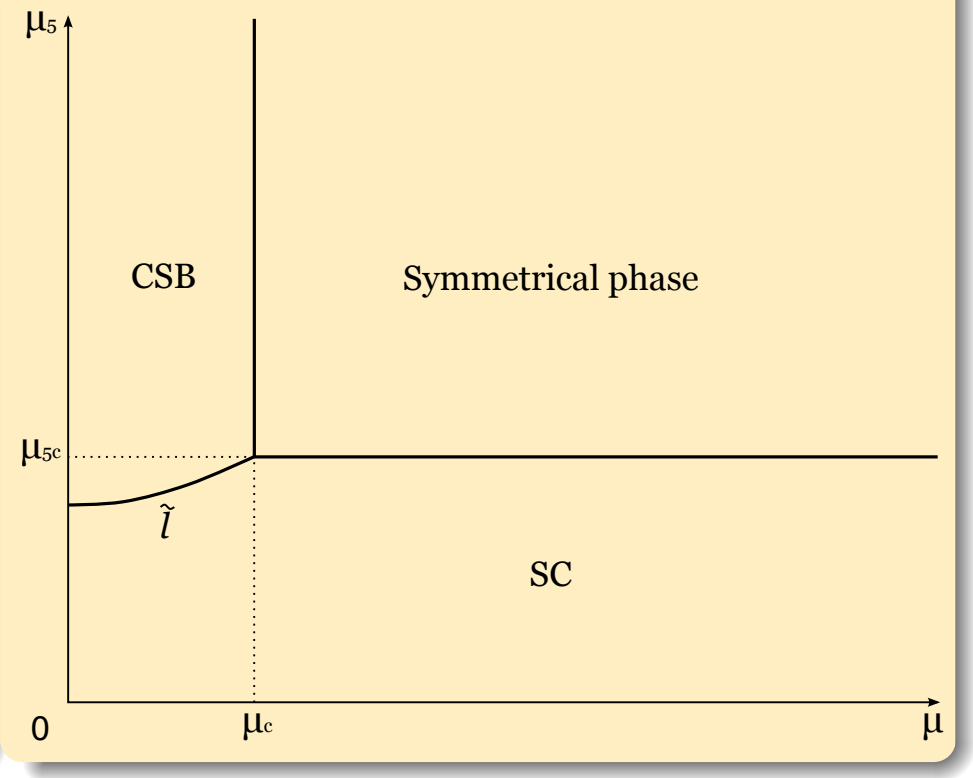
$$\begin{aligned} A &= \alpha^2 - 4\gamma = 16\Big[\mu_5^2\Delta^2 + M^2\mu^2 + \Delta^2 M^2 + \mu_5^2\mu^2 + p_1^2(\mu^2 + \mu_5^2)\Big], \\ B &= -2\alpha = 4(M^2 + \Delta^2 + \mu^2 + \mu_5^2 + p_1^2), \quad C = \beta^2 = (8\mu_5\mu p_1)^2. \end{aligned}$$

Phase diagram in the general homogeneous case

$M_1 > M_2$ $(G_1 > G_2)$

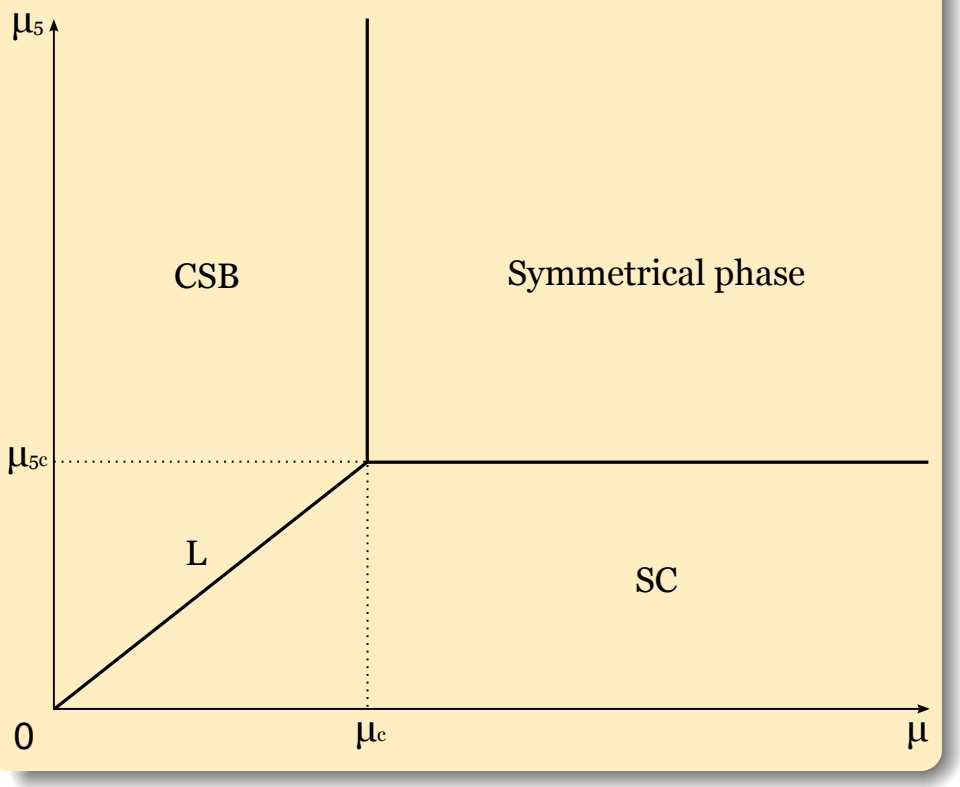


$M_1 < M_2$ $(G_1 < G_2)$



Phase diagram in the general homogeneous case

$M_1 = M_2 \ (G_1 = G_2)$

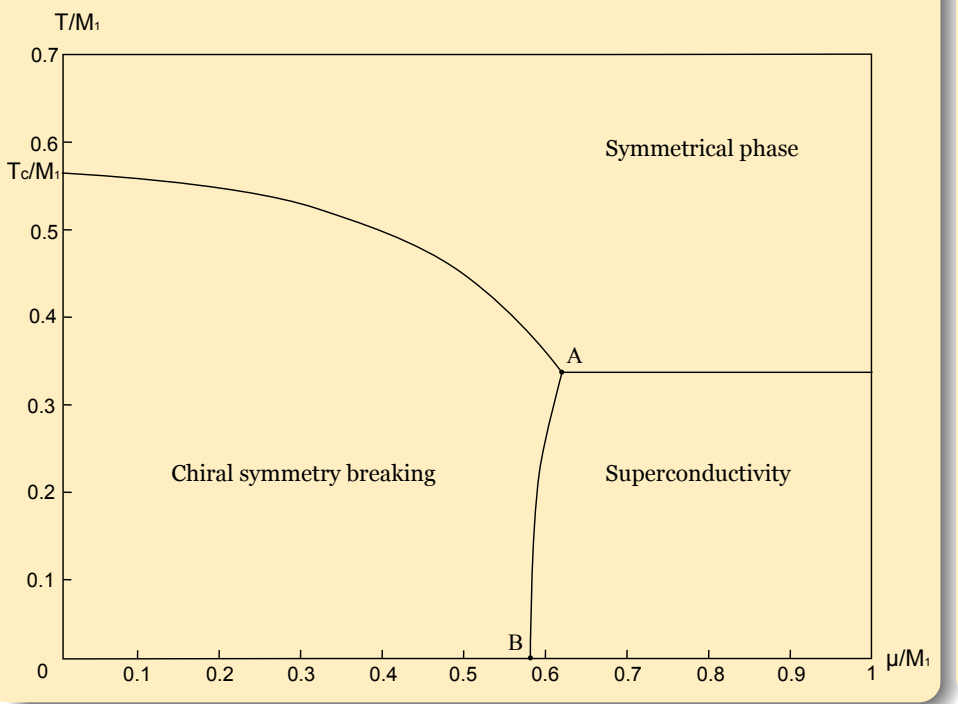


M.Thies, *Phys. Rev., D90:105017* (2014)[1].

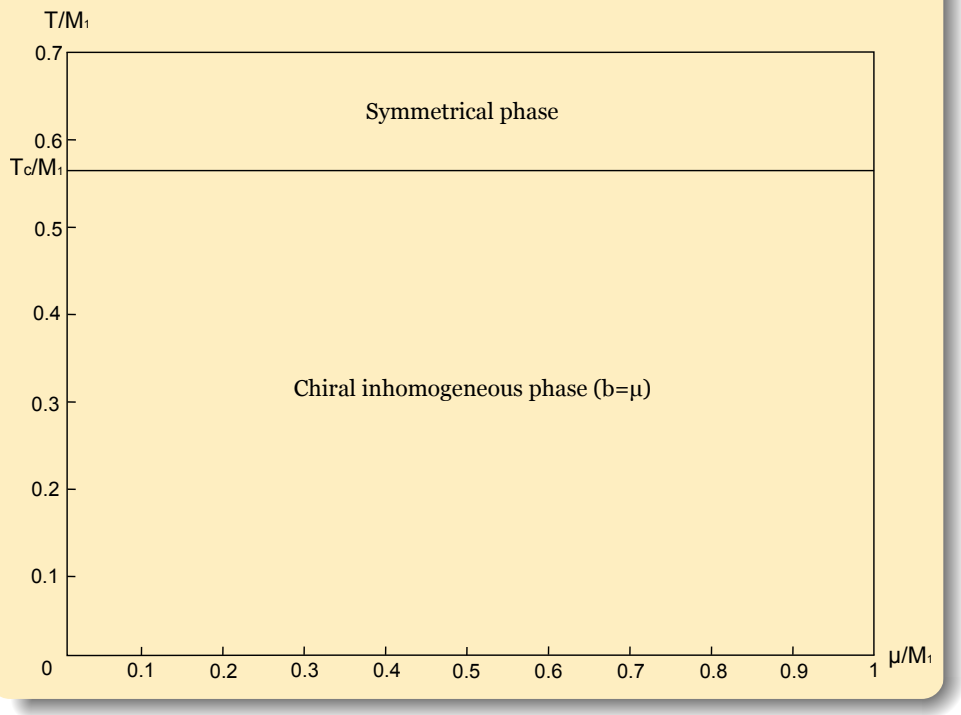
[1] Michael Thies called this case self-dual GN model (sdGN)

Phase diagram (μ, T) in inhomogeneous case $M_1 > M_2$

Homogeneous case $(b = 0)$



Inhomogeneous case $(b \neq 0)$



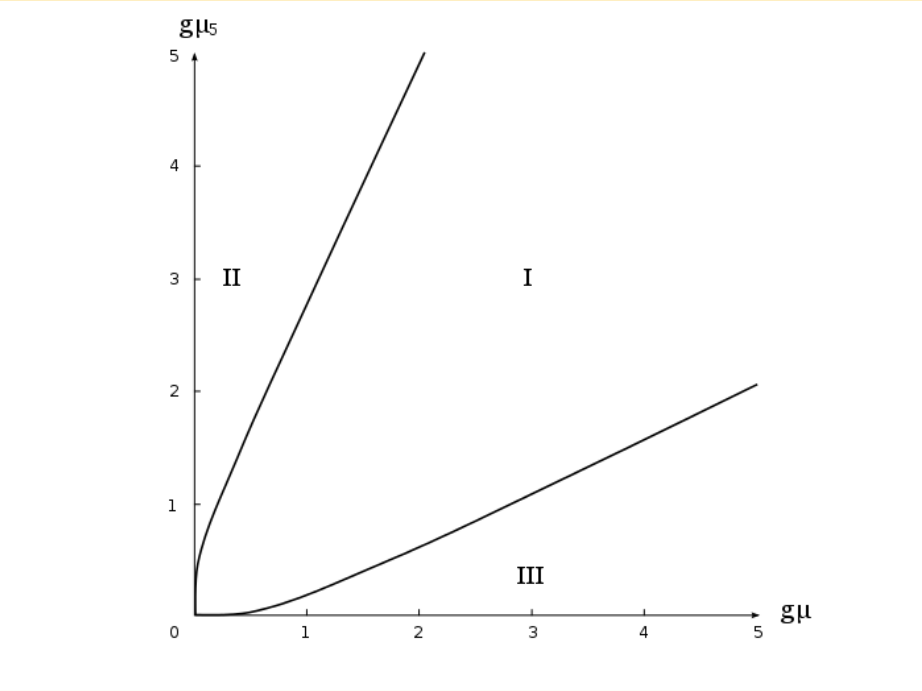
Discrete symmetry (duality) in (2+1)-dim NJL model

D. Ebert, T. G. Khunjua, K. G. Klimenko, V. Ch. Zhukovsky
Phys. Rev., D93:105022 , 2016.

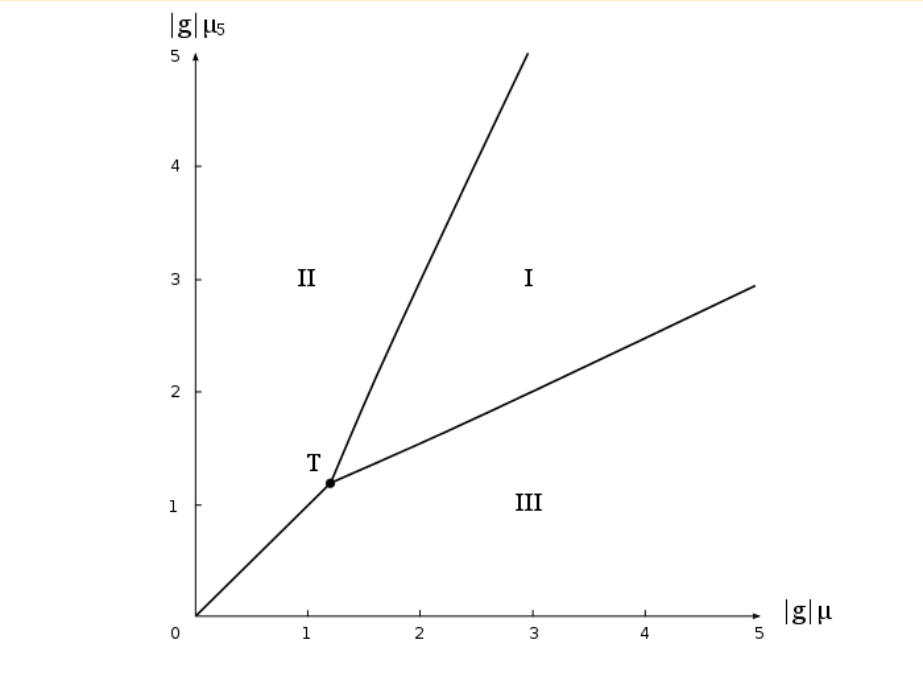
Self-dual case: $(G_1 = G_2)$

The (μ, μ_5)-phase portraits at fixed coupling constants:

$G_1 = G_2 \equiv G > 0$



$G_1 = G_2 \equiv G < 0$

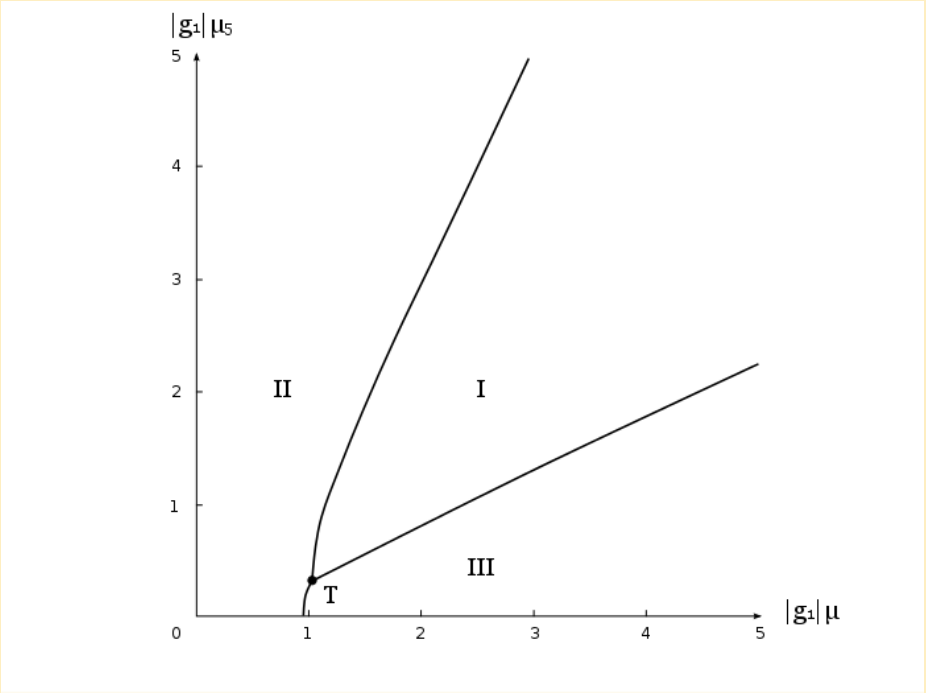


The notations I, II and III mean the symmetric, the chiral symmetry breaking (CSB) and the superconducting (SC) phases, respectively. T denotes a triple point.

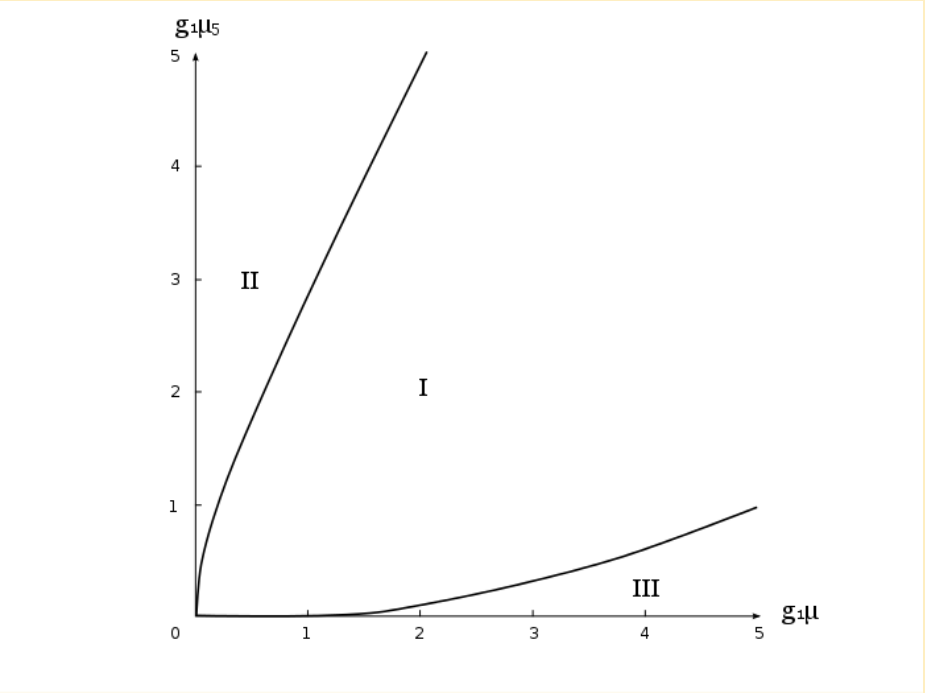
General case: $(G_1 \neq G_2)$

The (μ, μ_5)-phase portraits at fixed coupling constants:

$G_1 > 0$ and $G_2 = 0.2G_1$



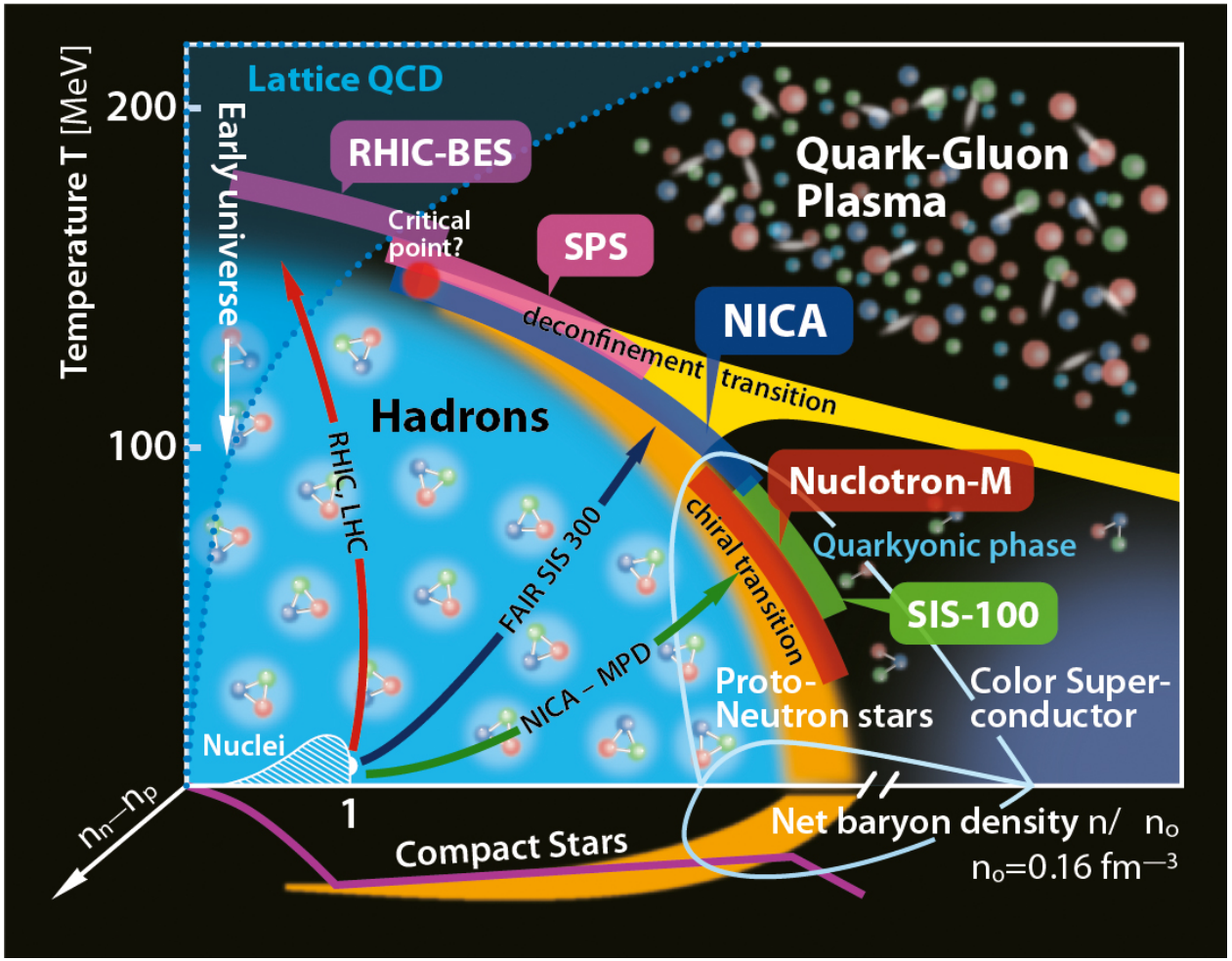
$G_1 < 0$ and $G_2 = -2G_1$



The notations I, II and III mean the symmetric, the chiral symmetry breaking (CSB) and the superconducting (SC) phases, respectively. T denotes a triple point.

Pion condensation in the framework of (3+1)-dim GN model

QCD phase diagram



Introduction

Models with four-fermion interactions

Isospin asymmetry is the well-known property of dense quark matter, which exists in the compact stars and is produced in heavy ion collisions. On the other hand, the chiral imbalance between left- and right- handed quarks is another highly anticipated phenomenon that could occur in the dense quark matter.

To investigate dense quark under these conditions we use Nambu–Jona-Lasinio (NJL) model and take into account:

- Baryon – μ_B chemical potential to investigate non-zero density
- Isospin – μ_I chemical potential to investigate non-zero isotopic imbalance
- Chiral isospin – μ_{I5} chemical potential to investigate chiral isotopic imbalance
- Non-zero bare quark mass ($m_0 \neq 0$) to promote real threshold to pion condensation phase
- Non-zero temperature ($T \neq 0$) in order to make our investigation applicable to hot dense quark matter and compare our NJL-model analysis with the known lattice results

Lagrangian of the model

$$\mathcal{L} = \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu - m_0 + \frac{\mu_B}{3}\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 + \frac{\mu_{I5}}{2}\tau_3\gamma^0\gamma^5\Big]q + \frac{G}{N_c}\Big[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\vec{\tau}q)^2\Big]$$

Definitions

- q is the flavor doublet $q = (q_u, q_d)^T$
- q_u and q_d are four-component Dirac spinors as well as color N_c-plets[a]
- τ_k $(k = 1, 2, 3)$ are Pauli matrices
- m_0 is the diagonal matrix in flavor space with bare quark masses (from the following $m_u = m_d = m_0$)

[a] The summation over flavor, color, and spinor indices is implied

Lagrangian of the model

$$\mathcal{L} = \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu - m_0 + \frac{\mu_B}{3}\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 + \frac{\mu_{I5}}{2}\tau_3\gamma^0\gamma^5\Big]q + \frac{G}{N_c}\Big[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\vec{\tau}q)^2\Big]$$

Notations

- μ_B is a baryon number chemical potential
- μ_I is taken into account to promote non-zero imbalance between u and d quarks
- μ_{I5} is stands to promote chiral isospin imbalance between $u_{L(R)}$ and $d_{L(R)}$
- G is coupling constant

Semi-bosonized version of the Lagrangian

Let us introduce the semi-bosonized version of the Lagrangian that contains only quadratic powers of fermionic fields as well as auxiliary bosonic fields $\sigma(x)$, $\pi_a(x)$, :

$$\widetilde{\mathcal{L}} = \bar{q}\Big[\gamma^\rho \mathrm{i}\partial_\rho - m_0 + \mu\gamma^0 + \nu\tau_3\gamma^0 + \nu_5\tau_3\gamma^0\gamma^5 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\Big]q - \frac{N_c}{4G}\Big[\sigma\sigma + \pi_a\pi_a\Big]$$

Bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}\mathrm{i}\gamma^5\tau_a q)$$

The new notations of chemical potentials

$$\mu \equiv \frac{\mu_B}{3};\ \nu \equiv \frac{\mu_I}{2};\ \nu_5 \equiv \frac{\mu_{I5}}{2}$$

Note that the composite bosonic field $\pi_3(x)$ can be identified with the physical $\pi^0(x)$-meson field, whereas the physical $\pi^\pm(x)$-meson fields are the following combinations of the composite fields, $\pi^\pm(x) = (\pi_1(x) \pm i\pi_2(x))/\sqrt{2}$.

Calculation of the TDP

After all possible analytical calculations, we have the following form for the TDP:

$$\Omega(M,\Delta) = \frac{(M-m_0)^2+\Delta^2}{4G} - \sum_{i=1}^{4}\int_0^{\Lambda}\frac{p^2dp}{2\pi^2}|\eta_i| - T\sum_{i=1}^{4}\int_0^{\Lambda}\frac{p^2dp}{2\pi^2}\Big\{\ln(1+e^{-\frac{1}{T}(|\eta_i|-\mu)})+\ln(1+e^{-\frac{1}{T}(|\eta_i|+\mu)})\Big\},$$

where η_i are the roots of the following polynomial:

$$\begin{aligned}
&\left(\eta^4-2a\eta^2-b\eta+c\right)\left(\eta^4-2a\eta^2+b\eta+c\right)=0,\\
&a=M^2+\Delta^2+|\vec{p}|^2+\nu^2+\nu_5^2;\\
&b=8|\vec{p}|\nu\nu_5;\\
&c=a^2-4|\vec{p}|^2(\nu^2+\nu_5^2)-4M^2\nu^2-4\Delta^2\nu_5^2-4\nu^2\nu_5^2.
\end{aligned}$$

Fiiting parameters

Since the NJL model is a non-renormalizable theory we have to use fitting parameters for the quantitative investigation of the system. We use the following, widely used parameters:

$$m_0 = 5,5\,\mathrm{MeV}; \qquad G = 15.03\,\mathrm{GeV}^{-2}; \qquad \Lambda = 0.65\,\mathrm{GeV}.$$

In this case at $\mu = \nu = \nu_5 = 0$ one gets for constituent quark mass the value $M = 311\,\mathrm{MeV}$.

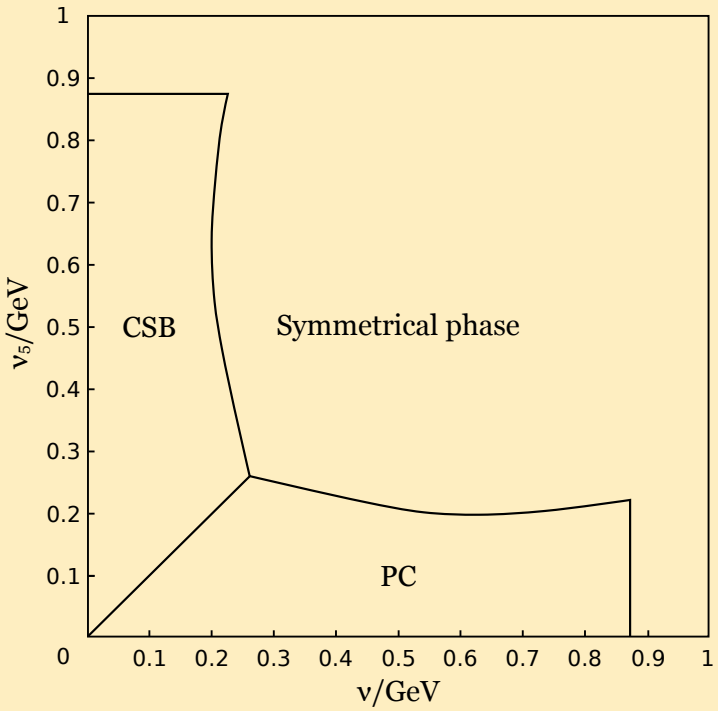
Phases

To define the ground state of the system one should find the coordinates (M_0, Δ_0) of the global minimum point (GMP) of the TDP. We also interested in the quark number density: $n_q = -\frac{\partial \Omega(M_0, \Delta_0)}{\partial \mu}$. We have found the following phases in the system:

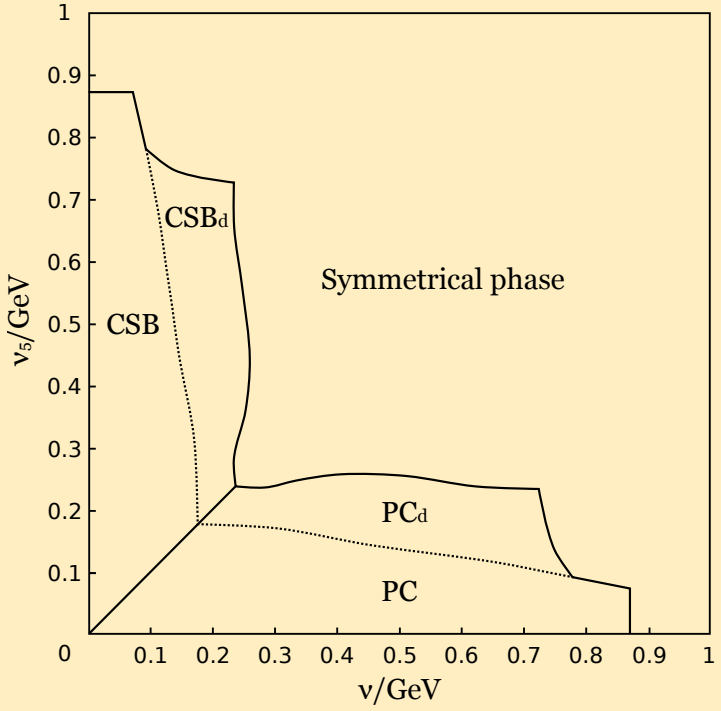
- $M = 0; \Delta = 0; n_q = 0$ – symmetrical phase (it could be realized only in chiral limit $m_0 = 0$)
- $M \neq 0; \Delta = 0; n_q = 0$ – chiral symmetry breaking phase (**CSB**)
- $M \neq 0; \Delta \neq 0; n_q = 0$ – pion condensation phase with zero quark density (**PC**) ($M = 0$ in chiral lim.)
- $M \neq 0; \Delta = 0; n_q \neq 0$ – chiral symmetry breaking phase with non-zero quark density ($\mathbf{CSB_d}$)
- $M \neq 0; \Delta \neq 0; n_q \neq 0$ – pion condensation phase with non-zero quark density ($\mathbf{PC_d}$)
- $M \approx m_0; \Delta = 0; n_q \neq 0$ – partially restored (**CSB**) phase with non-zero quark density ($\mathbf{CSB_{dr}}$)

(ν, ν_5)-phase portraits in the chiral limit

$\mu = 0$ MeV

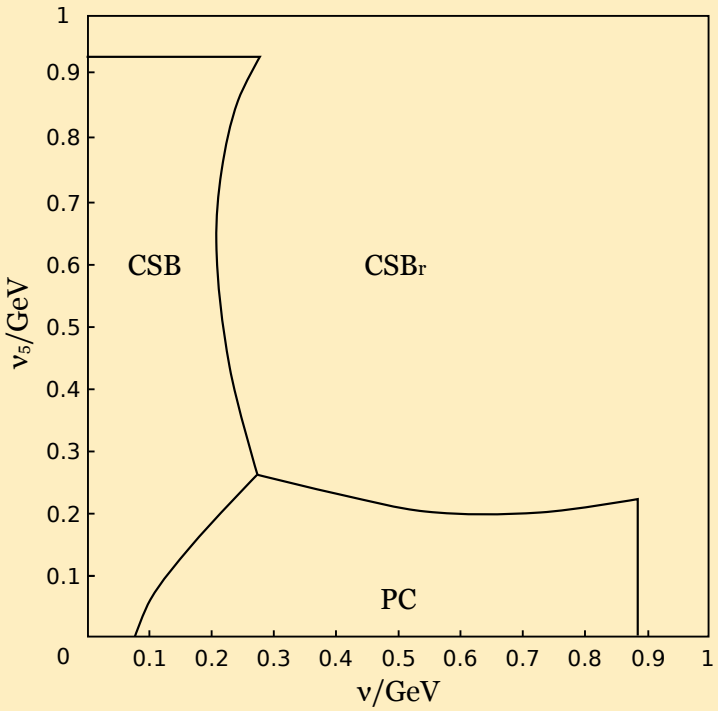


$\mu = 150$ MeV

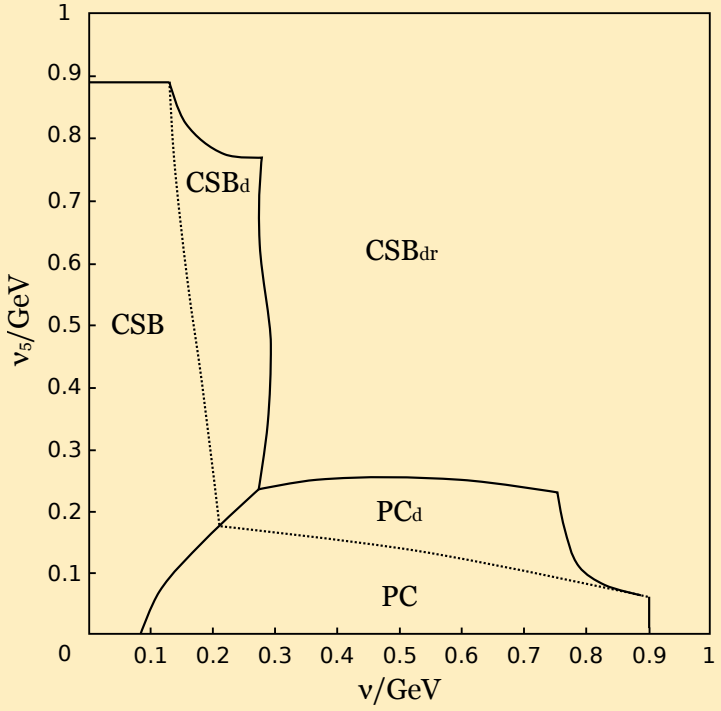


(ν, ν_5)-phase portraits in the physical point

$\mu = 0$ MeV



$\mu = 150$ MeV

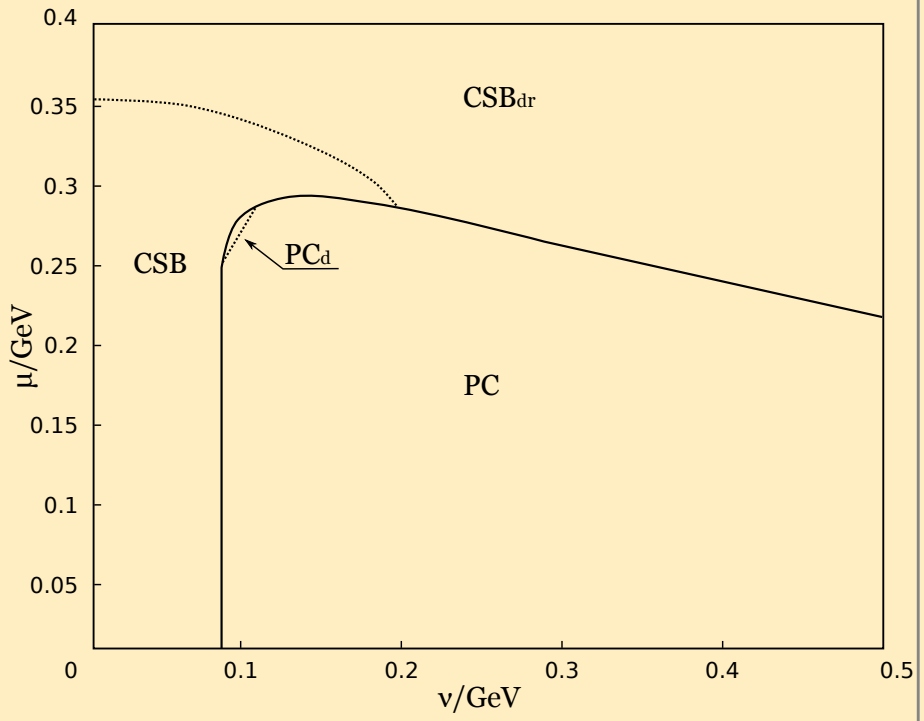


Useful applications of the model and duality-symmetry

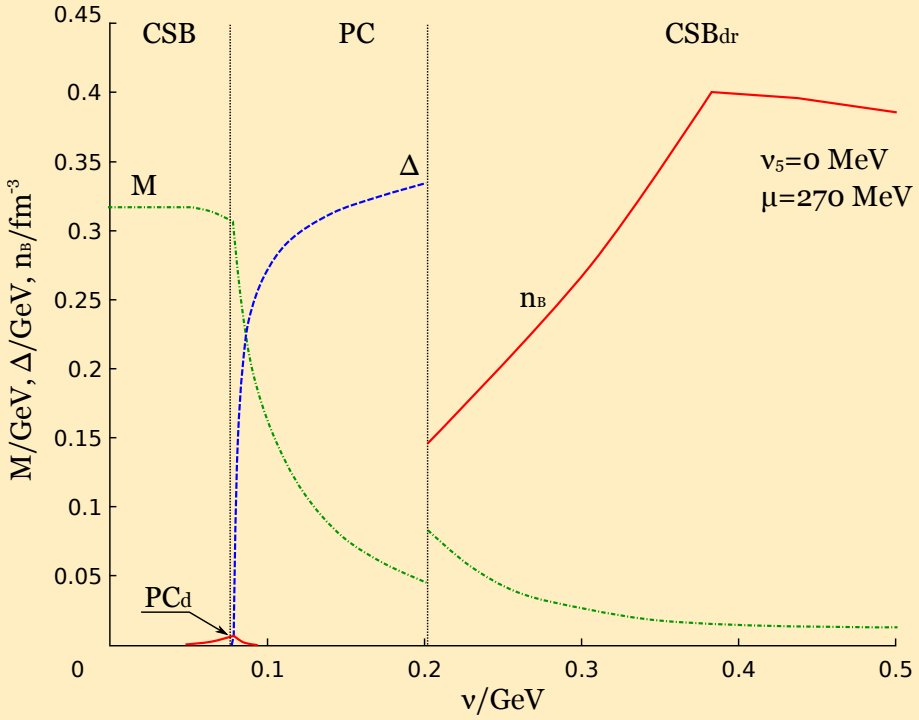
PC_d phase existence

$$\nu_5 = 0 \text{ MeV}$$

(μ, ν)-phase portrait



Slice at $\mu = 270$ MeV

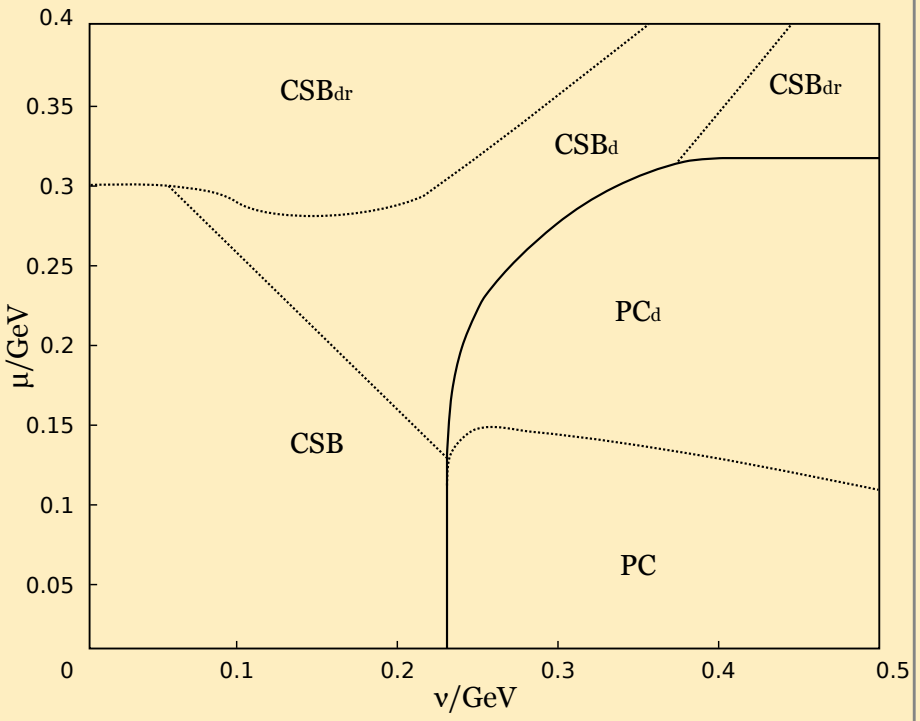


It is evident from the figures that PC_d phase exist in the very small region of the phase portrait.

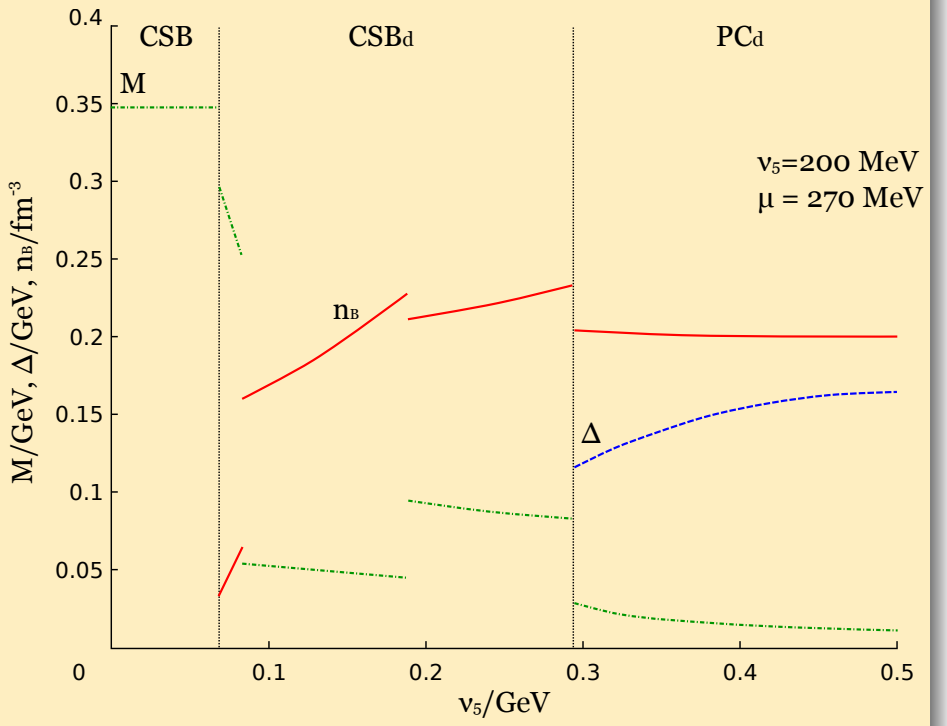
PC_d phase existence

$$\nu_5 = 200 \text{ MeV}$$

(μ, ν)-phase portrait



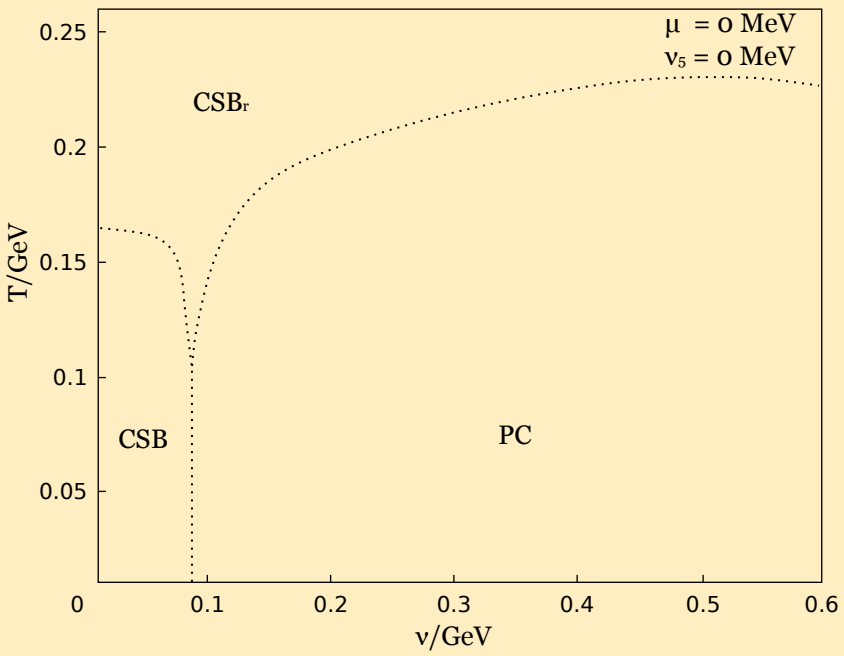
Slice at $\mu = 270$ MeV



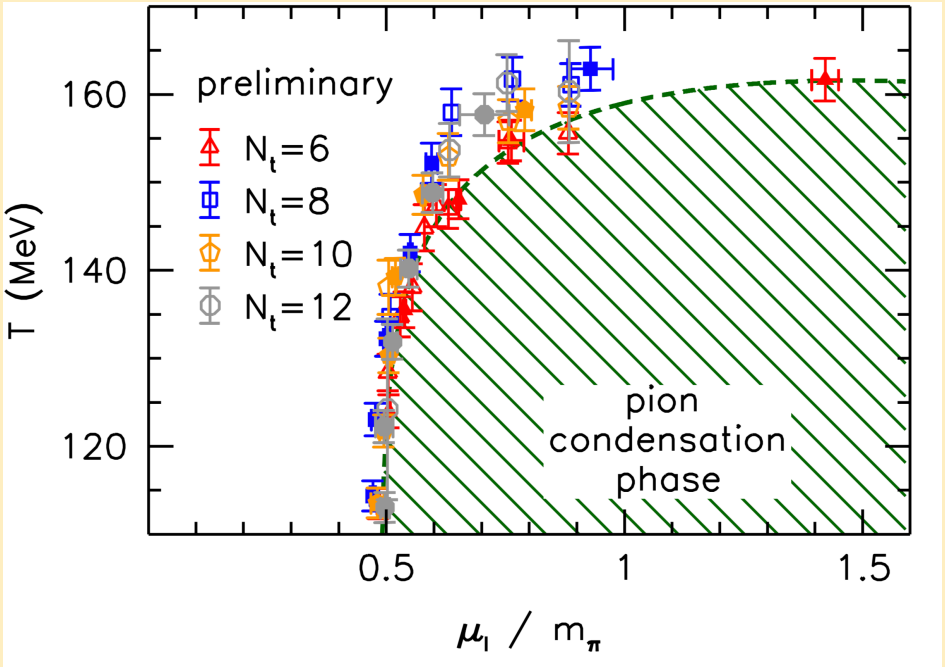
Non-zero isospin chiral potential ν_5 does promote the PC_d phase in a wide range of the parameters.

Comparison with lattice (PC condensate)

(ν, T)-phase portrait



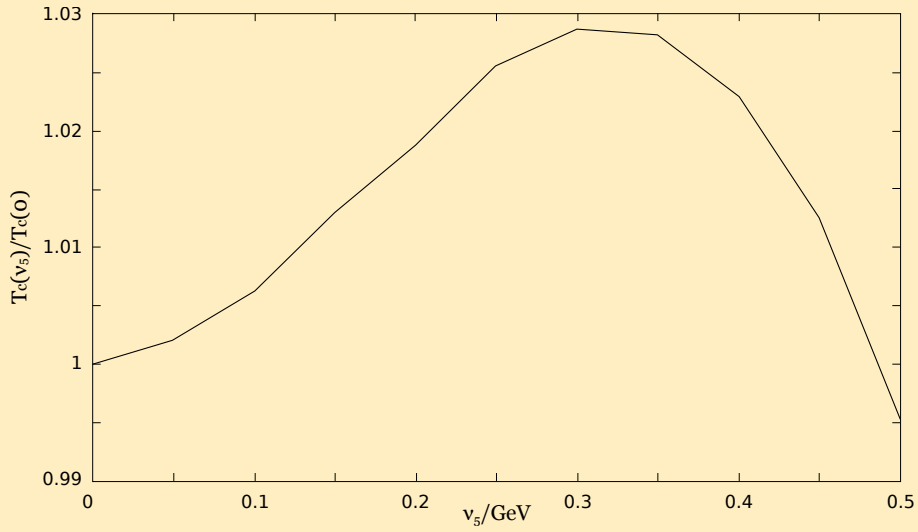
Lattice [1611.06758] Brandt et al.



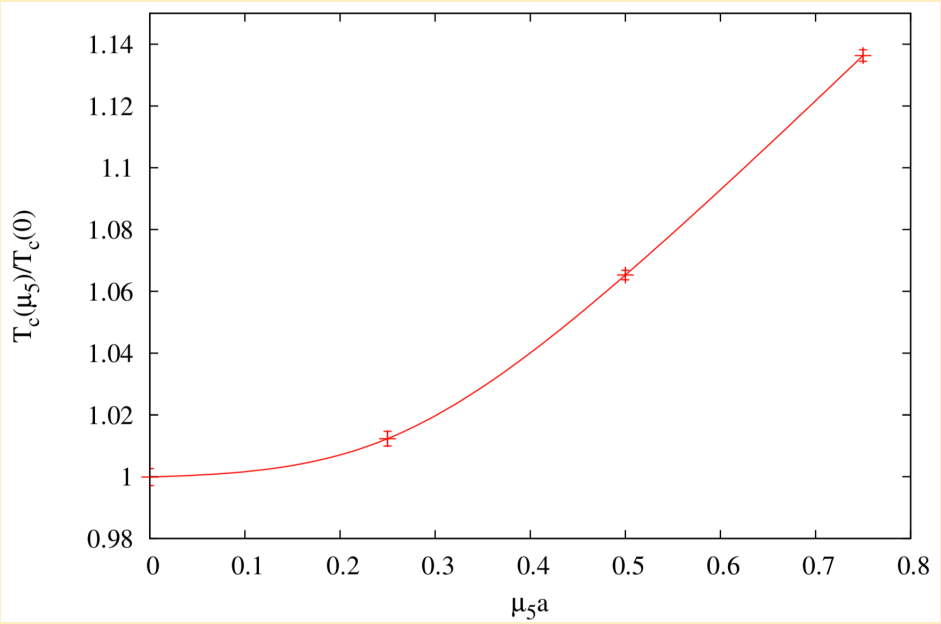
Qualitatively comparable with the first principle lattice simulation.

Comparison with lattice (chiral cataysis)

Critical temperature T_c within NJL



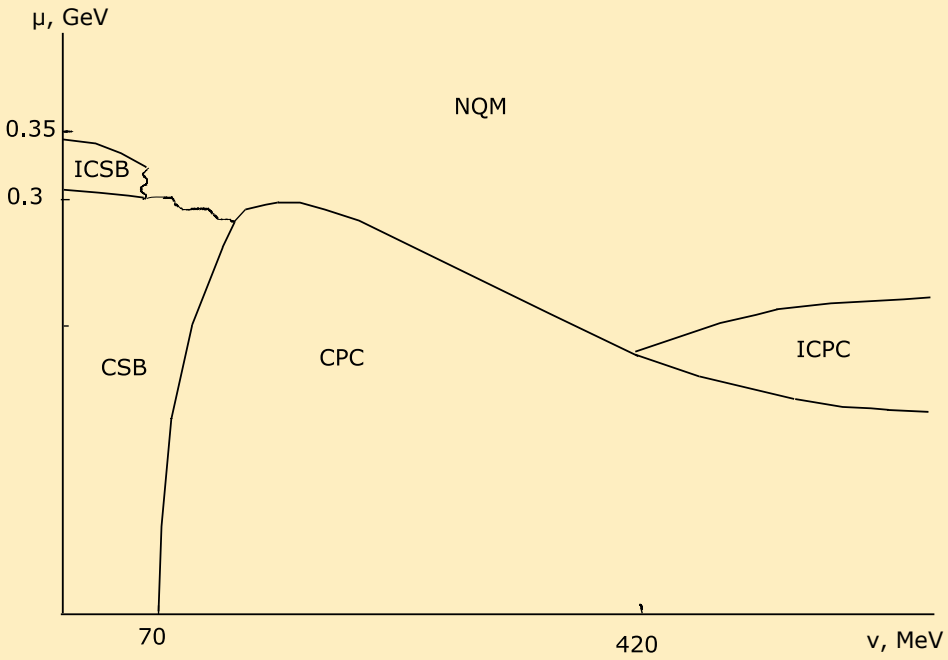
Critical temperature T_c within Lattice



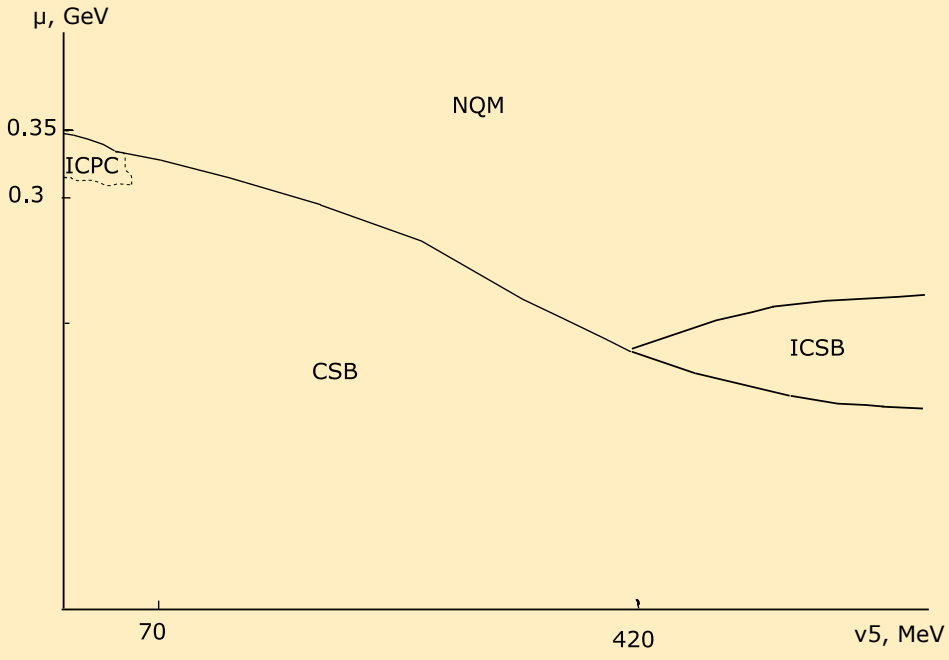
We have recently shown that introduction of the chiral chemical potential μ_5 into consideration (with the following term in the Lagrangian: $\frac{\mu_5}{2}\bar{q}\gamma^0\gamma^5 q$) leads to an additional dual-symmetry between $\mu_{I5} \longleftrightarrow \mu_5$ in the region where $\Delta = 0$. In other words, in the NJL model (1) we can certainly consider μ_{I5} as a μ_5 (only in the pure CSB phase). So we can compare our results with the known lattice calculations with μ_5 [1512.05873] (Braguta et al.).

Inhomogeneous condensates

Critical temperature T_c within NJL



Critical temperature T_c within Lattice



We have shown that there is duality even in inhomogeneous case in CDW anzatz:

$$\mathcal{D}_I : \quad M \longleftrightarrow \Delta, \quad \nu \longleftrightarrow \nu_5, \quad k \longleftrightarrow k'.$$

T. G. Khunjua, K. G. Klimenko, and R. N. Zhokhov.
JHEP, 06:006, 2019.

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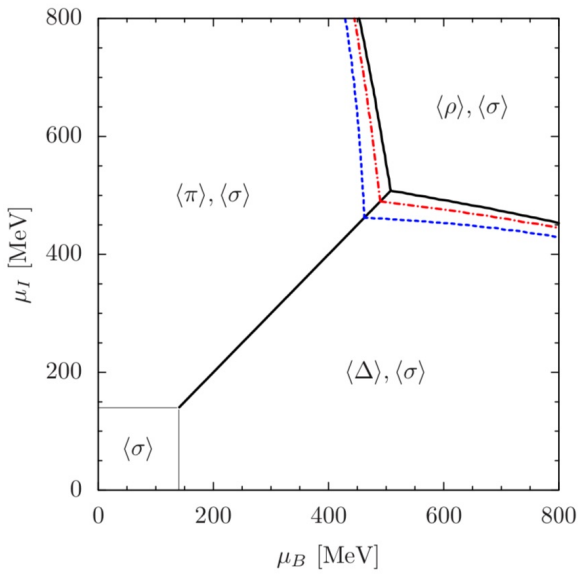
Present day interests (more dualities?)

2-color NJL model

$$L = \bar{q}\left[i\hat{\partial} - m_0 + \mathcal{M}\gamma^0\right]q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2\right] + H\left[\bar{q}i\gamma^5\sigma_2\tau_2 C\bar{q}^T\right]\left[q^T C i\gamma^5\sigma_2\tau_2 q\right]$$

$$\mathcal{M} = \frac{\mu_B}{2} + \frac{\mu_I}{2}\tau_3 + \frac{\mu_{I5}}{2}\gamma^5\tau_3 + \mu_5\gamma^5$$



J. Andersen, T. Brauner
PRD 81:096004, 2010.

Dual-symmetry in TDP

Massless QCD with $N_c = 2$ and $N_f = 2$ obeys the Pauli–Gürsey $U(2N_f) = U(4)$ symmetry. Its action is:

$$S_{QC_2D} = \int d^4x \sum_{f=1}^{N_f=2} i\bar{\psi}_f \widehat{D}\psi_f \equiv \int d^4x \; i\bar{q}\widehat{D}q,$$

where $\widehat{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - i\sigma_a A^a_\mu)$ and σ_a $(a = 1, 2, 3)$ are the generators of the color $SU_c(2)$ group in its fundamental representation. Note also that for each $f = 1, 2$ quark field ψ_f is the doublet with respect to color $SU_c(2)$ group, i.e., the spinor field $q \equiv q_{\alpha f}$ is the doublet both over the color index $\alpha = 1, 2$ and flavor index $f = 1, 2$. Taking into account that spinor fields anticommute we have:

$$\frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} i\bar{\psi}_f\gamma^\mu\partial_\mu\psi_f = -\frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} i\partial_\mu\bar{\psi}_f\gamma^\mu\psi_f$$

$$= \frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} i\psi_f^T(\gamma^\mu)^T\partial_\mu\bar{\psi}_f^T = \frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} i\overline{\psi_f^c}\gamma^\mu\partial_\mu\psi_f^c.$$

Dual-symmetry in TDP

In a similar way we have:

$$\frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} \bar{\psi}_f \gamma^\mu \sigma_a A^a_\mu \psi_f = \frac{1}{2}\int d^4x \sum_{f=1}^{N_f=2} \overline{\psi^c_f} \sigma_2 \gamma^\mu \sigma_a A^a_\mu \sigma_2 \psi^c_f.$$

So, it is possible to reduce S_{QC_2D} to the form:

$$S_{QC_2D} = \frac{i}{2}\int d^4x \left[\bar{\psi}_u \widehat{D} \psi_u + \bar{\psi}_d \widehat{D} \psi_d\right] + \frac{i}{2}\int d^4x \left[\overline{\psi^c_u} \sigma_2 \widehat{D} \sigma_2 \psi^c_u + \overline{\psi^c_d} \sigma_2 \widehat{D} \sigma_2 \psi^c_d\right].$$

Dual-symmetry in TDP

Supposing that quark fields ψ_u and ψ_d are transformed under fundamental representation of some flavor $SU(N_f = 2)$ group, whereas the fields ψ_u^c and ψ_d^c form the fundamental multiplet of another $SU(N_f = 2)$ group, we see that the action is formally invariant with respect to $SU(2) \times SU(2)$ flavour group. However, introducing the Nambu-Gorkov spinor Ψ, where

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_u \\ \psi_d \\ \sigma_2 \psi_u^c \\ \sigma_2 \psi_d^c \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} q \\ \sigma_2 q^c \end{pmatrix},$$

we see that $S_{QC_2D} = \int d^4x \; i\overline{\Psi}\widehat{D}\Psi$, i.e. this action is indeed invariant with respect to an enlarged $SU(4)$ flavor group, when Nambu-Gorkov spinor Ψ.

M. Hanada and N. Yamamoto, PoS LATTICE **2011**, 221 (2011)

M. Hanada and N. Yamamoto, JHEP **1202**, 138 (2012).

Outlook

- Duality correspondence that was established in 1977 by Ojima and Fukuda exist not only low-dimensional models, but also in (3+1)-dimensional models
- It is not only interesting mathematical property of the model, but useful tool of investigation of the phase diagram.
- Duality correspondence is inherit property not only models, but also fundamental theories and it maybe could helps to obtain model-independent results.

[a]

[a] R.N.Z. is grateful for support of the Foundation for the Advancement of Theoretical Physics and Mathematics BASIS grant and Russian Science Foundation under the grant No 19-72-00077

ν_5 generation

Let us suppose, for simplicity, that dense quark matter consists of two massless u and d quarks, whose chemical potentials, $\mu_u = \mu + \nu$ and $\mu_d = \mu - \nu$. Moreover, we suppose that quarks do not interact, and there is an external magnetic field $\vec{B} = (0, 0, B)$ directed along z axis. In this case in the equilibrium state of quark matter there is a nonzero and nondissipative axial current

$$\vec{j}_{5f} \equiv \langle \bar{q}_f \vec{\gamma} \gamma^5 q_f \rangle = \frac{Q_f \mu_f \vec{B}}{2\pi^2}$$

for each quark flavor $f = u, d$. In this formula Q_f is an electric charge of quark-flavor f, i.e. $Q_u = 2/3$, $Q_d = -1/3$. It means that axial currents of u and d quarks are opposite in their directions. Since $\vec{j}_{5f} = \langle \bar{q}_{fR} \vec{\gamma} q_{fR} \rangle - \langle \bar{q}_{fL} \vec{\gamma} q_{fL} \rangle$, where

$$q_{fR} = \frac{1 + \gamma^5}{2} q_f, \quad q_{fL} = \frac{1 - \gamma^5}{2} q_f,$$

As a result, a spatial separation of quark chiralities for each flavor f occurs. In other words, one can say that in the upper half of the three-dimensional space, i.e. at $z > 0$, the density, e.g., $n_{uR} \equiv \langle \bar{q}_{uR} \gamma^0 q_{uR} \rangle$ of the right-handed u quarks is greater than the density $n_{uL} \equiv \langle \bar{q}_{uL} \gamma^0 q_{uL} \rangle$ of the left-handed u quarks. Hence, in this case we have at $z > 0$ the positive values of the chiral charge density $n_{u5} \equiv n_{uR} - n_{uL}$ for u quarks.