# TENSOR RENORMALIZATION GROUP FOR BOSONIC FIELDS

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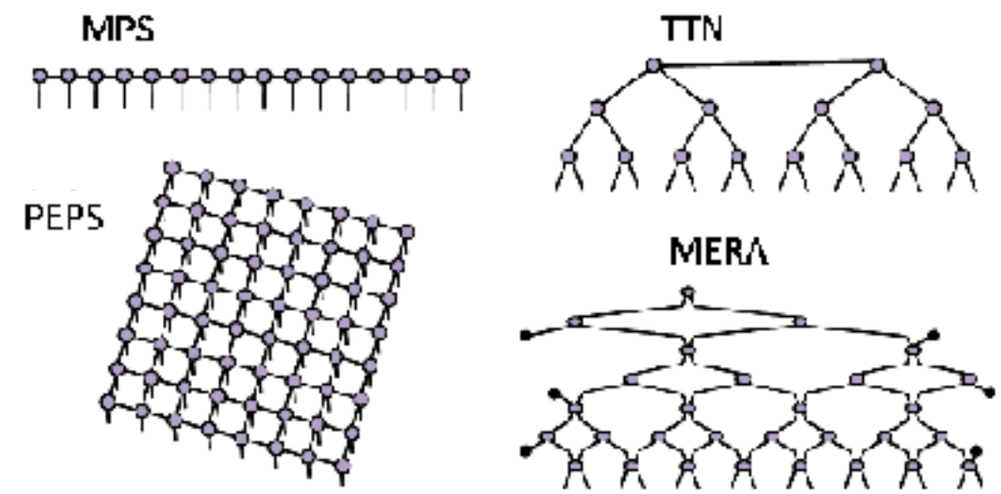
tensor network techniques are very successful

variational ansatz for ground states and low lying states

real space renormalization group

TRG, TNR




main ingredient ENTANGLEMENT PROPERTIES

APPLY TENSOR NETWORK TECHNIQUES TO QUANTUM FIELD THEORIES

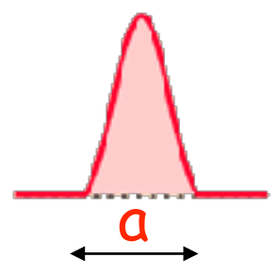
QFT extensively studied in momentum space

MPS, PEPS, TRG for lattice gauge field theories

truncated Hilbert space

continuous tensor networks: cMPS, cPEPS, cMERA

preserve QFT symmetries

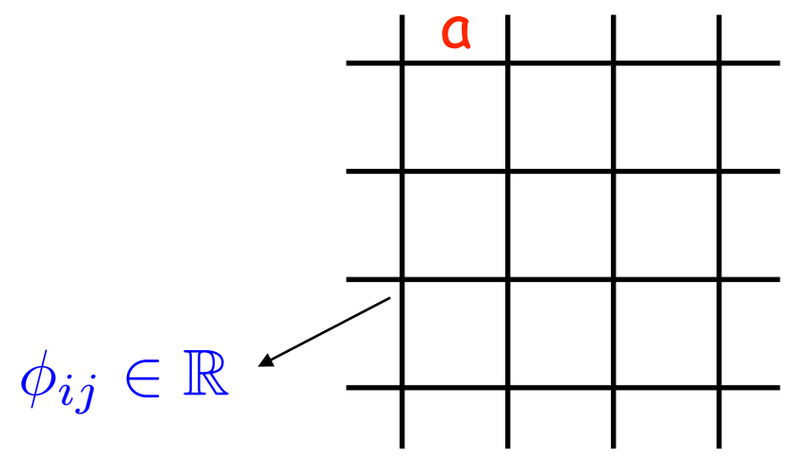



requiere a UV regulator

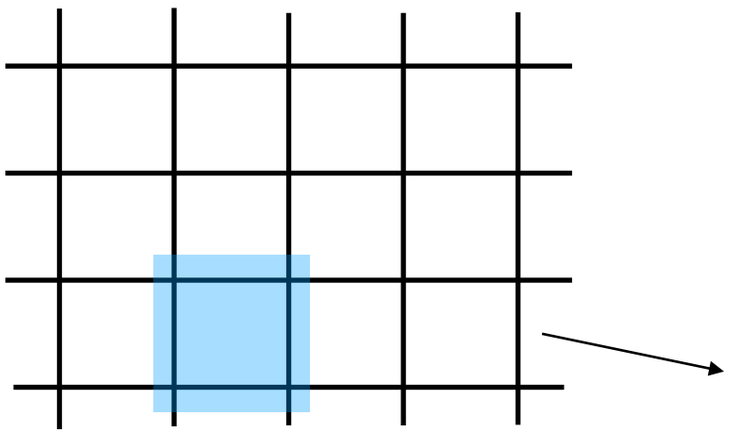
<u>in this talk</u>

discretise space-time: simple regulator

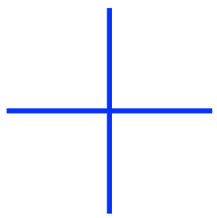
lattice variables: **CONTINUOUS FIELDS**



## REAL SPACE RENORMALIZATION GROUP (finite systems)



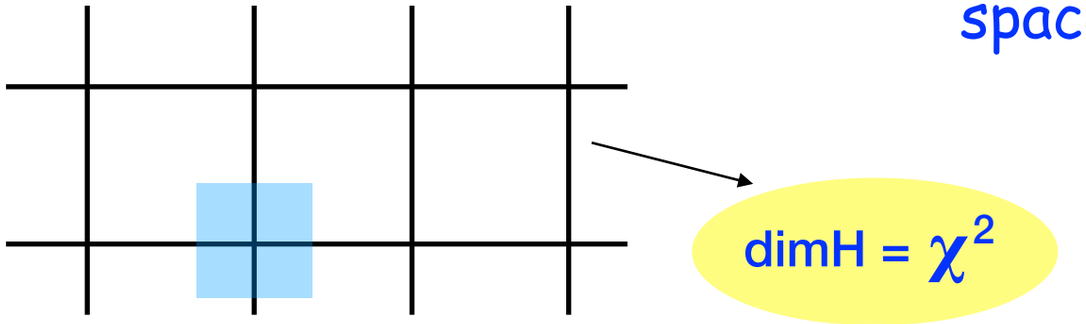
interaction takes place at vertices



= W

$\chi$ dimension of the Hilbert space on links

coarser lattice



dimH = $\chi^2$

space-time dof are traded by link variables

# link variables grows exponentially

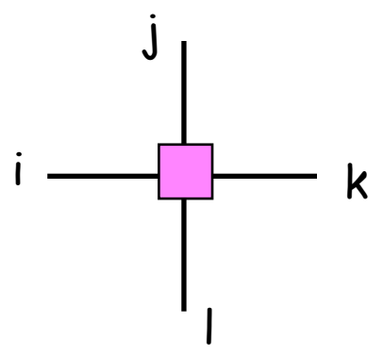
tensor renormalization group

how to truncate?

entanglement based truncation protocol

# TENSOR RENORMALIZATION GROUP (TRG)

(Levin and Nave, 2007)




$W_{ijkl} = M_{ij,kl}$

tensor matrix

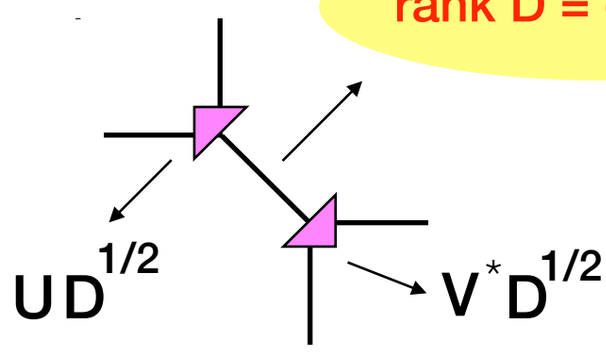
singular value decomposition

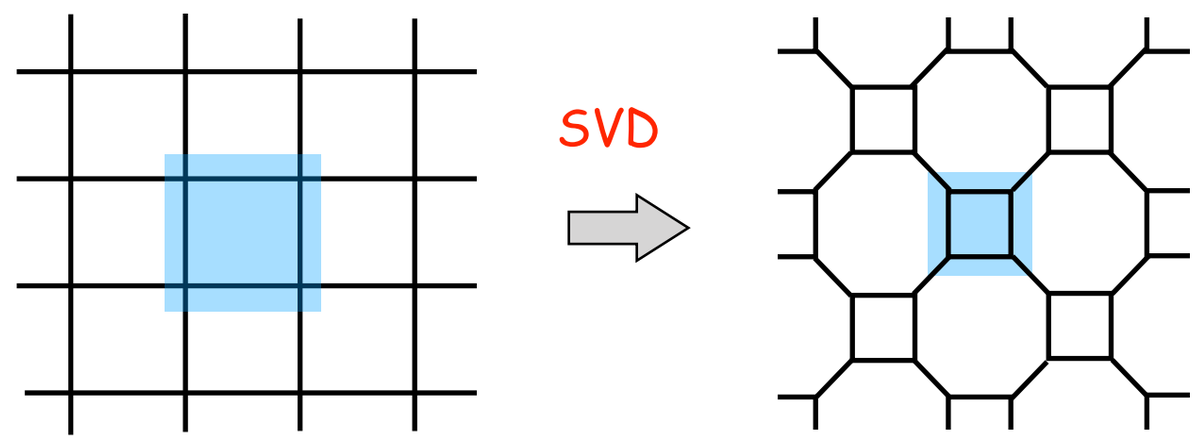
$M = U D V^{+}$

unitaries

diagonal, positive

rank D = dim $H_{new}$

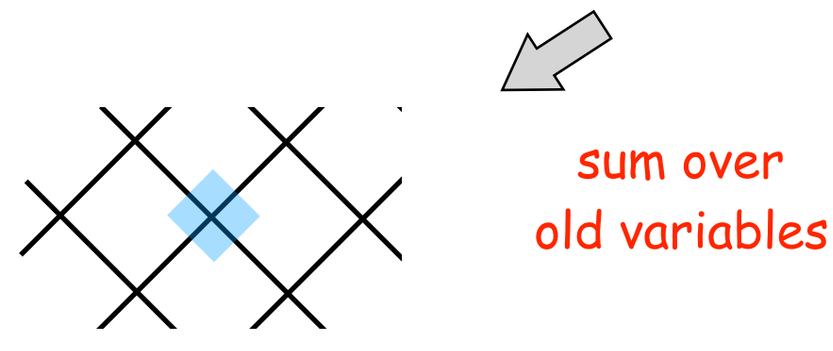


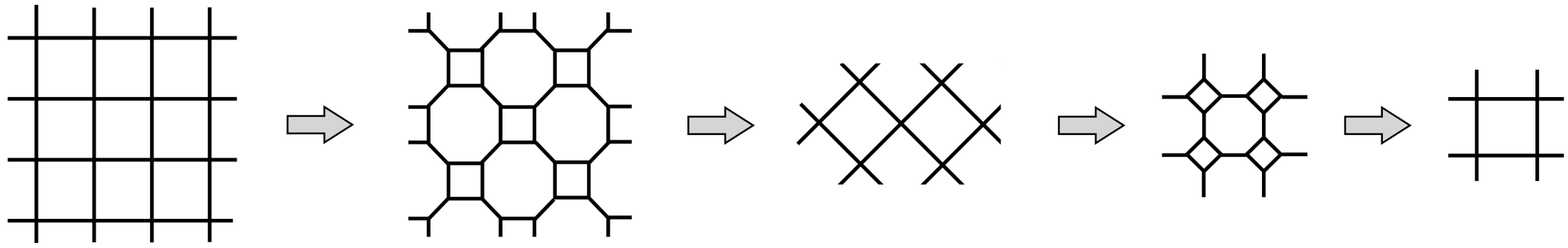

N sites

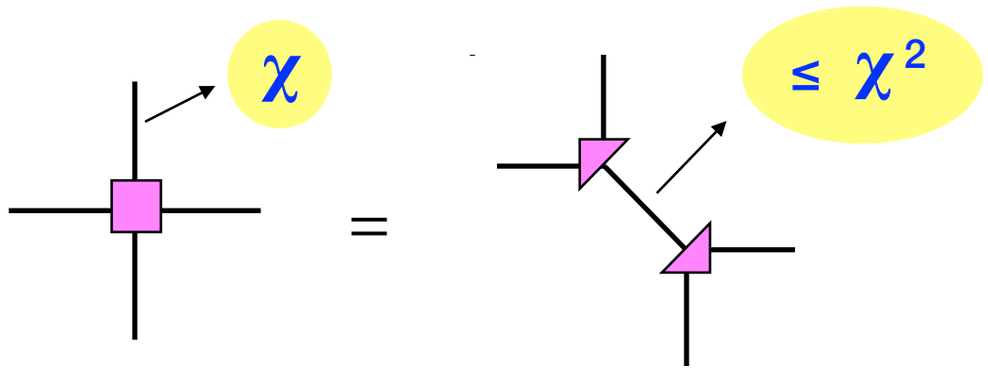
45° tilted lattice

N/2 sites



sum over old variables



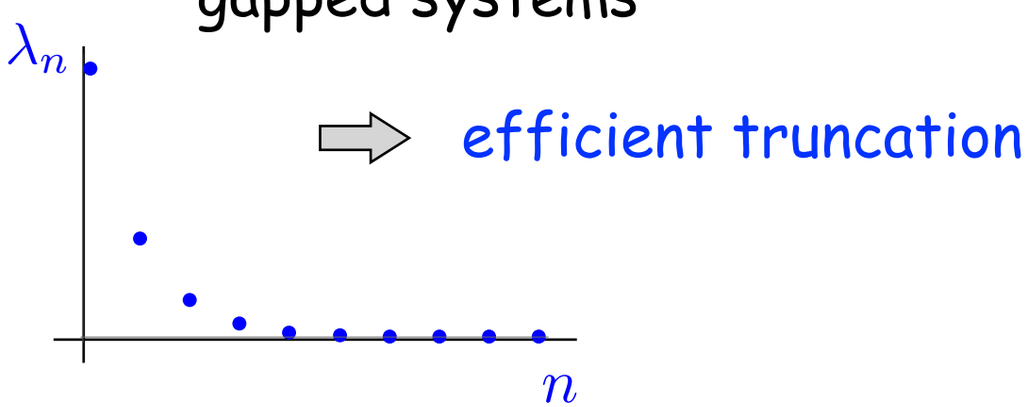



truncation: neglect small singular values

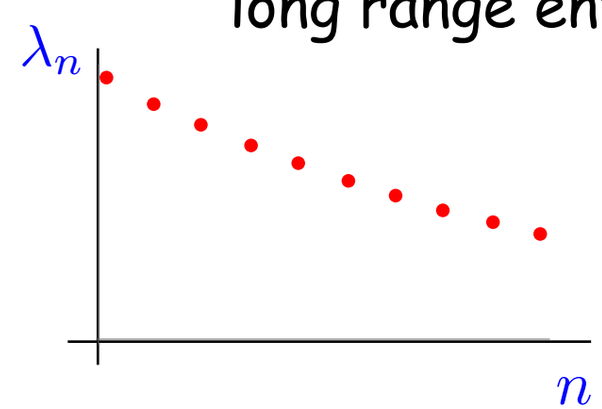
new singular values represent

entanglement at large scales

gapped systems




long range entanglement



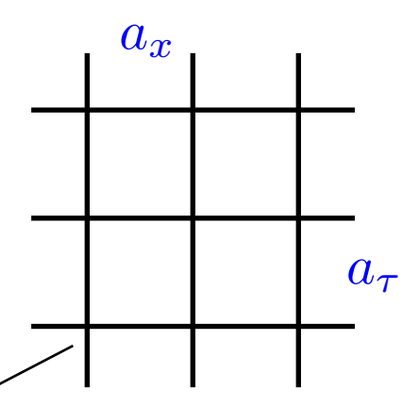
# USE TRG TO EVALUATE PATH INTEGRALS OF A SIMPLE QFT

free boson in 2d $\quad L = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$

starting point for perturbation theory

<u>euclidean partition function</u>

$$Z = \int \prod_{ij} d\phi_{ij} e^{-\frac{1}{2}\sum_{ij} a_\tau a_x \left( \frac{(\phi_{i+1j}-\phi_{ij})^2}{a_x^2} + \frac{(\phi_{ij+1}-\phi_{ij})^2}{a_\tau^2} + m^2\phi_{ij}^2 \right)}$$



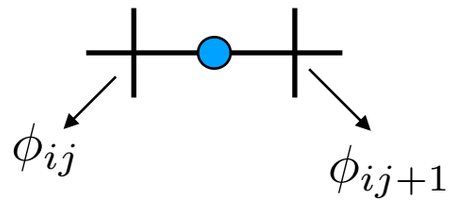

$\phi_{ij} \in \mathbb{R}$

$a_\tau = a_x = 1$

**REQUIREMENT:** link variables are always fields

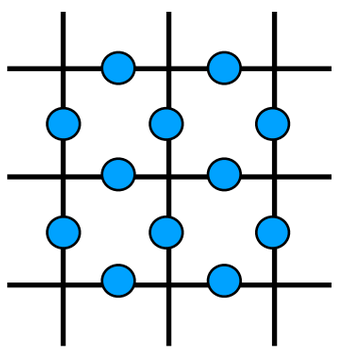
periodic boundary conditions

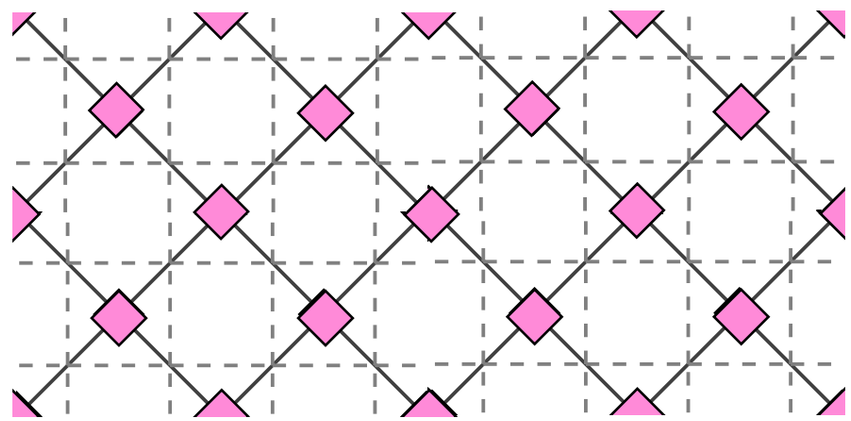
interaction takes place on links

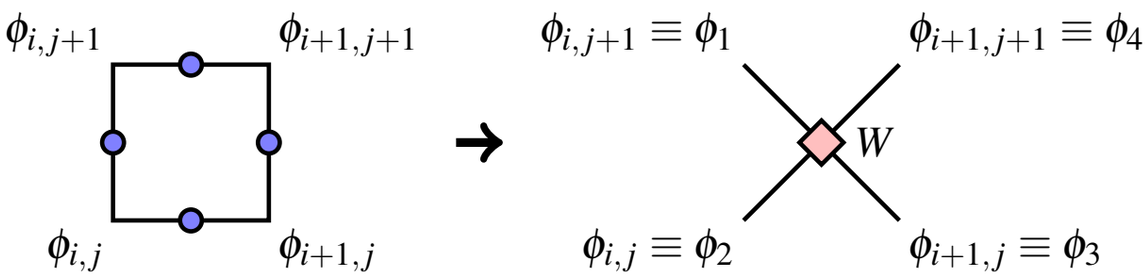



$$= \quad e^{-\frac{1}{2}[(\phi_{ij}-\phi_{ij+1})^2+\frac{m^2}{4}(\phi_{ij}^2+\phi_{ij+1}^2)]}$$

change to a vertex model

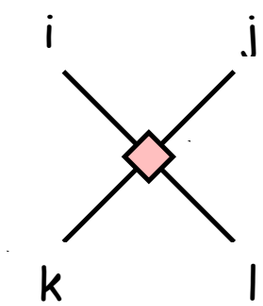






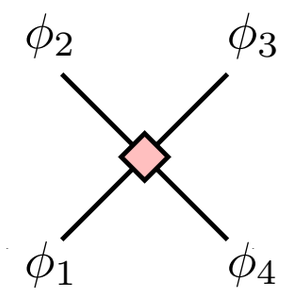

$$W(\phi_i) = e^{-\frac{1}{2}\sum_{i=1}^{4}[(\phi_i-\phi_{i+1})^2+\frac{m^2}{2}\phi_i^2]}$$

finite dim. Hilbert space




tensors

QFT




functions of continuos variables

how to implement the TRG?

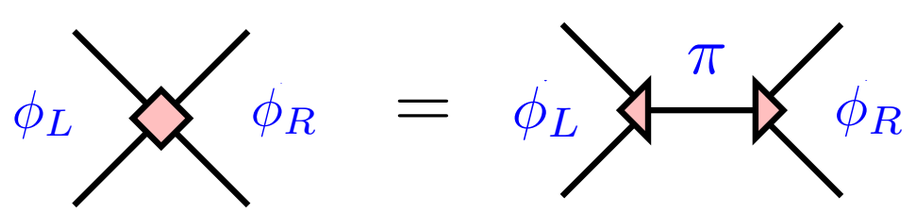
standard TRG for QFT

(Shimizu, 2012)

$W(\phi) = U\, D\, V^{+}$

rank D = ∞

search for a SVD-type decomposition




$\pi$: fields

W always gaussian

$\phi_L = \{\phi_1, \phi_2\}$

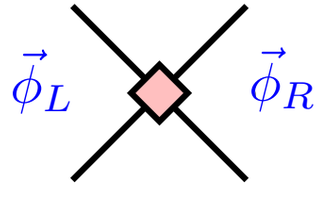
$\phi_R = \{\phi_3, \phi_4\}$

WORK ON THE QUADRATIC FORMS OF THE GAUSSIAN EXPONENT

several fields per lattice link $\vec{\phi}_i = \{\phi_{i1}, \phi_{i2} \ldots, \phi_{i\chi}\}$

bond dimension $\chi$

gaussian vertex

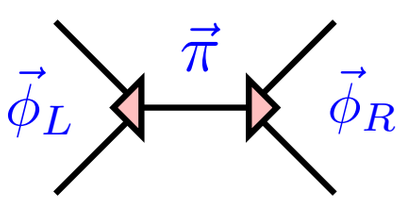



$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L A_L \vec{\phi}_L - \vec{\phi}_R A_R \vec{\phi}_R - 2\vec{\phi}_L B \vec{\phi}_R}$$

normalisation constant

$A_{L,R}$, B: real matrices

$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L \tilde{A}_L \vec{\phi}_L}\, e^{-\sum_\alpha \lambda_\alpha (\vec{v}_\alpha . \vec{\phi}_L + \vec{w}_\alpha . \vec{\phi}_R)^2}\, e^{-\vec{\phi}_R \tilde{A}_R \vec{\phi}_R}$$




$$\frac{1}{\sqrt{\prod \lambda_\alpha}} \int d\vec{\pi}\, e^{\sum_\alpha -\frac{\pi_\alpha^2}{\lambda_\alpha} + 2i\pi_\alpha (\vec{v}_\alpha . \vec{\phi}_L + \vec{w}_\alpha . \vec{\phi}_R)}$$

$\vec{\pi} = \{\pi_1, \ldots, \pi_{\tilde{\chi}}\}$

new bond dimension $\tilde{\chi}$

large freedom in choosing the LR decomposition

⟹ what is the optimal way?

$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L A_L \vec{\phi}_L - \vec{\phi}_R A_R \vec{\phi}_R - 2\vec{\phi}_L B \vec{\phi}_R}$$

use the SVD of the B matrix

$$B = U\, D\, V^{+}$$

$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L \tilde{A}_L \vec{\phi}_L}\, e^{-\sum_\alpha \lambda_\alpha (\vec{v}_\alpha . \vec{\phi}_L + \vec{w}_\alpha . \vec{\phi}_R)^2}\, e^{-\vec{\phi}_R \tilde{A}_R \vec{\phi}_R}$$

singular values of B

$\vec{v}_\alpha$ , $\vec{w}_\alpha$ : eigenvectors of U, V

$\tilde{\chi}$ = rank B

GAUSSIAN SVD

**TRUNCATION** <u>requirement</u>: acts on the number of fields

$\lambda_\alpha$ small $\Rightarrow$ $\lambda_\alpha \to 0$

$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L \tilde{A}_L \vec{\phi}_L}\, e^{-\sum_\alpha \lambda_\alpha (\vec{v}_\alpha . \vec{\phi}_L + \vec{w}_\alpha . \vec{\phi}_R)^2}\, e^{-\vec{\phi}_R \tilde{A}_R \vec{\phi}_R}$$

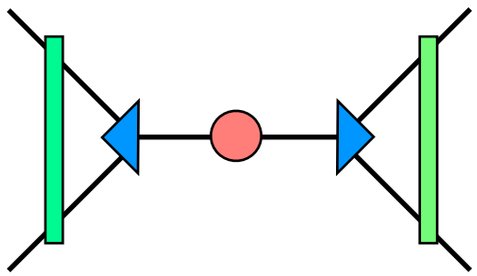
$$\frac{1}{\sqrt{\prod \lambda_\alpha}} \int d\vec{\pi}\, e^{\sum_\alpha -\frac{\pi_\alpha^2}{\lambda_\alpha} + 2i\pi_\alpha (\vec{v}_\alpha . \vec{\phi}_L + \vec{w}_\alpha . \vec{\phi}_R)}$$

$$e^{-\lambda x^2} \approx 1$$

$$\frac{1}{\sqrt{\lambda_\alpha}} e^{-\frac{\pi_\alpha^2}{\lambda_\alpha}} \to \delta(\pi_\alpha)$$

important to suppress
the large field limit

new ingredient of the gSVD



$M = U\, D\, V^+$

$W = G_L\, U\, S\, V^+ G_R$

## exact free energy

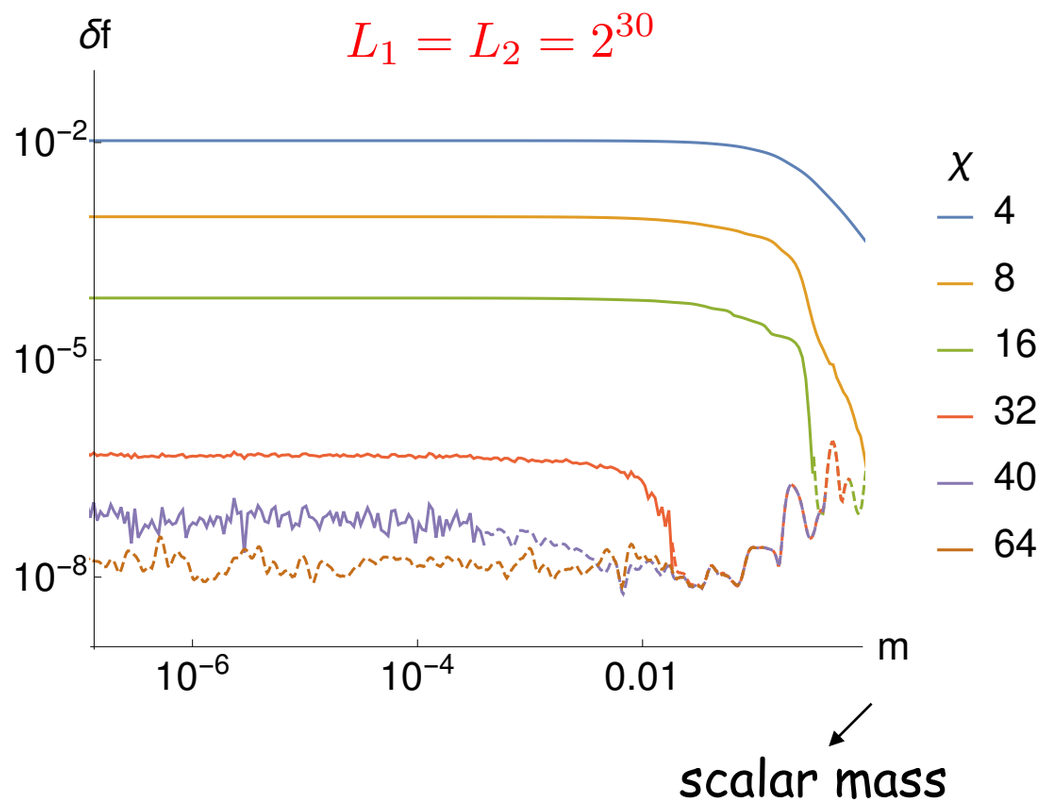
$$f_{ex} = -\log\sqrt{\frac{\pi}{2}} + \frac{1}{2L_1L_2}\sum_{n_1,n_2}\log\left(\sin^2\frac{\pi n_1}{L_1} + \sin^2\frac{\pi n_2}{L_2} + \frac{m^2}{4}\right)$$

$L_1 \times L_2$ lattice

$$f = -\frac{\log Z}{L_1 L_2}$$

## numerical results

$$\delta f = \frac{f - f_{ex}}{f_{ex}} \longrightarrow \sum_{n_1,n_2} \to \int d^2p$$




$\chi = 32$ ⇨ $\delta f < 10^{-6}$

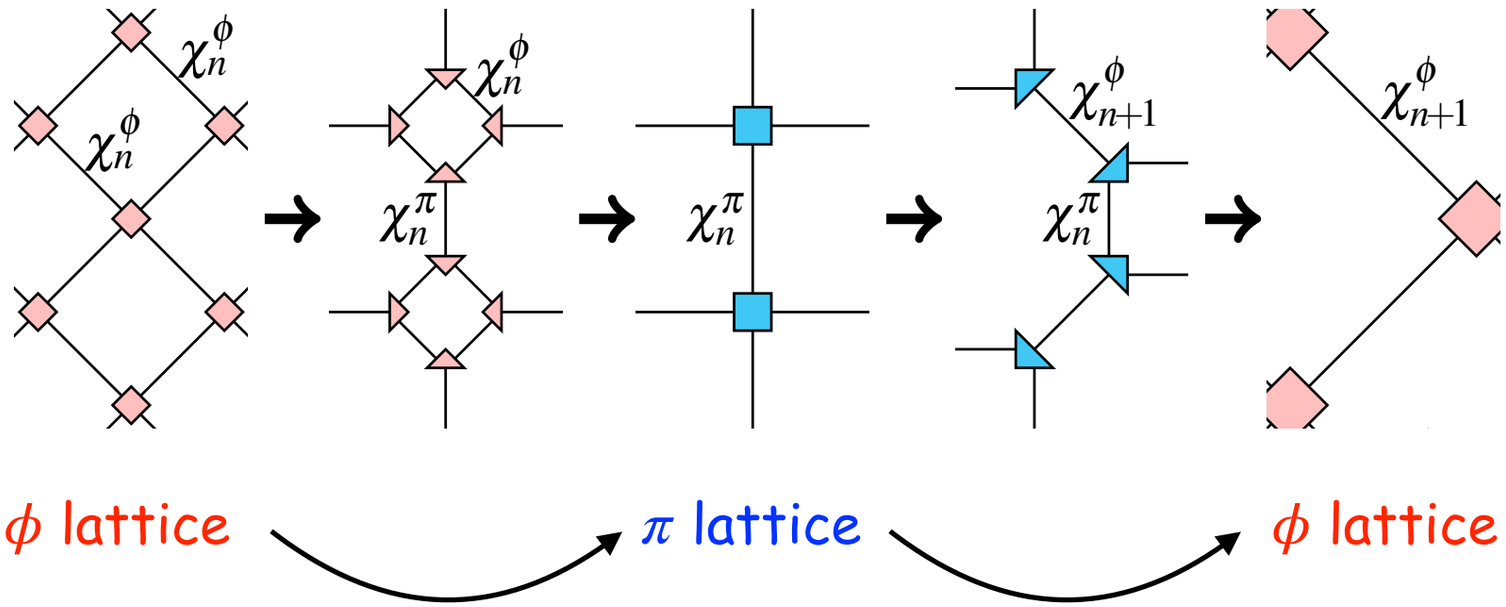
$\chi = 64$ ⇨ average precision $10^{-8}$

**SMALL MASS DOES NOT REQUIRE LARGER $\chi$**

$$\frac{1}{\sqrt{\lambda_\alpha}} e^{-\frac{\pi_\alpha^2}{\lambda_\alpha}}$$

fields efficiently group singular values

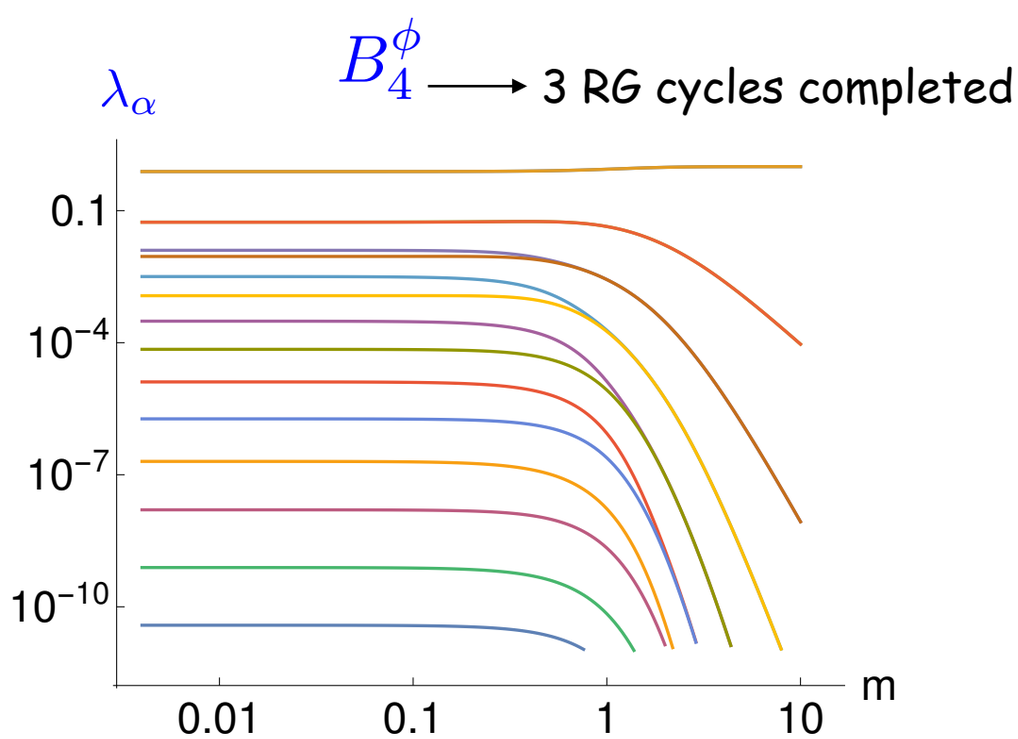
## DETAILS OF TRUNCATION




truncation

$$\chi_n^\pi = 2\chi_n^\phi$$

$$\chi_{n+1}^\phi = \chi_n^\pi$$




strongly decreasing

even for very small masses

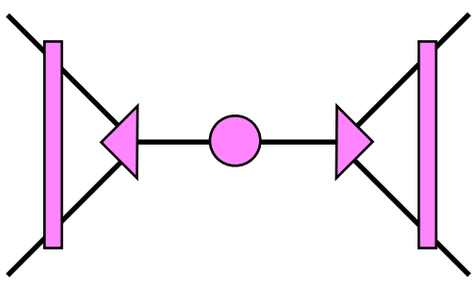
⇒ efficient truncation

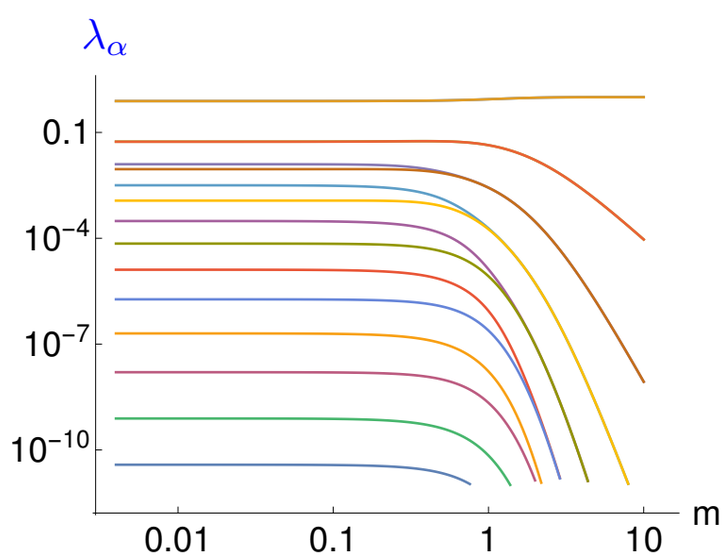
large freedom in choosing the LR decomposition

$$W(\vec{\phi}_i) = \rho\, e^{-\vec{\phi}_L A_L \vec{\phi}_L - \vec{\phi}_R A_R \vec{\phi}_R - 2\vec{\phi}_L B \vec{\phi}_R}$$

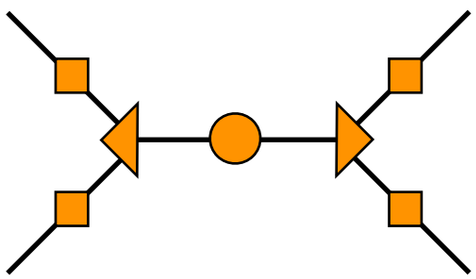
what is the optimal way?

use the SVD of the B matrix






possible alternative



fast convergence destroyed

deserves better understanding

## MASSLESS LIMIT : leading finite size effects

$$Z^{\text{exact}}_{L_1 L_2} \simeq \frac{e^{-f_\infty L_1 L_2}}{m(L_1 L_2)^{1/2}} \; Z_{\text{CFT}}(\tau)$$

$L_1, L_2 \gg 1$

$L_2/L_1$ fixed

$\sum_{n_1,n_2} \to \int d^2p$

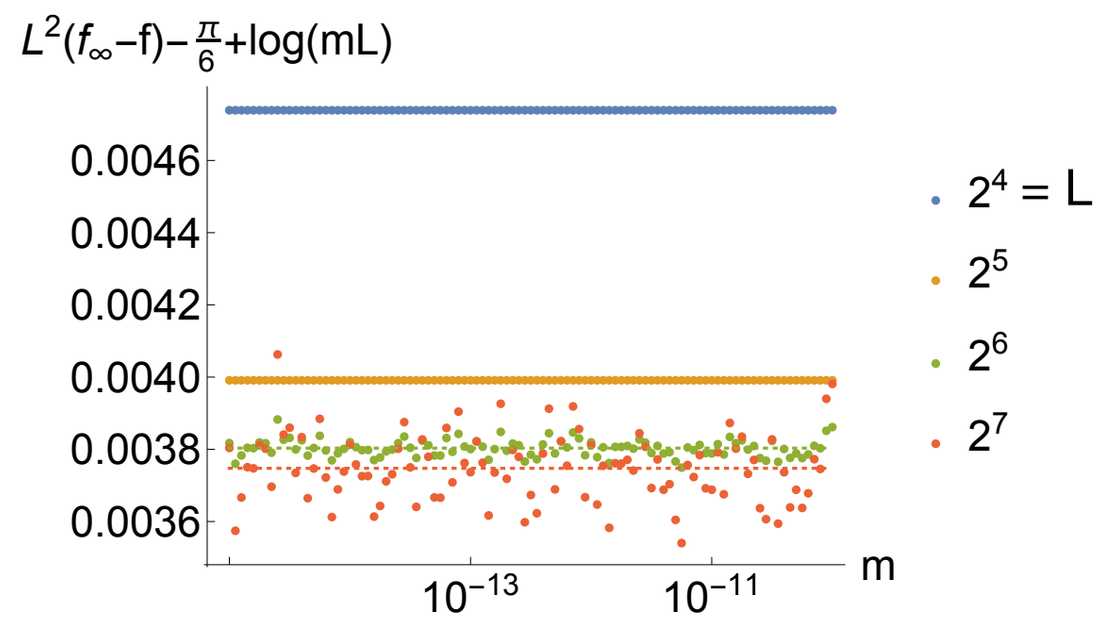
massless boson on a torus

$\tau = iL_2/L_1$

test on medium size lattices $L_1 = L_2 = L$

$$L^2(f_\infty - f) - \frac{\pi}{6} + \log(mL) =$$

$$= -2\sum_{n=1}^{\infty} \log(1 - e^{-2\pi n}) \approx 0.00375$$

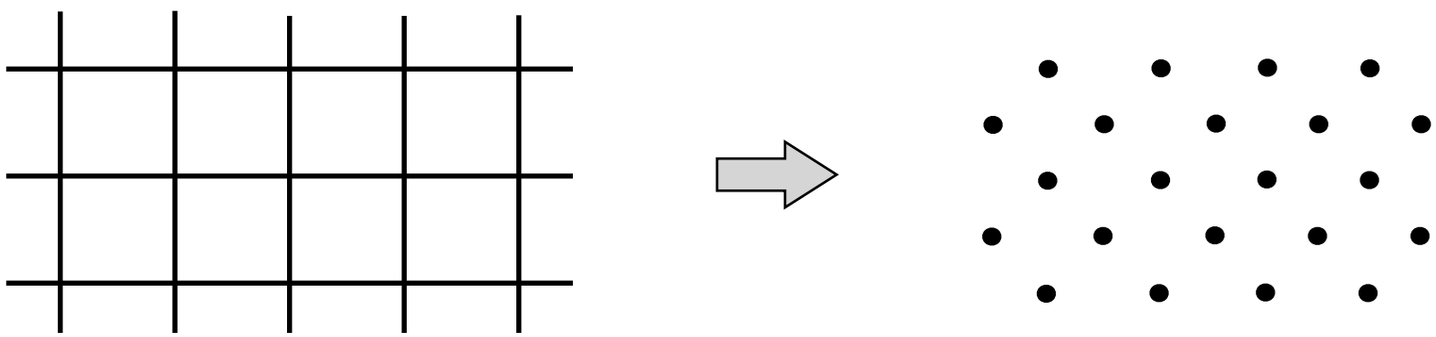



$$c_{\text{gTRG}} - 1 \approx 10^{-5} \quad L = 2^6$$
$$\approx 10^{-6} \quad L = 2^7$$

# RENORMALIZATION GROUP

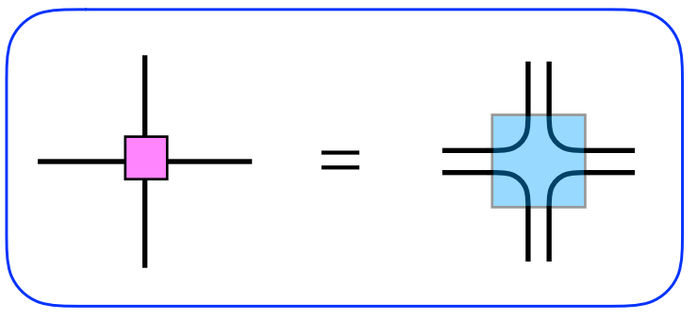
mass becomes O(1) in lattice units after $n(m) \sim -\frac{\log m}{\log 2}$ RG cycles

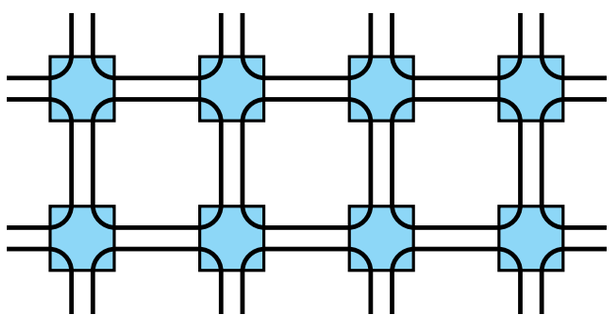
$n \gtrsim n(m)$ : trivial infrared fixed point



decoupled link variables

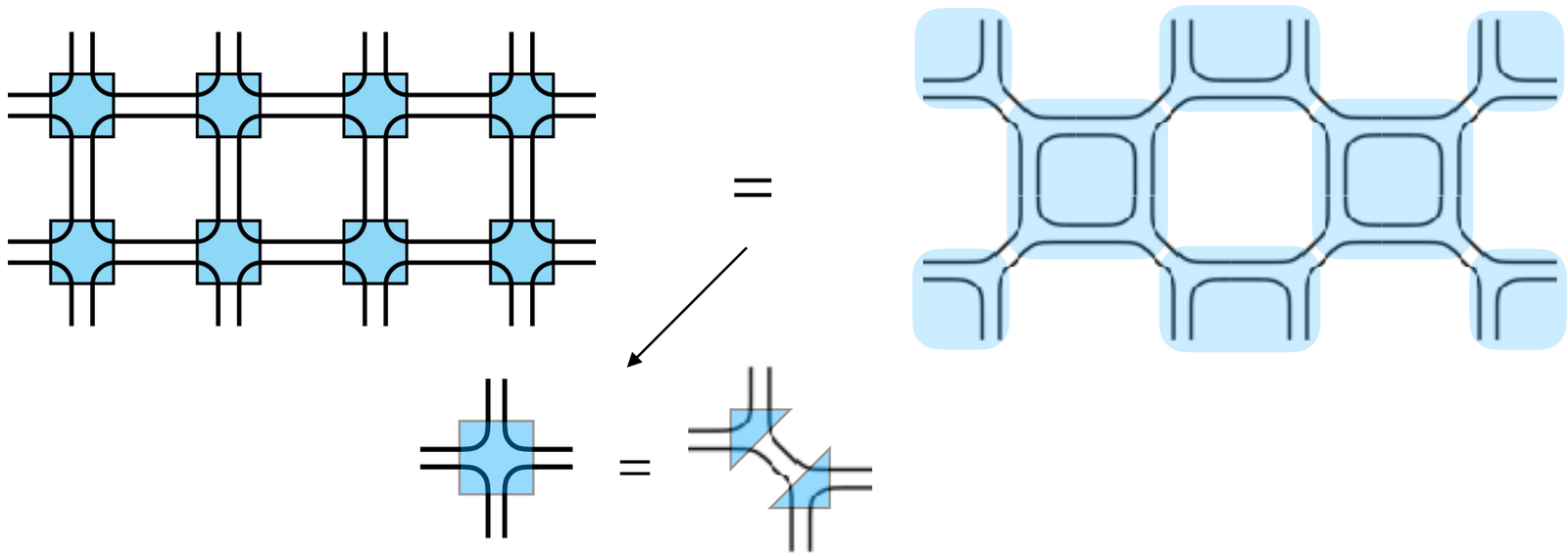
corner double line structure (CDL)



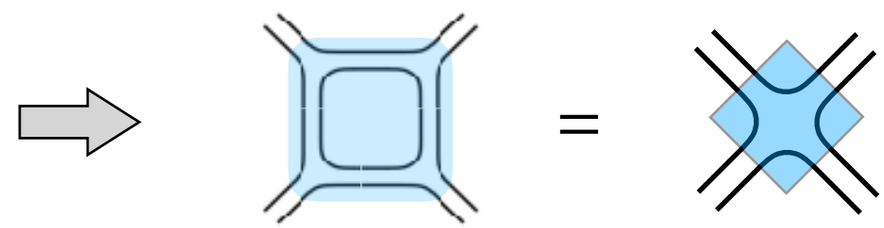


correlations confined to a single plaquette

# TRG does not remove ultralocal entanglement



CDL is the IR fixed point of TRG



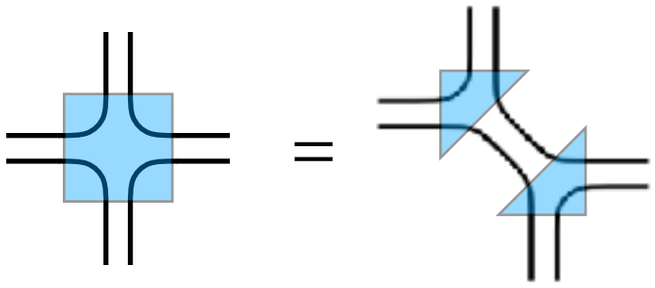
CDL fixed point expected

also for gTRG

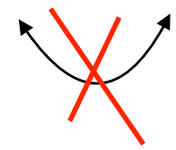
after $n \gtrsim n(m)$ RG cycles

is CDL a problem? TNR

(Vidal and Evenbly, 2015)



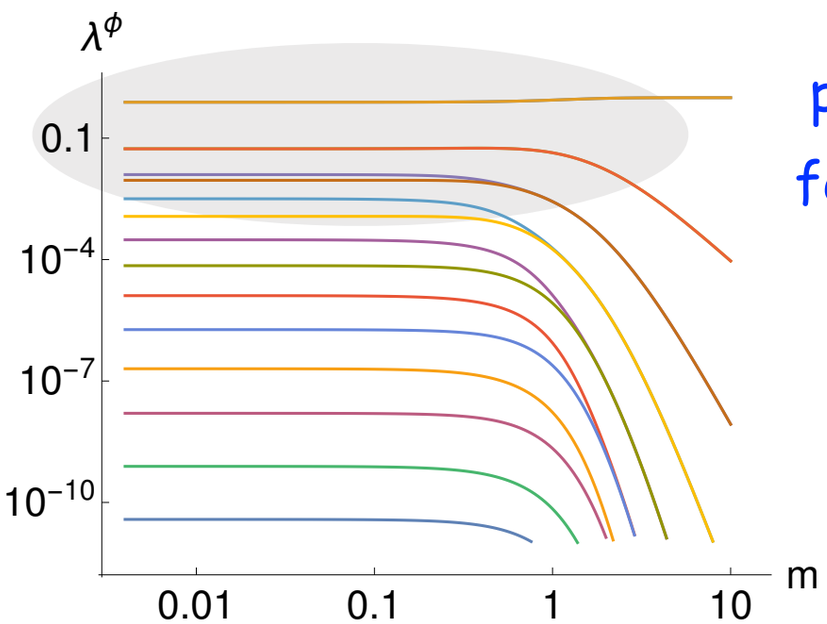
$$\mathcal{H} = \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$$



no interaction

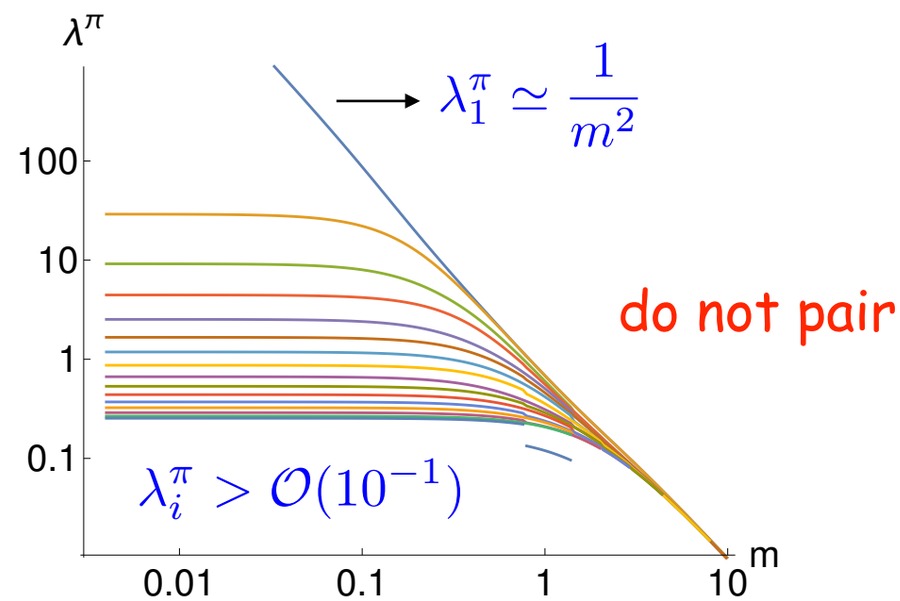
⇨ singular values of B should come in pairs

## $\phi$ lattices versus $\pi$ lattices




pair up after few RG cycles

3 RG cycles

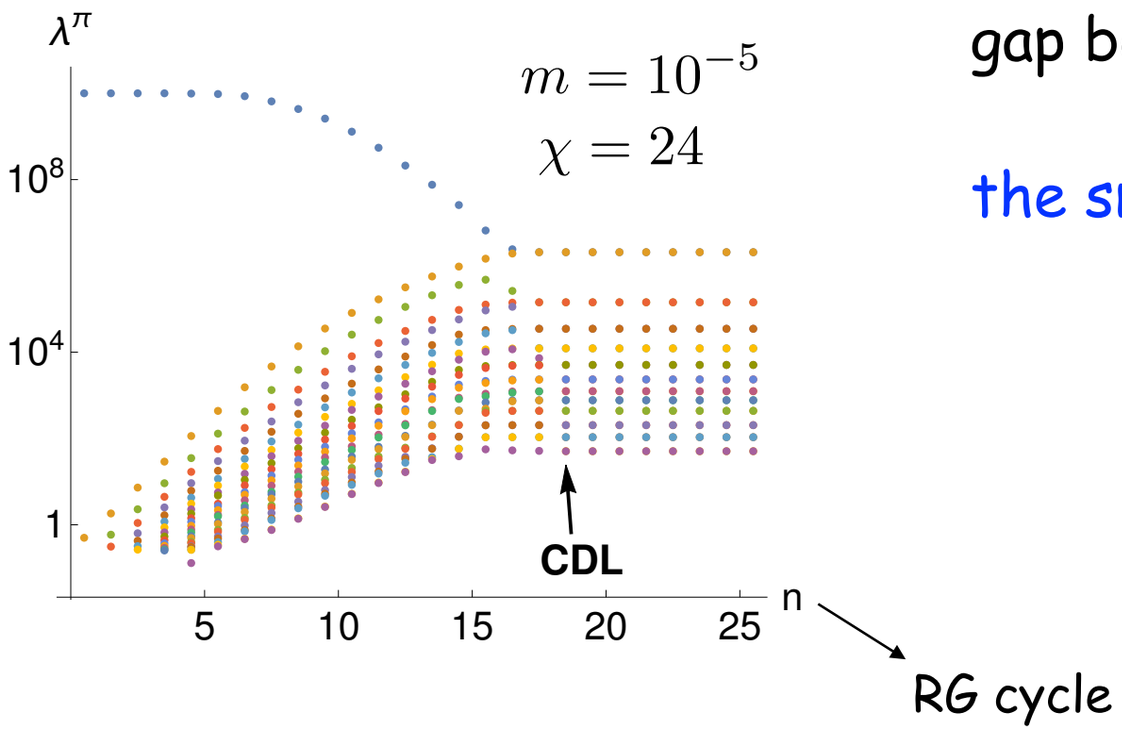



do not pair

control truncation

control RG flow

## RG flow




gap between $\lambda_1^\pi$ and $\lambda_2^\pi$ closes up along RG

the smaller m, the more RG cycles are needed

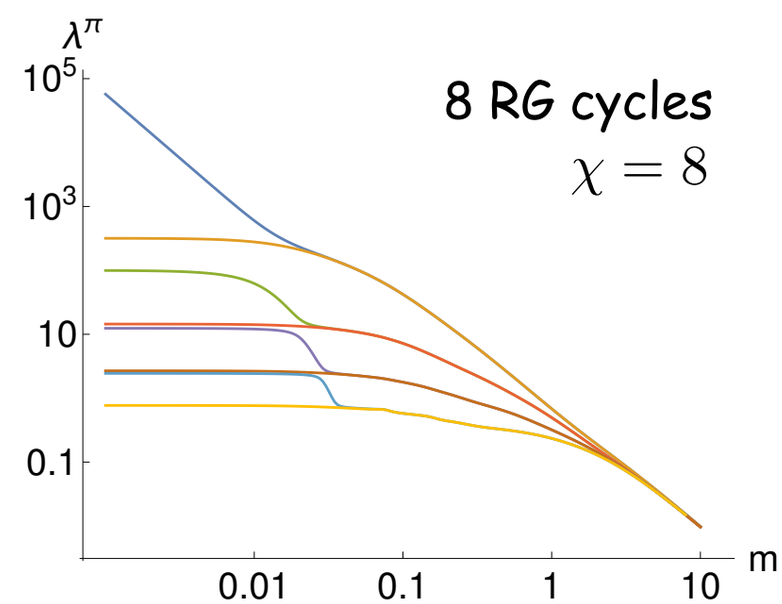
$n_{\rm CDL} \simeq 19$ consistent with

$n(m{=}10^{-5}) \simeq 16 - 17$

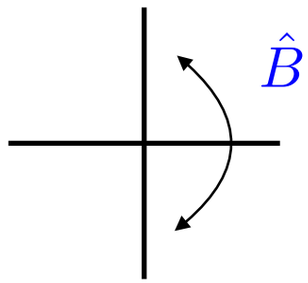
singular values pair up for m > 0.03

$0.03 \times 2^8 \simeq 8$ in lattice units

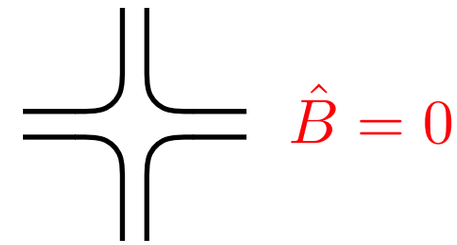
CDL not before scales larger than $\xi = 1/m$



pairing criterium not enough for CDL



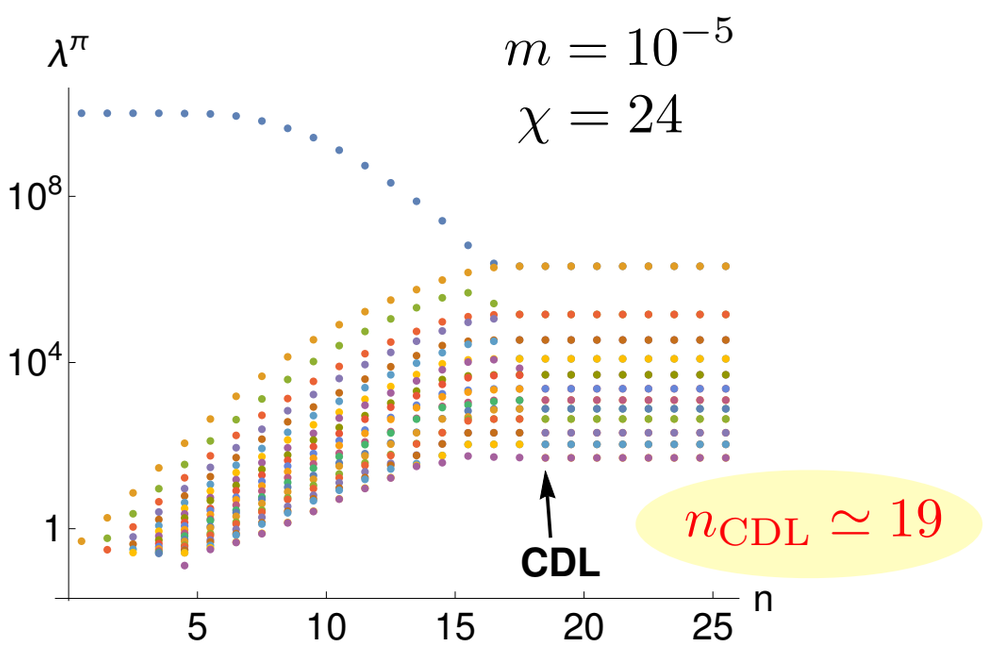

submatrix of B connecting fields on opposite links

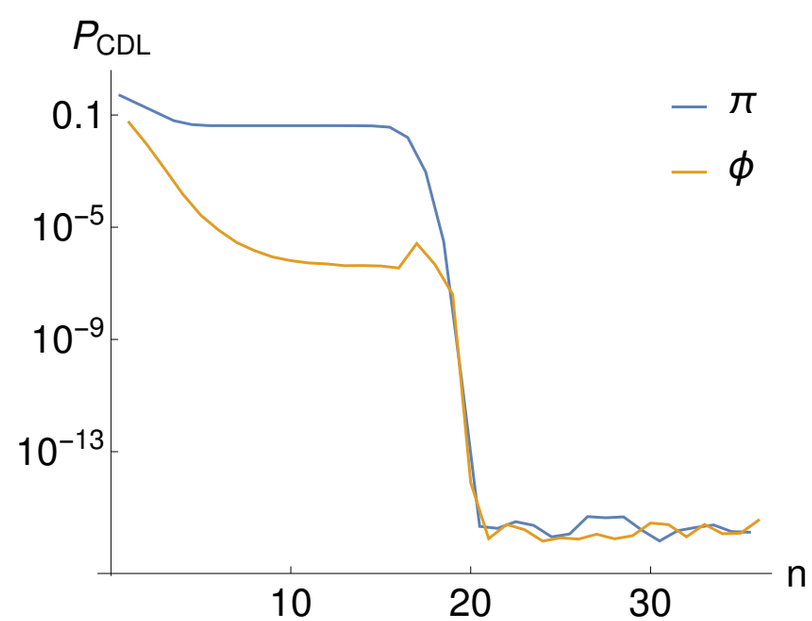



$$P_{\rm CDL} = \frac{1}{\chi}\frac{||\hat{B}||}{\lambda_1}$$

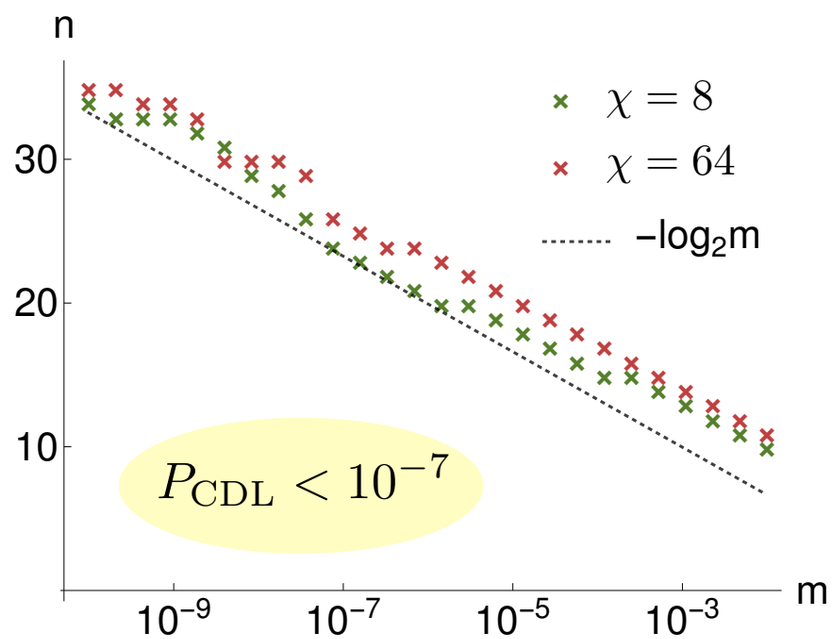
vanishing of $P_{\rm CDL}$ on 2 successive gTRG iterations

complete CDL structure



# RG cycles necessary to reach a CDL IR fixed point




similar results for large and small $\chi$

consistent with the scaling argument

long distance info kept for any $\chi$

⇩

arbitrarily small singular values always kept

$$e^{-\frac{1}{2\lambda}\pi^2}, \qquad |\pi| \to \infty$$

fields efficiently group singular values

THANKS!

<u>main question</u>: interaction

starting point for perturbation theory