A curvature bound from gravitational catalysis

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1. Motivations

- Any candidate theory of quantum gravity (QG) has to be compatible with the existence of **light fermionic particle** matter.
- Under the assumption of **chiral symmetry breaking** being triggered by QG one would expect fermions to have masses of the order of the Planck mass.
- Previous investigations on curved spacetimes (Sachs, Wipf 1994)(Elizalde, Leseduarte, Odinstov, Sil'nov 1996) pointed out a non trivial contribution to chiral phase transition in fermionic matter coming from the average curvature of the background manifold: in **gravitational catalysis** (Ebert, Tyukov, Zhukovsky 2009)(Gies, Lippoldt 2013) this contribution acts shifting the value of the RG scale k_{IR} at which the process is triggered, depending on the value of the (negative) Ricci scalar R.
- The aim of this work is to constraint the relation between these two quantities studying the chiral symmetry breaking in 4-fermions effective field theories implementing bosonization techniques.

2. Modelling the System

Chiral 4-fermion interactions have the generic form (Gies, Wetterich 2003)(Gies, Jaeckel 2006)(Eichhorn, Gies 2011):

$$S[\bar\psi,\psi]=\int d^Dx\sqrt{-g}\left\{\bar\psi\nabla\!\!\!/\,\psi-\frac{\bar\lambda}{4}\left[\left(\bar\psi^a\gamma_\mu\psi^a\right)^2-\left(\bar\psi^a\gamma_\mu\gamma_5\psi^a\right)^2\right]\right\}.$$

a and b being flavor indices.
It is possible to study a **bosonized** version of this action by implementing the Hubbard-Stratonovich trick and introducing an auxiliary field ϕ^{ab} whose equations of motion read:

$$\phi_{ab}=-2\bar\lambda\bar\psi_R^b\psi_L^a \qquad (\phi^\dagger)_{ab}=-2\bar\lambda\bar\psi_L^b\psi_R^a\,.$$

The vacuum expectation value of the bosonic field $\langle\phi\rangle=\phi_0$ will represent an **order parameter** for chiral symmetry breaking.
Integrating out the fermionic degrees of freedom one can write the effective potential as

$$U_{k_{IR}}(\phi)=U_\Lambda-\int_{k_{IR}}^{\Lambda}dk\,\partial_k U_k=\frac{N_f}{2\bar\lambda}\phi_0^2-\frac{N_f}{2}\int_{k_{IR}}^{\Lambda}dk\int_0^\infty\frac{dT}{T}\,e^{-\phi_0^2T}\partial_k f_k\mathrm{Tr}\left(e^{\nabla^2T}\right),\qquad(1)$$

where $\mathrm{Tr}\left(e^{\nabla^2T}\right)=K_T$ is the trace of the heat kernel for the Dirac covariant operator and

$$f_k=e^{-(k^2T)^p}$$

is a **regulator** function inserted to extract the flow of the potential with respect to the scale of the process.
In order to investigate a **second order** phase transition we will focus on the curvature of the potential at the origin in field space:

$$U_{k_{IR}}(\phi)\Big|_{\phi^2}=\frac{N_f}{2\bar\lambda}\phi_0^2+\frac{N_f\phi_0^2}{2}\int_{k_{IR}}^{\Lambda}dk\int_0^\infty dT\,\partial_k f_k\mathrm{Tr}\left(e^{\nabla^2T}\right),\qquad(2)$$

and check under which conditions it changes sign signaling a non trivial minimum for the effective potential.

3. Heat Kernel

The heat kernel of the covariant Dirac operator on hyperbolic D-dimensional spaces can be expressed in an integral representation as (Camporesi 1992)(Camporesi 1995):

$$K_T^{\mathrm{odd}}=\frac{2}{(4\pi T)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\int_0^\infty du\,e^{-u^2}\prod_{j=\frac12}^{\frac{D}{2}-1}(u^2+j^2\kappa^2T)\,,\qquad(3)$$

$$K_T^{\mathrm{even}}=\frac{2}{(4\pi T)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\int_0^\infty du\,e^{-u^2}u\coth\left(\pi\frac{u}{\kappa\sqrt{T}}\right)\prod_{j=1}^{\frac{D}{2}-1}(u^2+j^2\kappa^2T)\,,\qquad(4)$$

with $\kappa^2=\frac{|R|}{D(D-1)}$ being the inverse curvature radius.
Expanding the products, each monomial depending on a non-zero power of u will lead to ultra-violet divergences indicating the necessity of renormalization.
In $D=4$ this fact translates into two kind of contributions:

- "cosmological constant" contribution proportional to $\frac{1}{(4\pi T)^{\frac{D}{2}}}$, which does not need a new unknown parameter
- a counterterm corresponding to the introduction of the microscopic operator:

$$\mathcal{O}=-N_f\xi\phi^2|R|\,.\qquad(5)$$

4. Analytic Result in $D=4$

The presence of the hyperbolic cotangent inside (4) requires the adoption of numerical methods. Nevertheless, an interpolation between the strong and weak curvature regimes of the heat kernel provides an approximation which is good enough to capture the qualitative behavior of the potential.
For a subcritical coupling one can find the following (approximated) condition for the chiral phase transition to be allowed:

$$\frac{\kappa^3}{k_{IR}^3}+\frac{4}{3}\frac{\pi^{\frac52}}{\Gamma\left(1+\frac{1}{2p}\right)}\xi_{k_{IR}}\frac{\kappa^2}{k_{IR}^2}>\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(1-\frac1p\right)}{\Gamma\left(1+\frac{1}{2p}\right)},\qquad(6)$$

where $\xi_{k_{IR}}$ represents the infrared value of the coupling introduced in (5) and p must satisfy $p>1$. For $\xi_{k_{IR}}=0$ this leads to:

$$\frac{\kappa^3}{k_{IR}^3}>1.733\quad\text{for}\quad p=2\,,$$
$$\frac{\kappa^3}{k_{IR}^3}>\frac{\sqrt{\pi}}{2}\quad\text{for}\quad p\to\infty\,.$$

5. Constraining QG

In asymptotic safety (AS) the background metric satisfies the following semiclassical equations of motion, where $\bar\Lambda_k$ is the scale dependent cosmological constant:

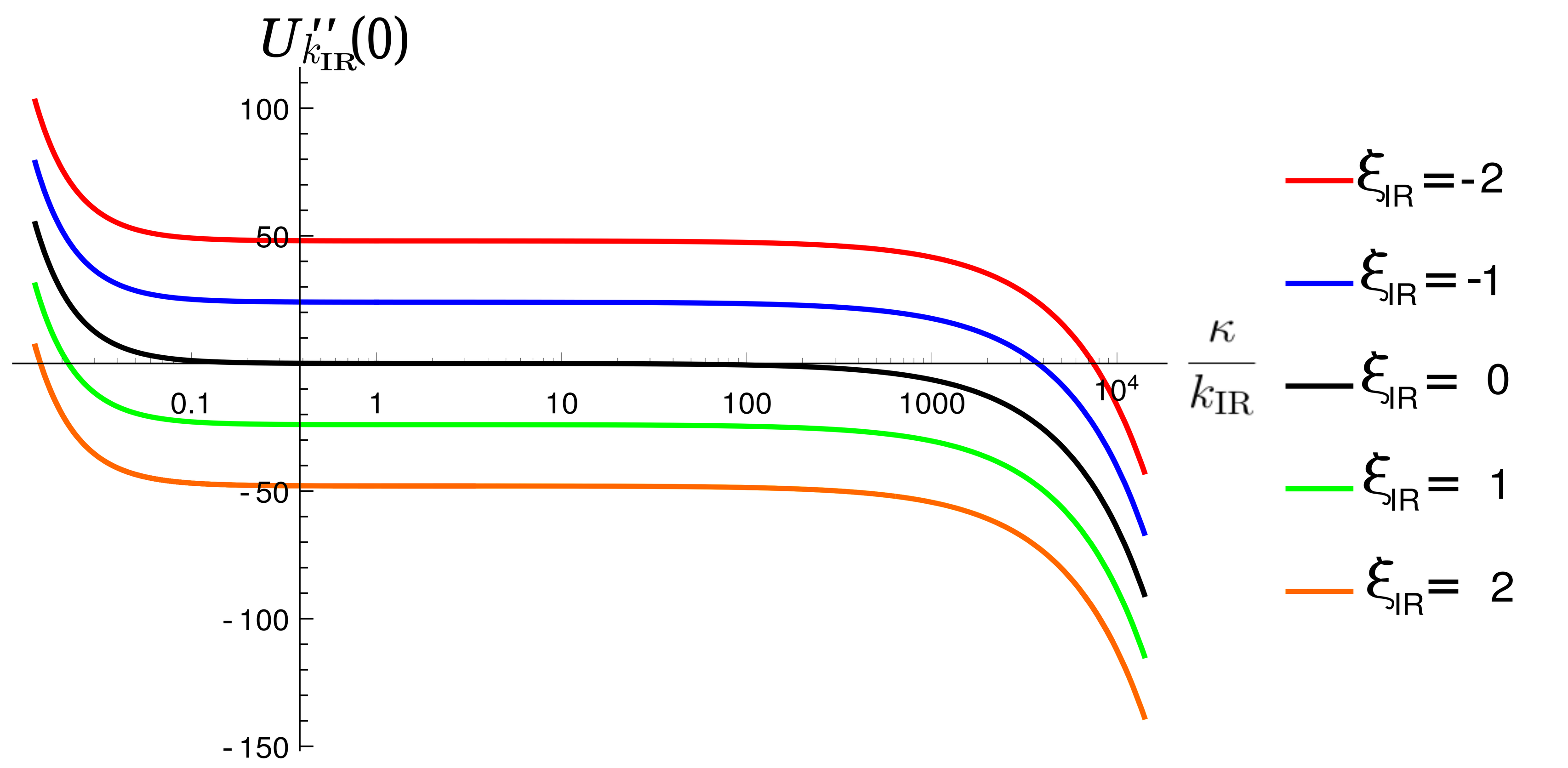
$$R_{\mu\nu}(\langle g\rangle_k)=\bar\Lambda_k\langle g_{\mu\nu}\rangle_k,\qquad\underset{\text{at UV f.p.}}{\Longrightarrow}\qquad\frac{R}{k^2}=4\lambda_*\,.\qquad(7)$$

Inclusion of fermionic degrees of freedom shift λ_* toward negative values.
By identifying the RG scale k_{IR} and the coarse graining scale k, one has:

$$\frac{\kappa^2}{k_{IR}^2}=\frac{|\lambda_*|}{3}\qquad\text{with}\quad\lambda_*=\lambda_*(N_S,N_f,N_V)\,.\qquad(8)$$

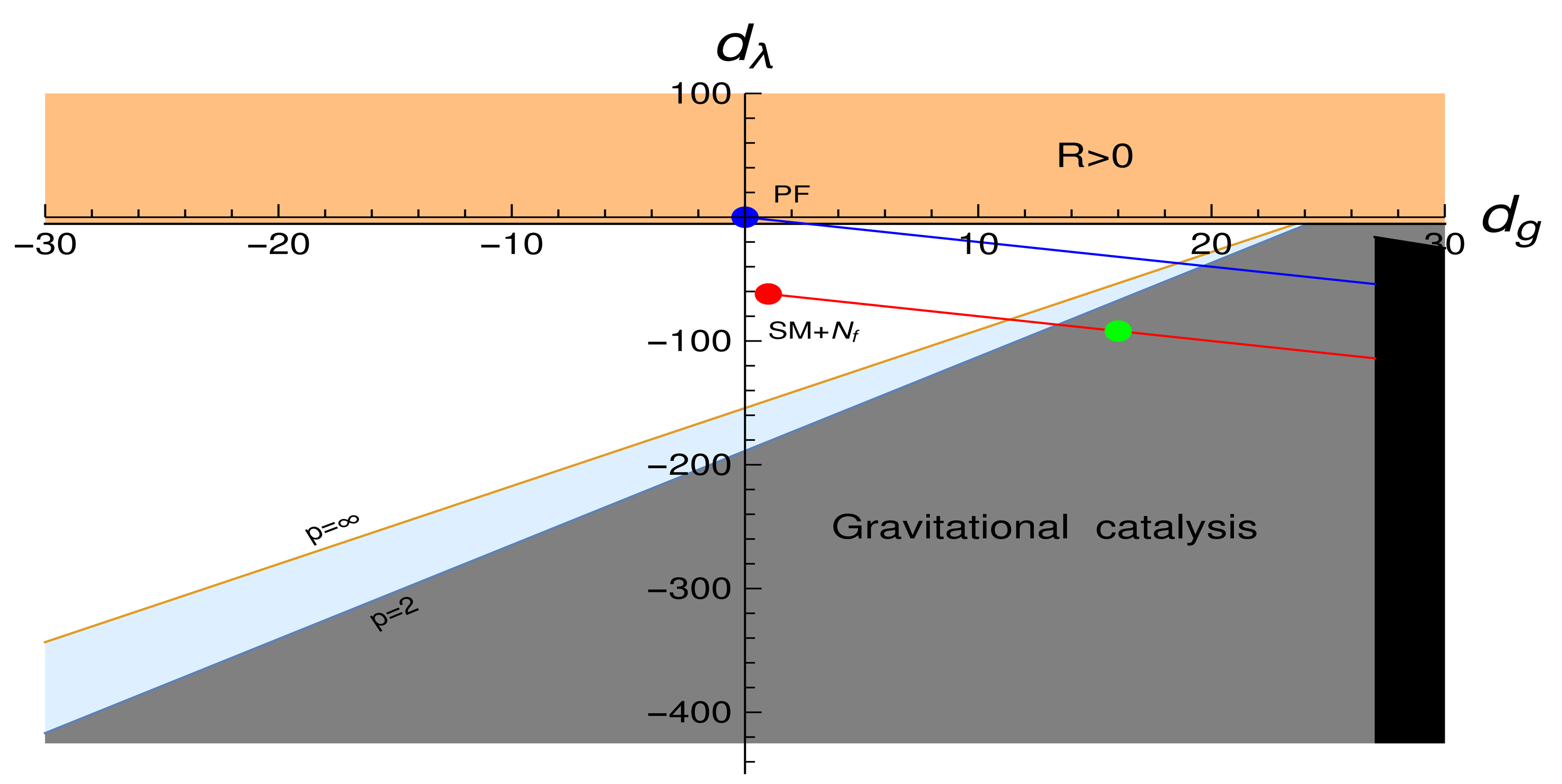
which allows to rephrase the bound as a critical fermion number for a given model, via the parameters $d_g=N_S-4N_V+2N_f$ and $d_\lambda=N_S+2N_V-4N_f$.

Curvature Bound in $D=4$ for Different Values of ξ_{IR}



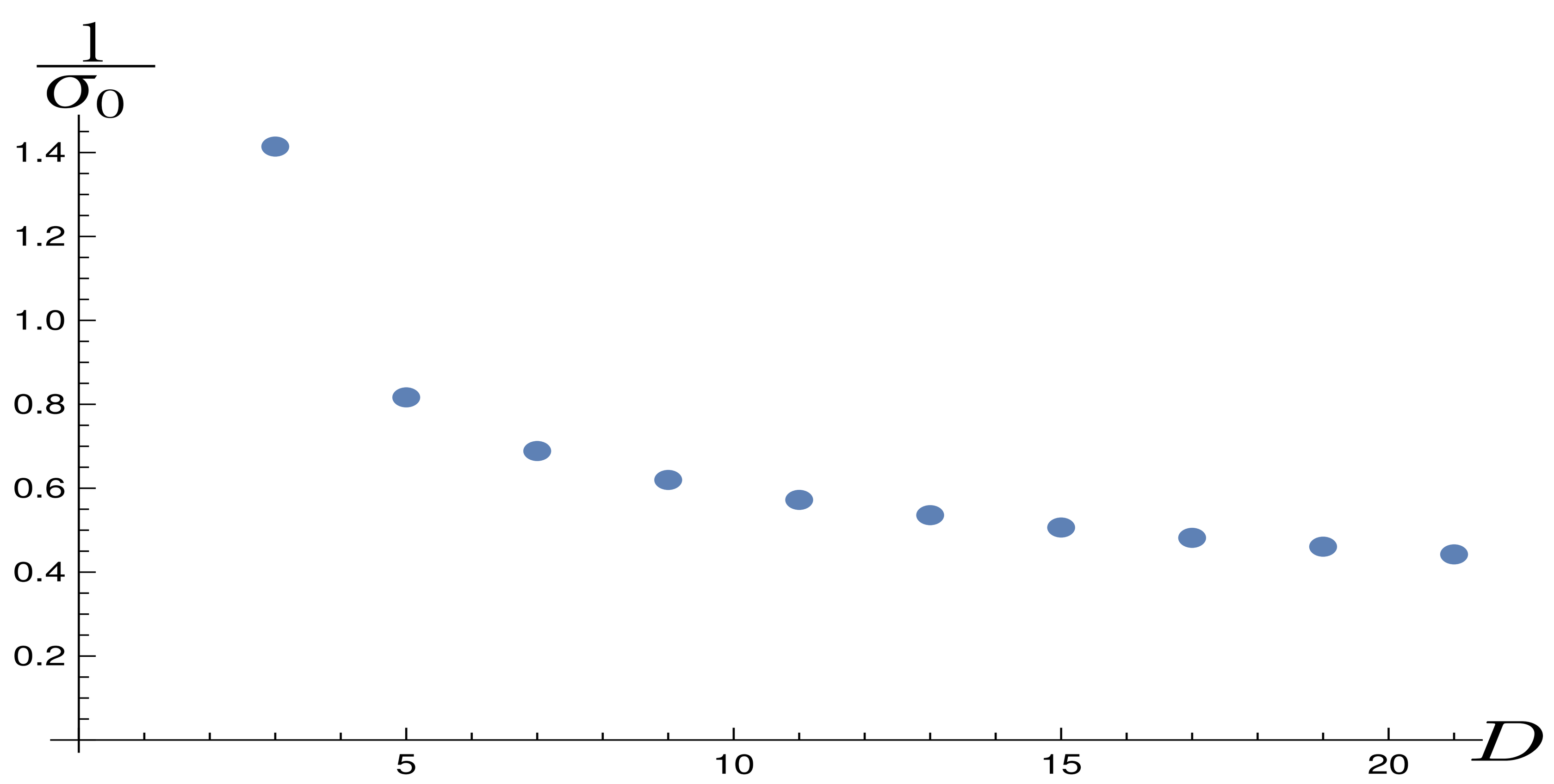
$p=2$. For large enough values of the curvature, the chiral symmetry is always broken at finite ξ_{IR}.

Constraining UV Predictions in Asymptotic Safety



$\xi_{k_{IR}}=0$. In asymptotic safety the bound can be translated into a critical number of fermion species (plot based on [Biemans, Platania and Saueressig, 2017]). The blue and red straight lines represent the evolution of, respectively, a purely fermionic model (blue dot) and the standard model (red dot). The green dot describes a standard model-like theory with a fourth matter generation.

Dimensional Dependence



$p\to\infty$. The leading contribution of the bound becomes stronger by increasing the dimensions of the background.

6. Numerical Results: $D=4$, $p=2$

From the plots it is possible to infer that:

- For every finite value of ξ_{IR} it is possible to observe a the generation of a non trivial minimum of the potential indicating chiral symmetry breaking.
- The transition is triggered only depending on the value of the ratio between κ and k_{IR}, in agreement with the picture of gravitational catalysis provided in (Gies, Lippoldt 2013).
- By applying the result to asymptotic safety we see that the standard model lies in the safe region of parameters. A fourth generation of fermions, though, seems to be exceeding the bound.
- In higher dimensions the bound seems to become stronger

8. Conclusions

We showed how the curvature of **local patches** of spacetime may trigger chiral symmetry breaking when being comparable with the energy scale of the process involved. It is always possible to find a set of values for the parameters k_{IR} and R **preventing** the generation of massive fermionic matter.
Any QG theory with a notion of averaged spacetime curvature has to satisfy our gravitational catalysis bound to be compatible with observations. A specific example was provided within the AS framework. We believe this result may provide a criterion to test the viability of such theories.

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