

TOPOLOGICAL CRYSTALLINE PHASE IN NUCLEAR MATTER UNDER STRONG MAGNETIC FIELDS

Tomáš Brauner

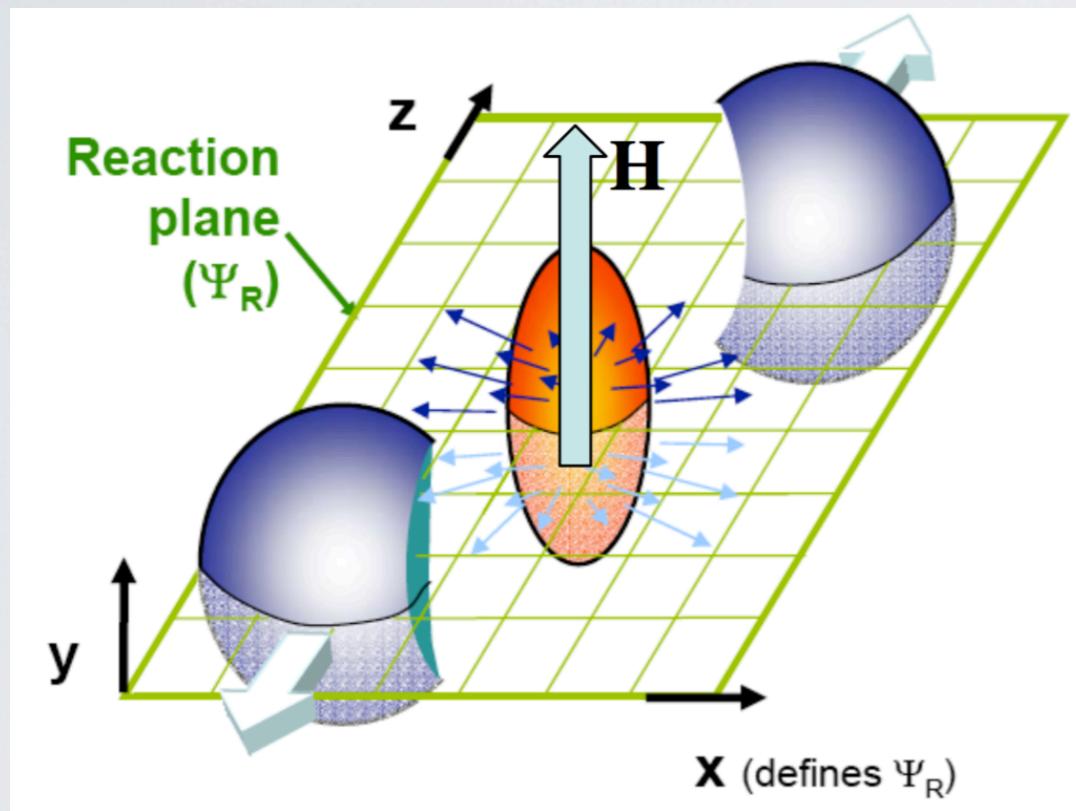
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Helena Kolešová
Georgios Filios



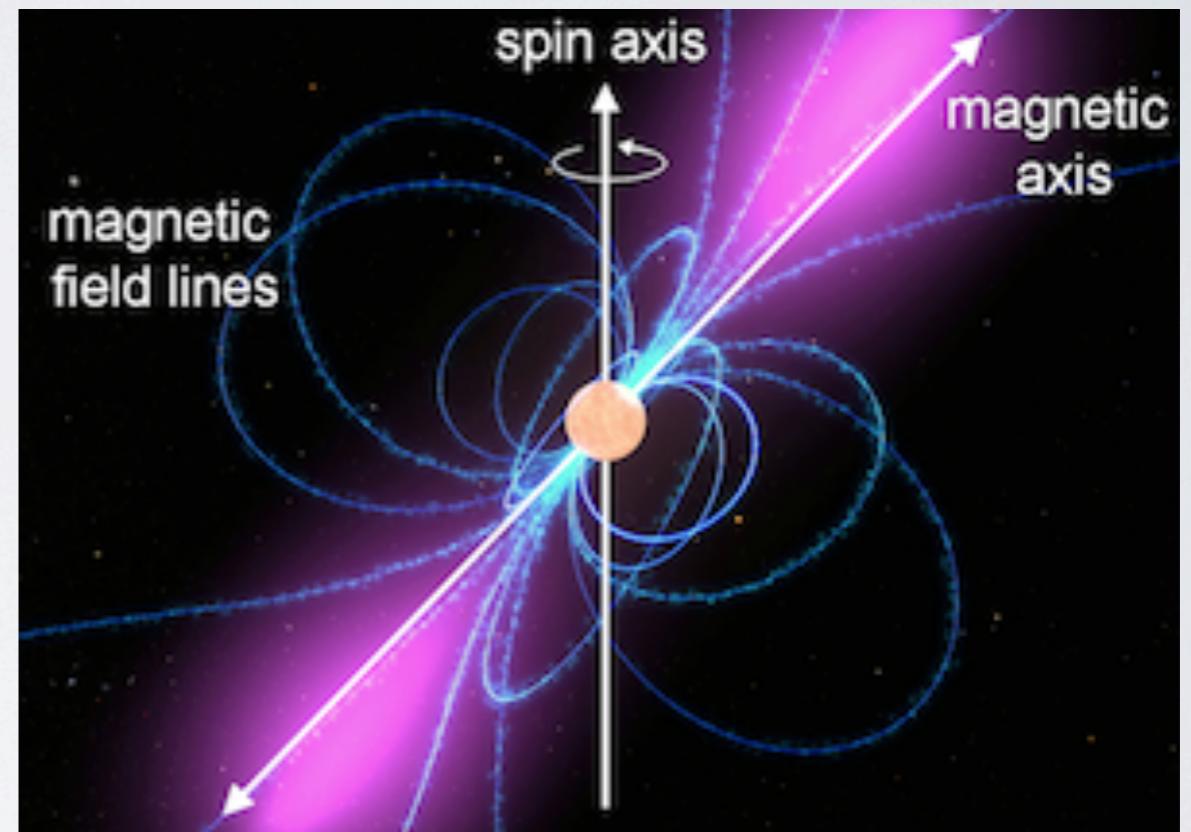
INTRODUCTION

QCD IN STRONG MAGNETIC FIELDS

Strong fields in both heavy-ion collisions and neutron stars!

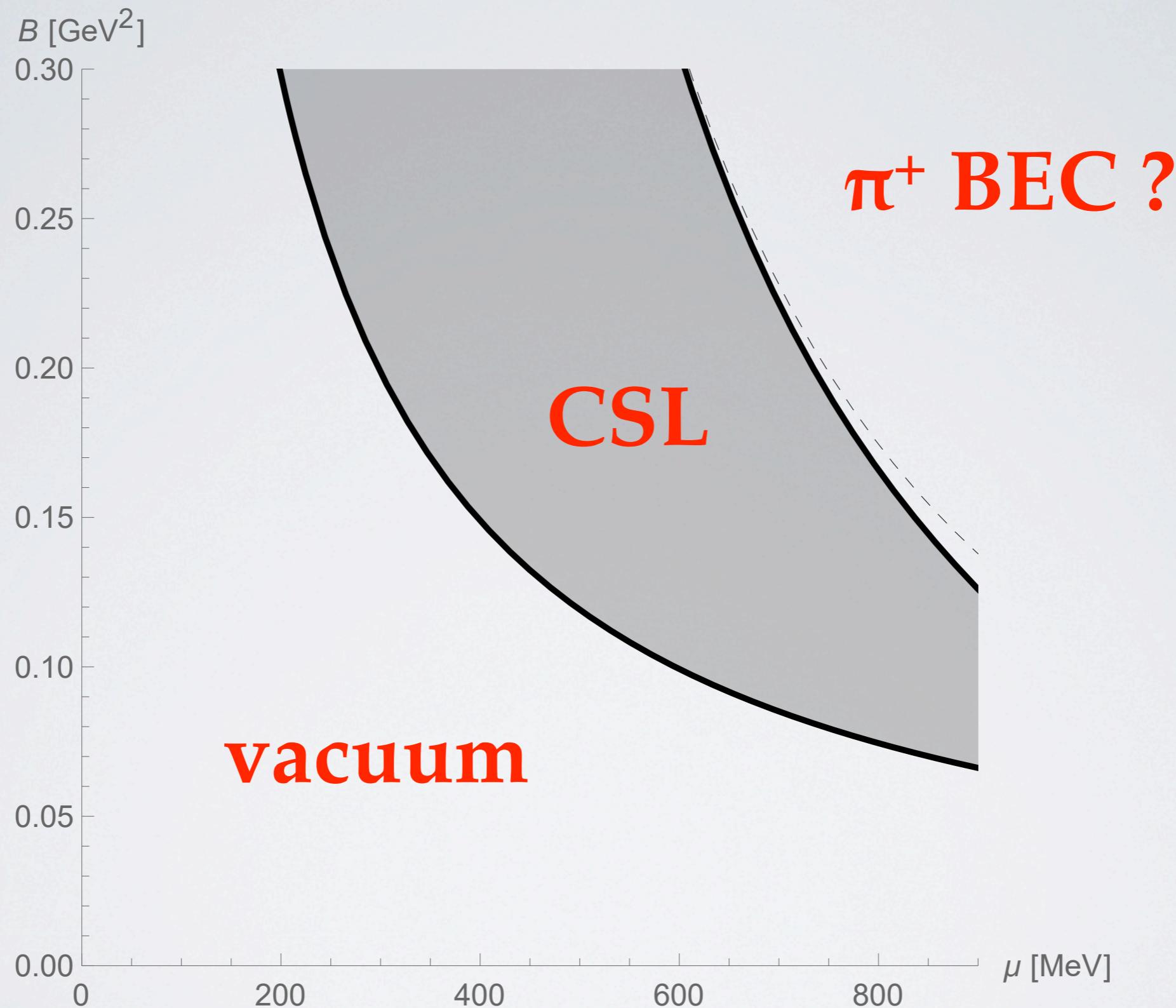


Credit: Kharzeev, PPNP 75 (2014)



Credit: NASA/Goddard Space Flight Center
Conceptual Image Lab

PHASE DIAGRAM



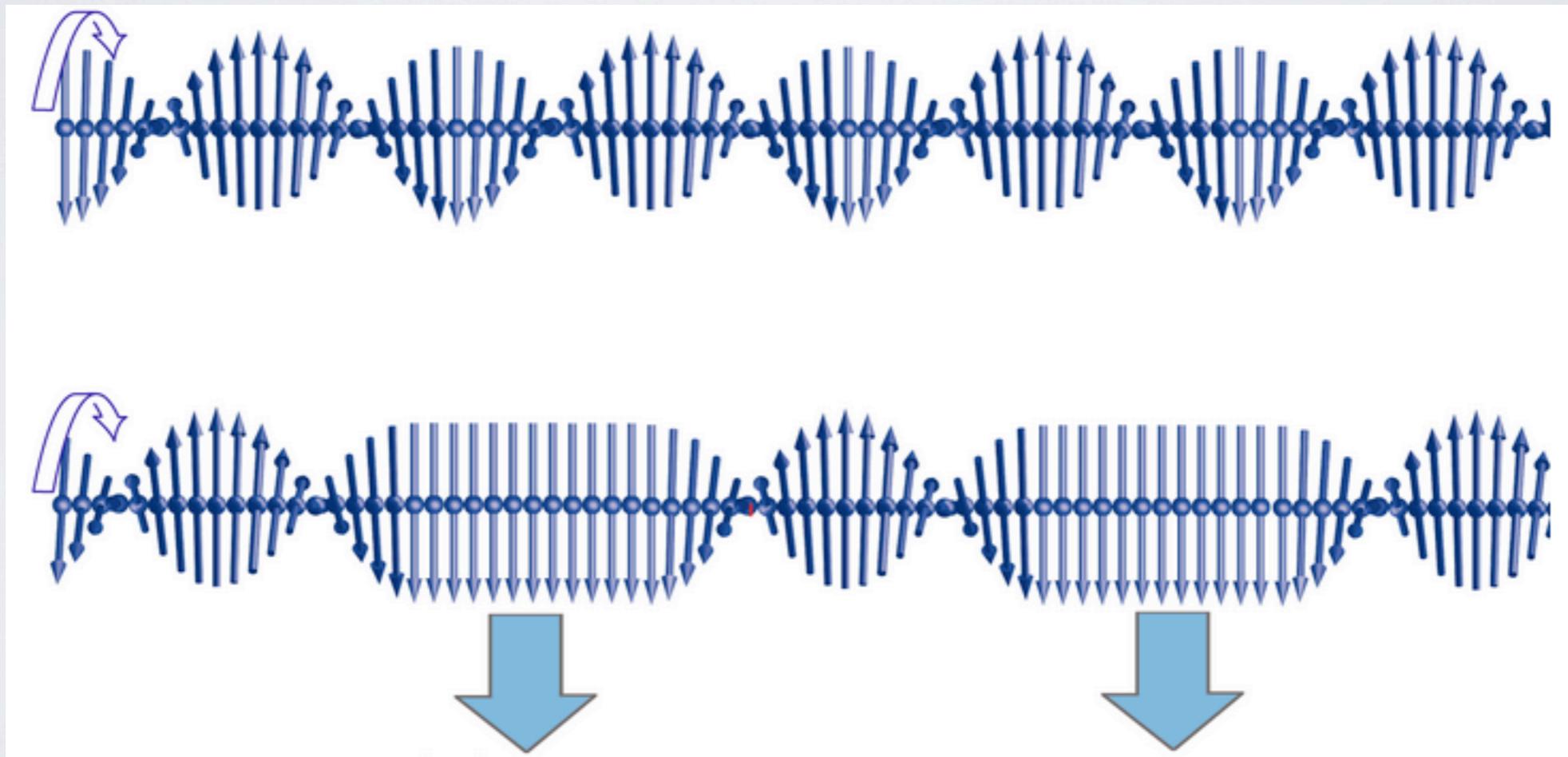
Conversion factor: $1 \text{ GeV}^2 \approx 1.7 \times 10^{20} \text{ G}$

TB, Yamamoto, JHEP 04 (2017)

CHIRAL SOLITON LATTICE

Periodic, parity-violating topological soliton, observed e.g. in:

- ♦ Liquid crystals.
- ♦ Chiral magnets.



Credit: Togawa *et al.*, PRL 108 (2012)

CSL IN QCD

SYMMETRY OF NUCLEAR MATTER

- ♦ Isospin symmetry:

$$\left. \begin{array}{c} u \\ d \end{array} \right\} \psi \implies \text{SU}(2)_V$$

- ♦ Chiral symmetry: $\psi_L \& \psi_R \implies \text{SU}(2)_L \times \text{SU}(2)_R$

- ♦ Ground state (vacuum): spontaneous symmetry breaking!

$$\text{SU}(2)_L \times \text{SU}(2)_R \rightarrow \text{SU}(2)_V$$

- ♦ 3 Nambu–Goldstone bosons: pions π^+, π^0, π^- .

LOW-ENERGY EFT: QCD

- ♦ Symmetry-breaking pattern:

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \rightarrow \mathrm{SU}(2)_V$$

- ♦ Parameterization of the coset space:

$$[\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R]/\mathrm{SU}(2)_V \implies \Sigma \in \mathrm{SU}(2)$$

- ♦ Leading-order effective Lagrangian:

$$\mathcal{L} = \frac{f_\pi^2}{4} [\mathrm{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2 \mathrm{tr}(\Sigma + \Sigma^\dagger)]$$

Explicit symmetry breaking

WESS-ZUMINO-WITTEN TERM

Anomalous coupling of pions has nontrivial consequences:

- ♦ Neutral pions can couple to electromagnetic fields!
- ♦ Pion fields can carry baryon number!

$$S_{WZW} = - \int d^4x \left(A_\mu^B - \frac{1}{2} A_\mu \right) j_B^\mu$$

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Baryon number gauge field Electromagnetic gauge field

Goldstone–Wilczek current

$$j_B^\mu = -\frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} \text{tr} \left[(\Sigma D_\nu \Sigma^\dagger)(\Sigma D_\alpha \Sigma^\dagger)(\Sigma D_\beta \Sigma^\dagger) + \frac{3i}{4} F_{\nu\alpha} \tau_3 (\Sigma D_\beta \Sigma^\dagger + D_\beta \Sigma^\dagger \Sigma) \right]$$

NEUTRAL PION BACKGROUND

$$\mathcal{L} = \frac{f_\pi^2}{4} [\text{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2 \text{tr}(\Sigma + \Sigma^\dagger)] + \mathcal{L}_{\text{WZW}}$$

$$\downarrow \quad \Sigma = e^{i\tau_3 \phi}$$

$$\mathcal{H} = \frac{f_\pi^2}{2} (\nabla \phi)^2 + m_\pi^2 f_\pi^2 (1 - \cos \phi) - \frac{\mu}{4\pi^2} \mathbf{B} \cdot \nabla \phi$$

- ♦ Sine-Gordon Hamiltonian with a topological term!
- ♦ Favors one-dimensional modulation in direction of \mathbf{B} .
- ♦ Equation of motion: simple pendulum!

$$\partial_z^2 \phi = m_\pi^2 \sin \phi$$

NEUTRAL PION BACKGROUND

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- ♦ Chiral limit ($m_\pi=0$): Hamiltonian minimized by

$$\phi(z) = \frac{\mu B z}{4\pi^2 f_\pi^2}$$

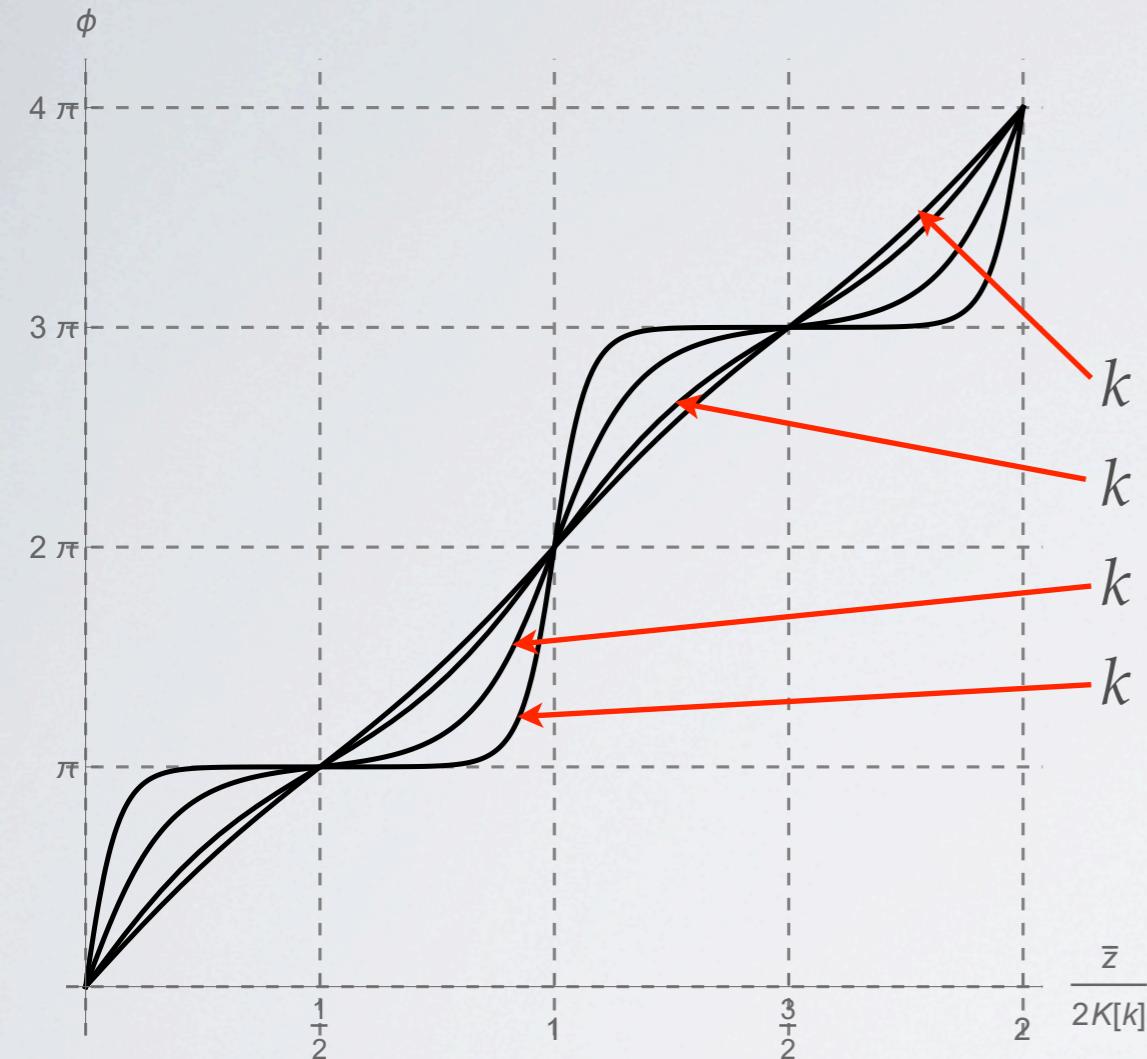
- ♦ Solution in the general case:

$$\cos \frac{\phi(\bar{z})}{2} = \text{sn}(\bar{z}, k), \quad \bar{z} = \frac{zm_\pi}{k}$$

Jacobi elliptic function Elliptic modulus

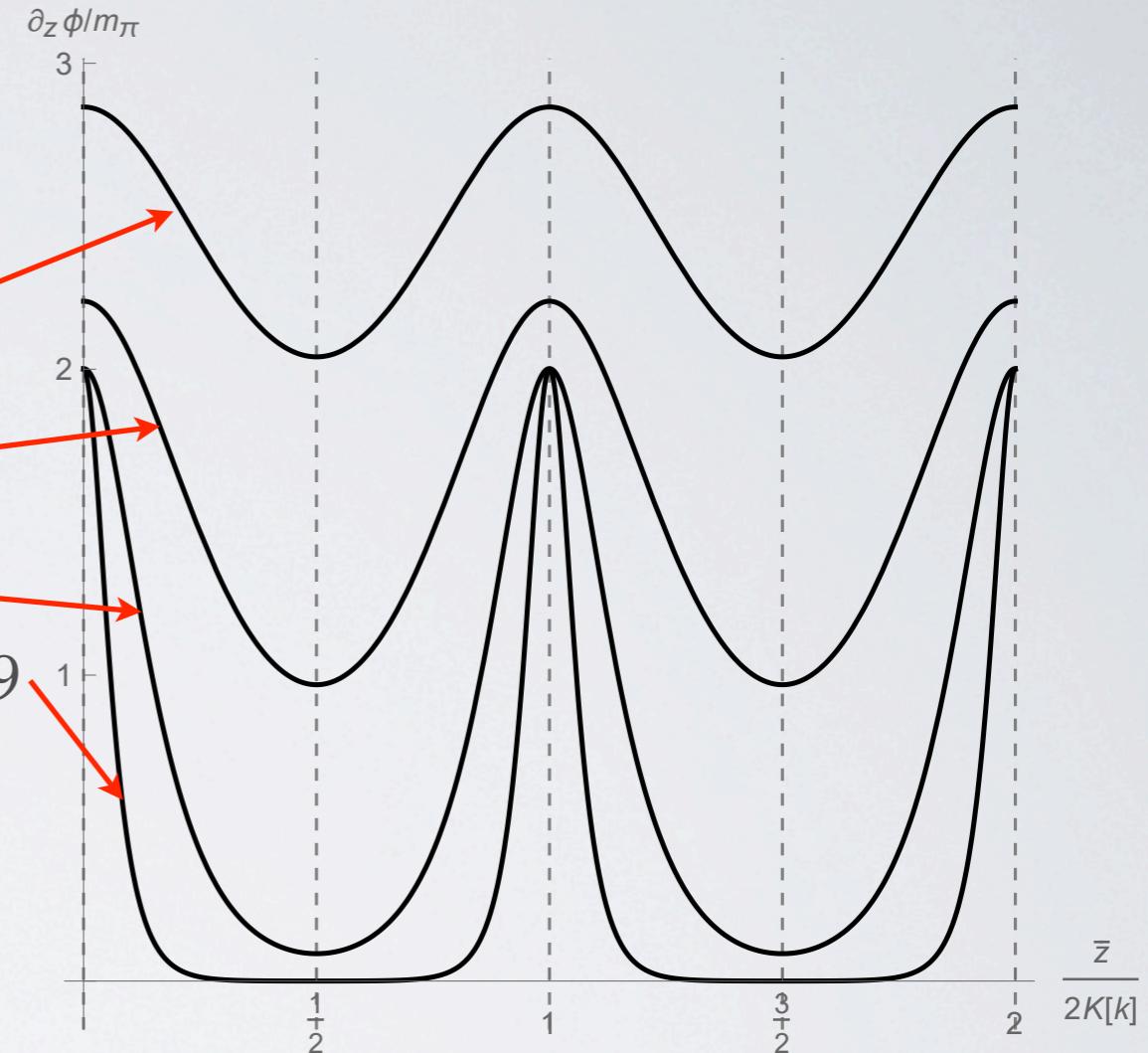
CHIRAL SOLITON LATTICE

The solution is a periodic lattice of topological solitons!



Phase of the solution

Kishine, Ovchinnikov,
Solid State Phys. 66 (2015)



Topological charge density

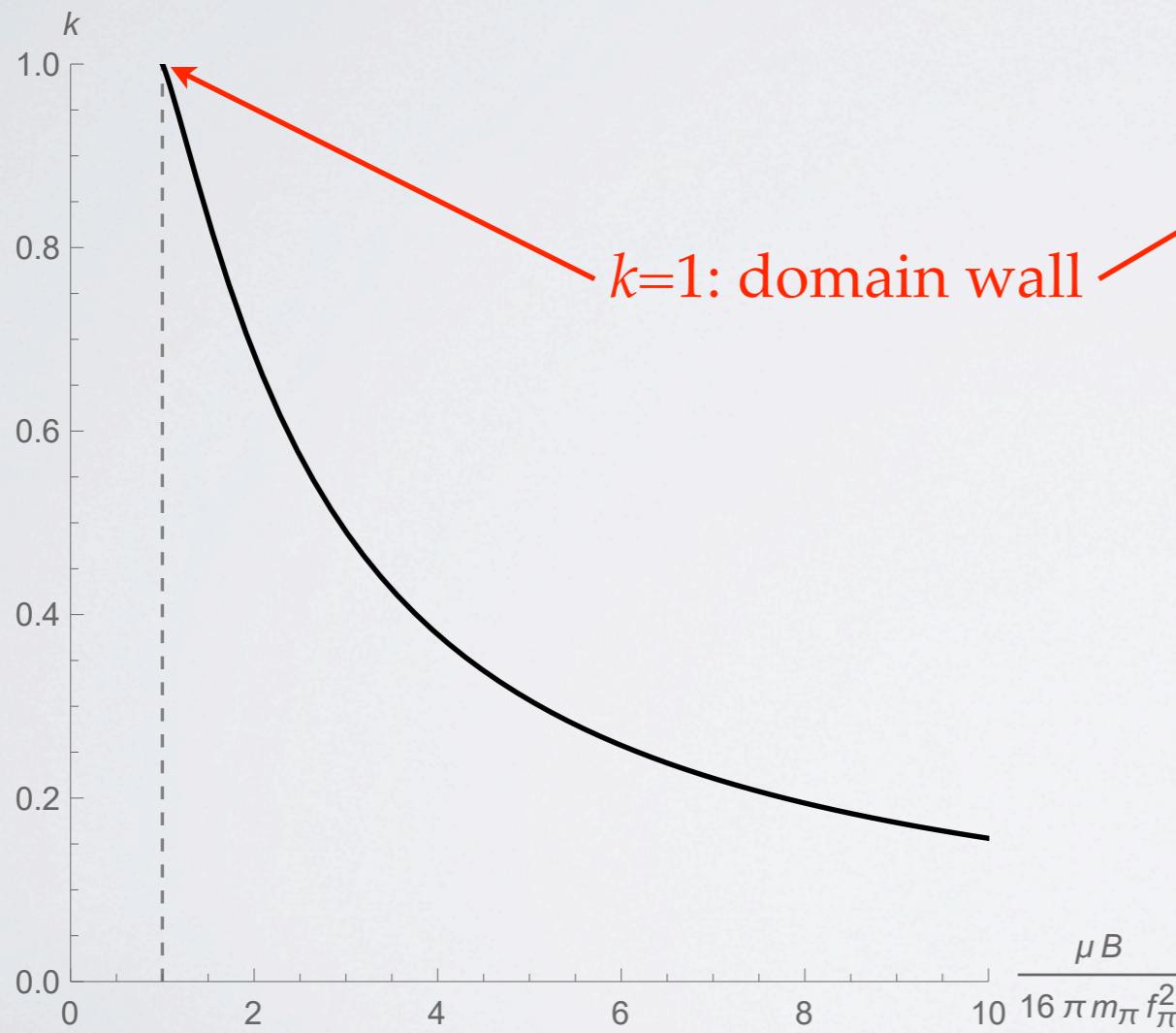
$$n_B(z) = \frac{B}{4\pi^2} \partial_z \phi(z)$$

$$m(z) = \frac{\mu}{4\pi^2} \partial_z \phi(z)$$

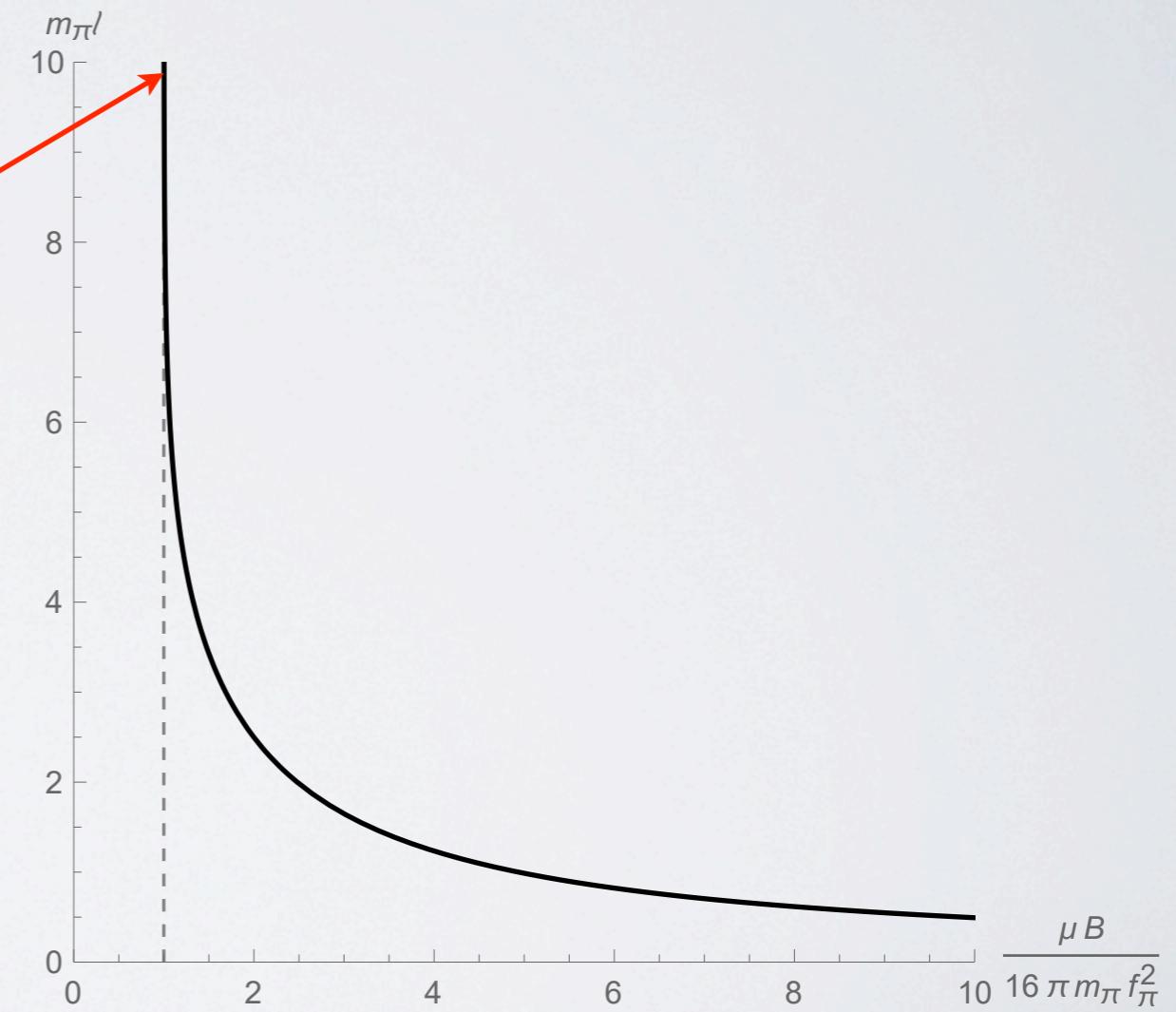
THE OPTIMUM CSL SOLUTION

The ground state is found by minimization of the Hamiltonian.

$$\frac{E(k)}{k} = \frac{\mu B}{16\pi m_\pi f_\pi^2}$$



Optimum value of elliptic modulus



Period of the lattice

CRITICAL MAGNETIC FIELD

CSL is energetically favored
above certain critical value of magnetic field.

$$\frac{E(k)}{k} = \frac{\mu B}{16\pi m_\pi f_\pi^2} \implies B_{\text{CSL}} = \frac{16\pi m_\pi f_\pi^2}{\mu}$$

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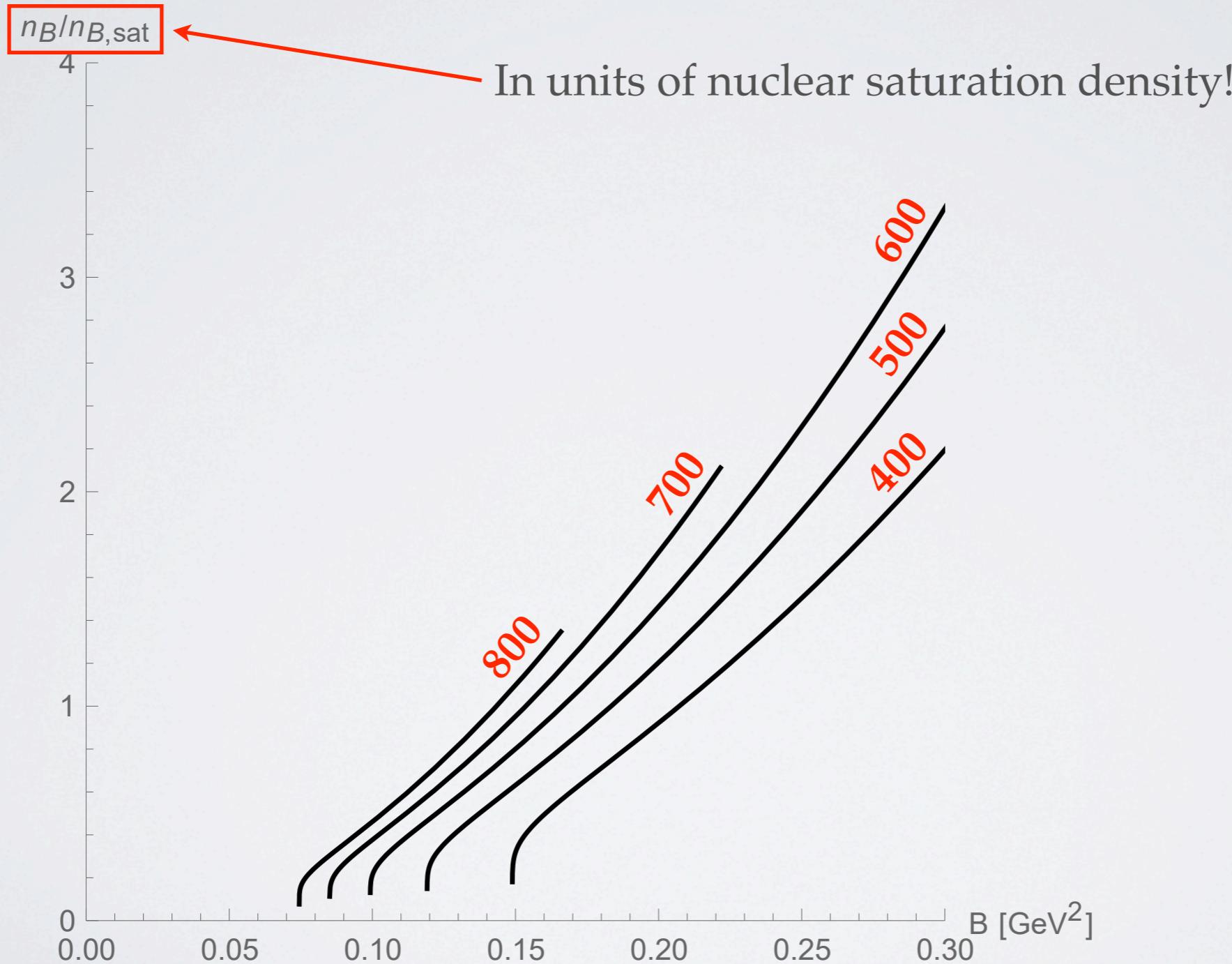
$$\left. \begin{array}{l} m_\pi \approx 140 \text{ MeV} \\ f_\pi \approx 92 \text{ MeV} \\ \boxed{\mu = 900 \text{ MeV}} \end{array} \right\} \implies B_{\text{CSL}} \approx 0.066 \text{ GeV}^2$$

Conversion factor: $1 \text{ GeV}^2 \approx 1.7 \times 10^{20} \text{ G}$

Could this possibly be relevant for neutron stars?

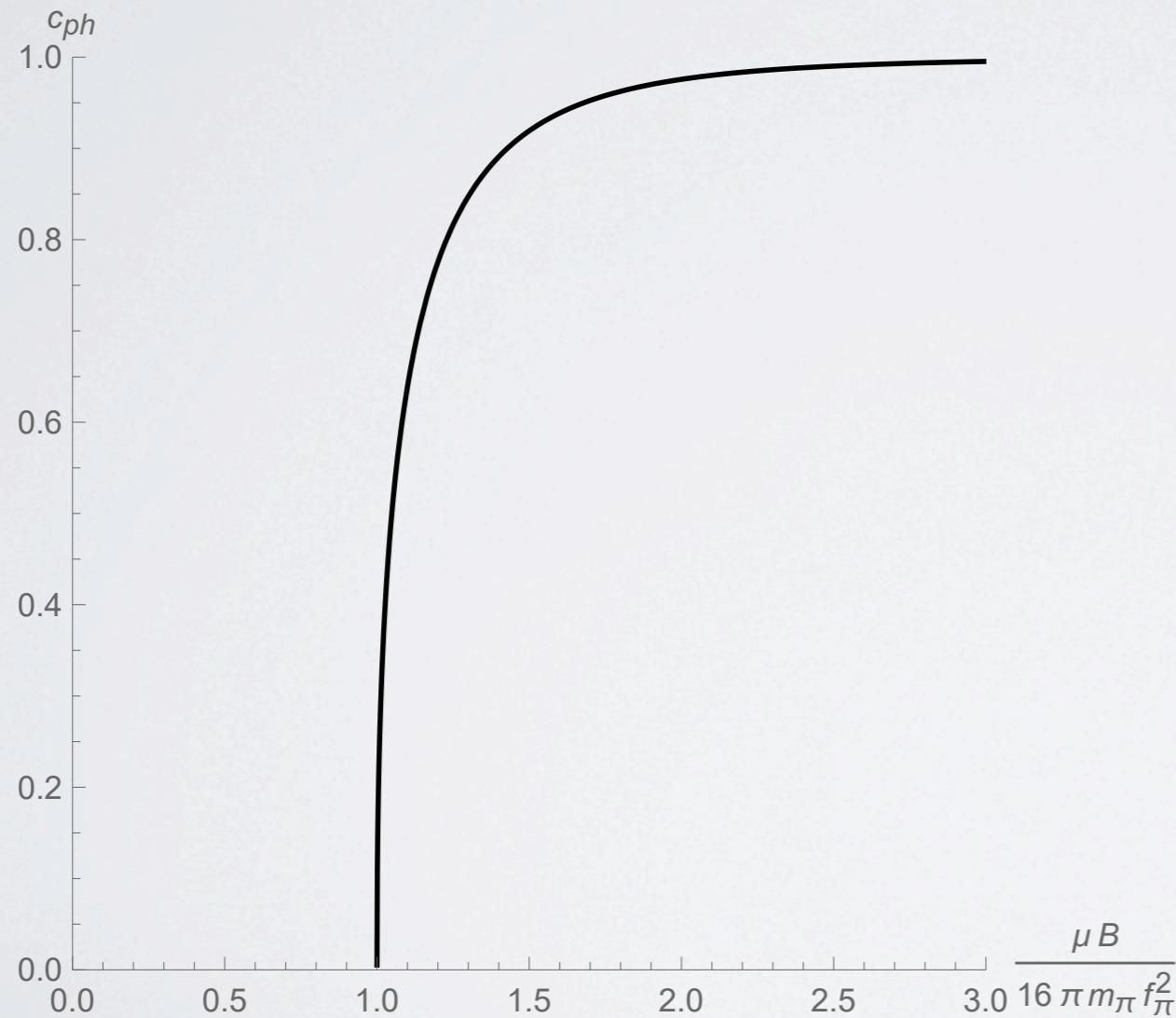
BARYON NUMBER DENSITY

What is the corresponding average baryon number density?



PHONONS OF THE SOLITON LATTICE

- ♦ Linearize the sine-Gordon equation around the CSL solution.
- ♦ Results in the **Lamé equation** ($n=1$) with known spectrum.
- ♦ Two-band, **gapless spectrum** in accord with the **Goldstone theorem**.



$$c_{ph} = \sqrt{1 - k^2} \frac{K(k)}{E(k)}$$

Phonon phase velocity

CHARGED PION FLUCTUATIONS

$$\mathcal{L} = \frac{f_\pi^2}{4} [\text{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2 \text{tr}(\Sigma + \Sigma^\dagger)]$$

$$\Sigma = e^{i\tau_3 \phi} e^{\frac{i}{f_\pi} \vec{\tau} \cdot \vec{\pi}}$$

↓
Expand to second order
in the field fluctuations

$$\mathcal{L}_{\text{bilin}} = \frac{1}{2} (\partial_\mu \vec{\pi})^2 + (A^\mu - \partial^\mu \phi)(\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1) + \frac{1}{2} A^\mu A_\mu (\pi_1^2 + \pi_2^2) - \frac{1}{2} m_\pi^2 \vec{\pi}^2 \cos \phi$$

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- ♦ CSL background acts as a chemical potential on charged pions!
- ♦ Chiral limit: easily solvable.
- ♦ General case: Lamé equation ($n=2$), solvable with some effort.

CHARGED PION SPECTRUM

- ♦ Chiral limit:

$$\omega = \sqrt{p_z^2 - \frac{\mu B p_z}{2\pi^2 f_\pi^2} + (2n+1)B}$$

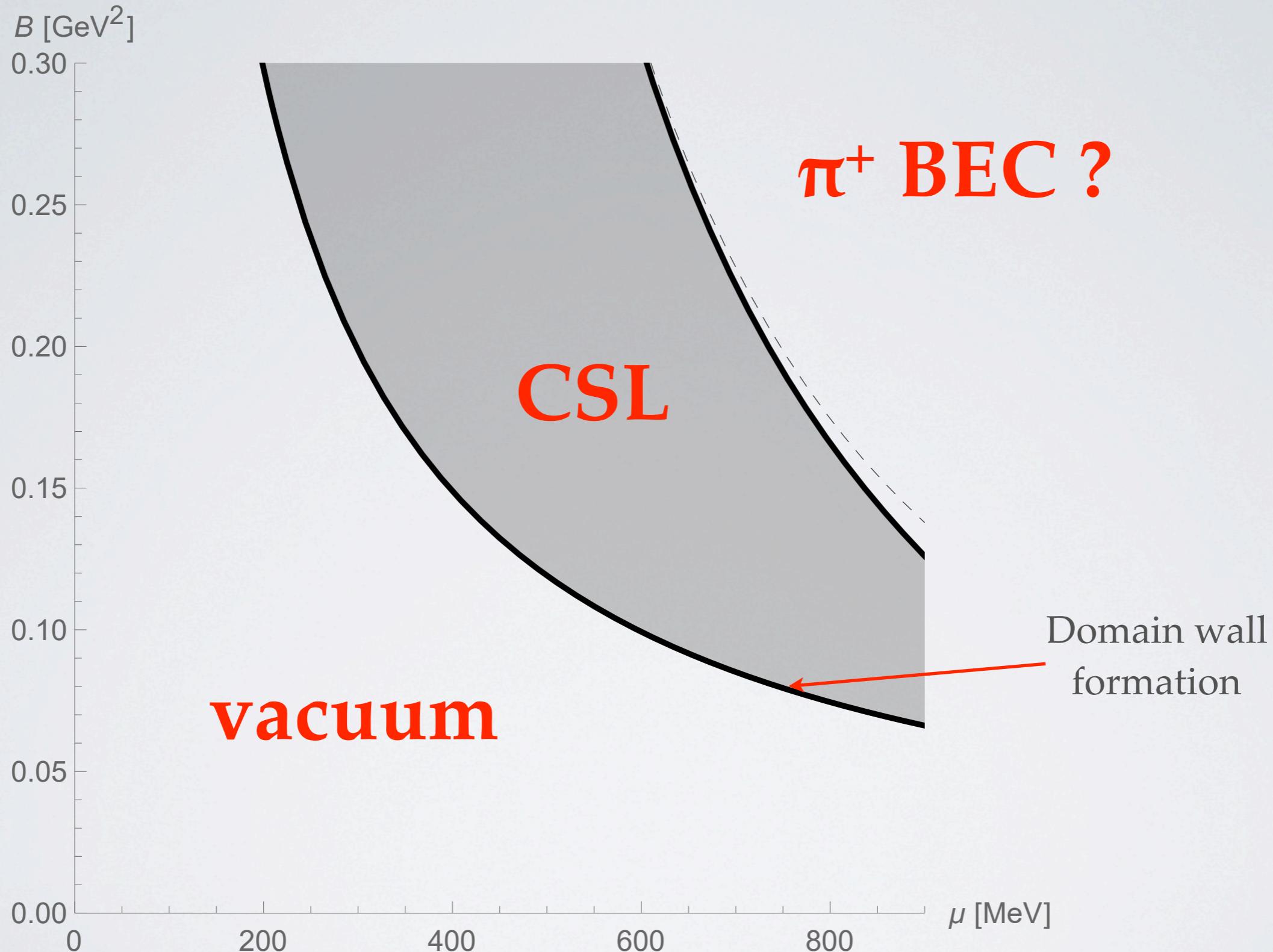
$$\min \omega^2 = B - \left(\frac{\mu B}{4\pi^2 f_\pi^2} \right)^2$$

- ♦ General case:

$$\min \omega^2 = B - \frac{m_\pi^2}{k^2} \left(2 - k^2 + 2\sqrt{1 - k^2 + k^4} \right)$$

The bottom of the lowest Landau level reaches zero
in sufficiently strong magnetic fields!

PHASE DIAGRAM



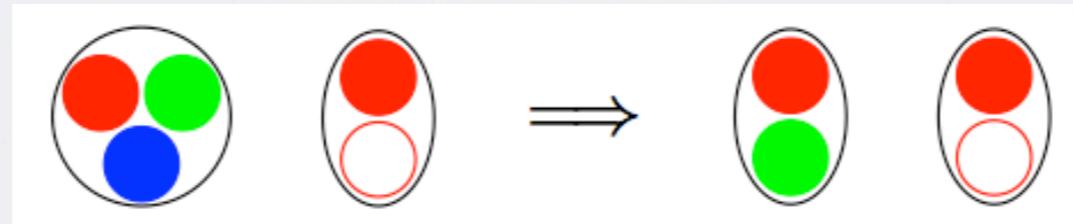
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TB, Yamamoto, JHEP 04 (2017)

CSL IN QCD-LIKE THEORIES

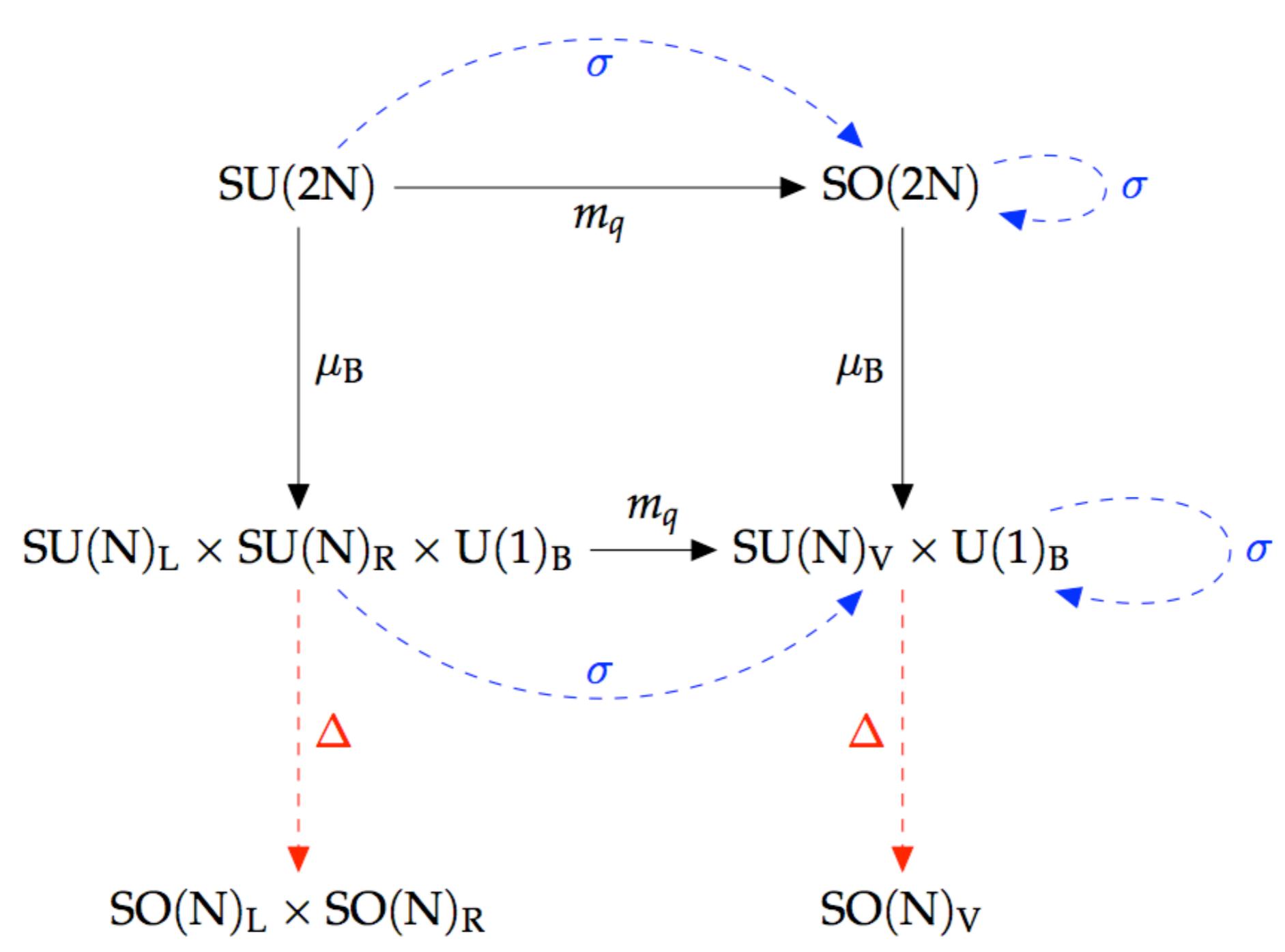
QCD-LIKE THEORIES AT $B=0$

- ♦ Quarks in a **(pseudo)real representation of gauge group**: theory invariant under the exchange $q_L \longleftrightarrow \bar{q}_R$.
 - ♦ **Real**: adjoint QCD, G_2 -QCD, ...
 - ♦ **Pseudoreal**: 2cQCD, ...
- ♦ Bosonic baryons in the spectrum:



- ♦ **Global flavor $SU(2N)$ symmetry** for N quark flavors.
- ♦ Low-energy spectrum determined by its spontaneous breaking by the chiral condensate.

SYMMETRY PATTERN: REAL THEORIES

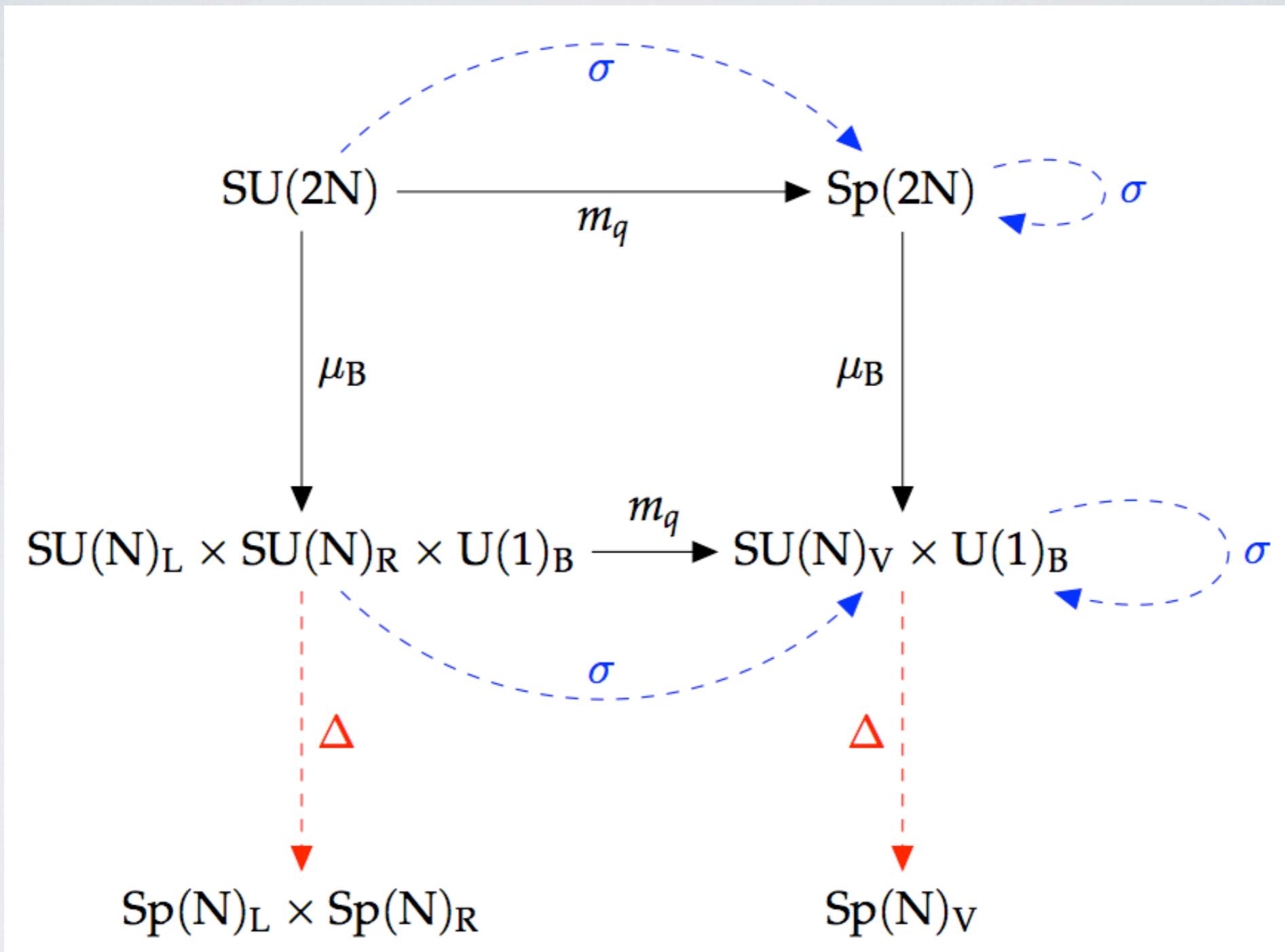


Pseudo-NG bosons in the vacuum:

- N^2-1 pions
- N^2+N diquarks

Kogut *et al.*, NPB 582 (2000)

SYMMETRY PATTERN: PSEUDOREAL THEORIES



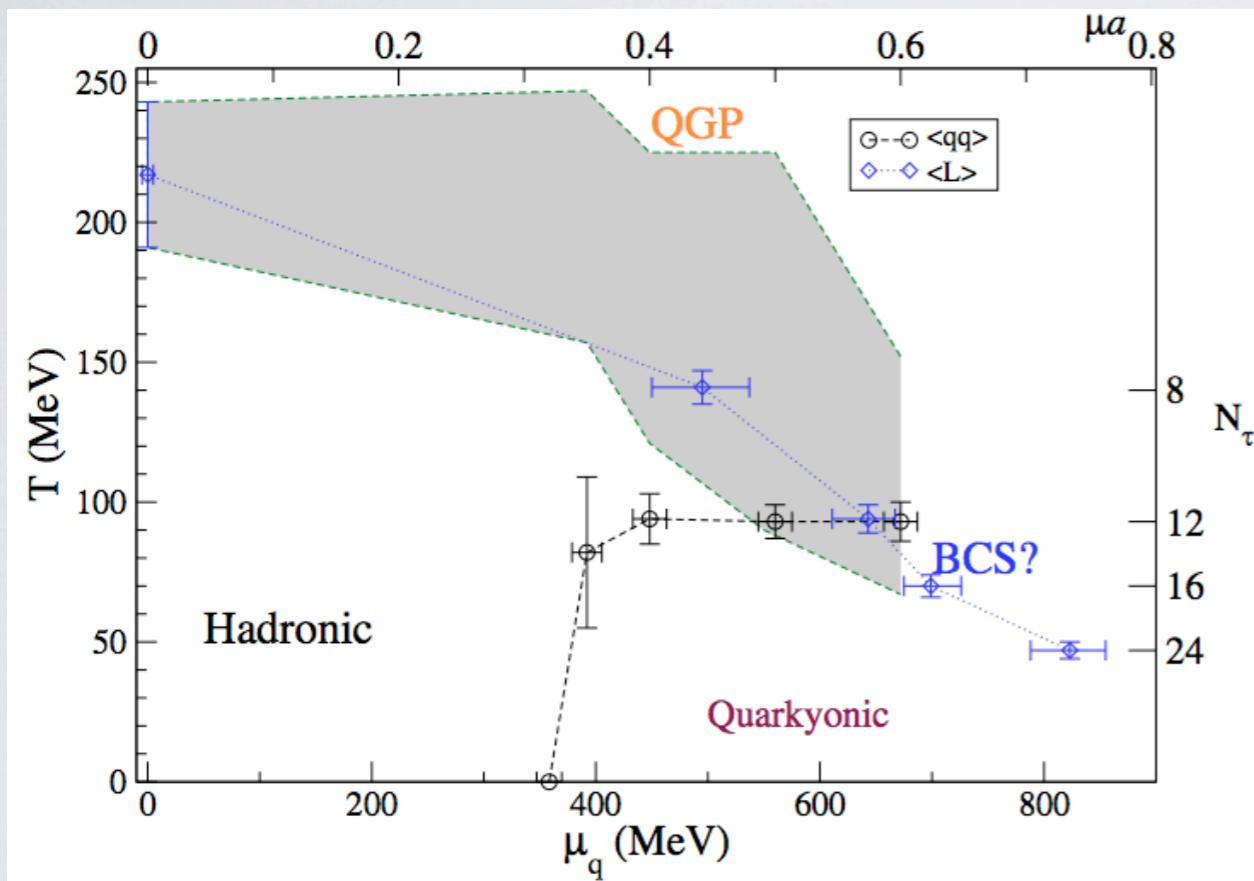
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Pseudo-NG bosons in the vacuum:

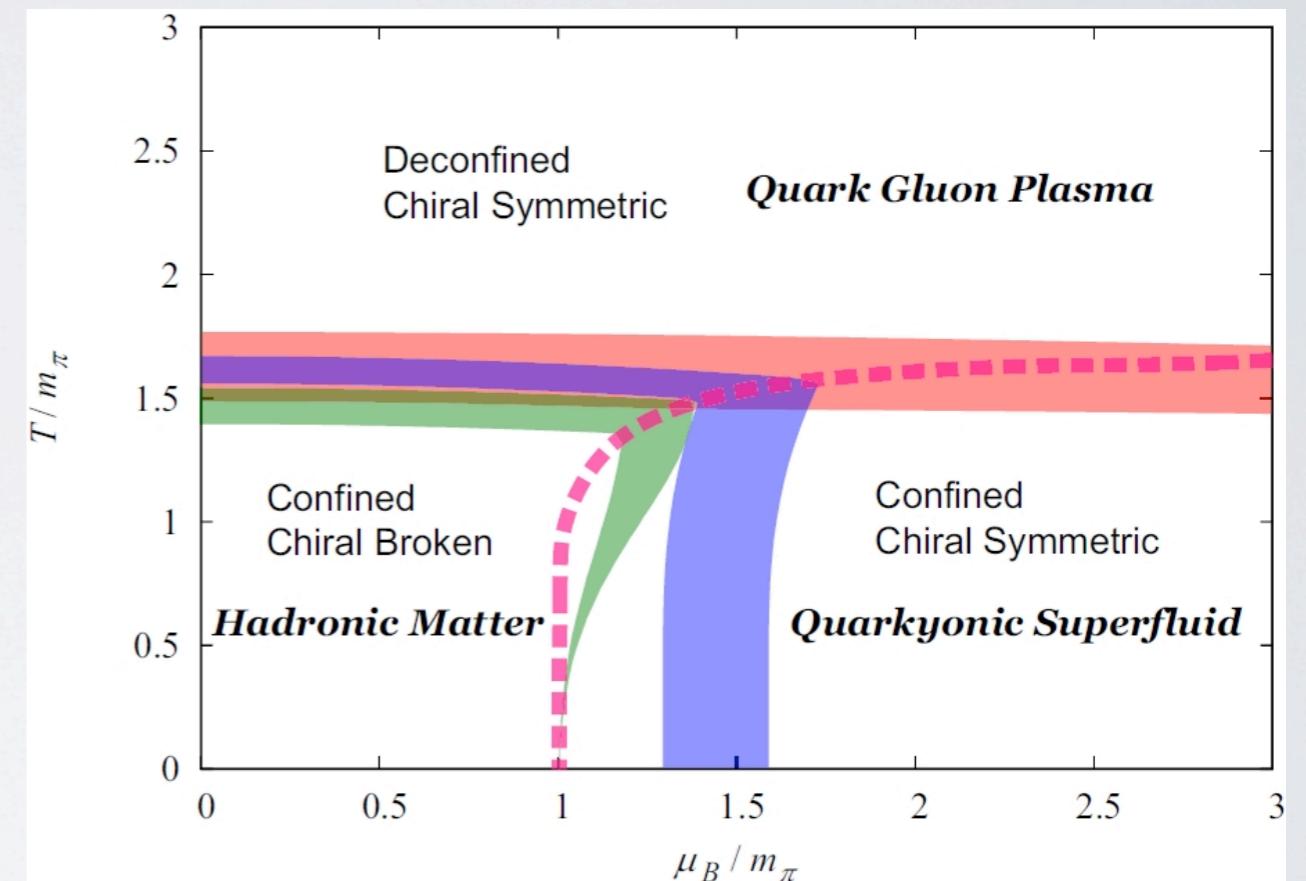
- N^2-1 pions
- N^2-N diquarks

PHASE DIAGRAM AT $B=0$ (2cQCD)

Boz *et al.*, EPJA **49** (2013)



TB, Fukushima, Hidaka, PRD **80** (2009)



ABSENCE OF SIGN PROBLEM

- ♦ Assume two flavors with $q_u = -q_d$.
- ♦ Conjugation property of Dirac operator:

$$(KC\gamma_5 \mathcal{P}) \mathcal{D}_u = \mathcal{D}_d (KC\gamma_5 \mathcal{P})$$

Dirac operator

Complex conjugation Conjugation of generators: $T_a^* = -\mathcal{P}T_a\mathcal{P}^{-1}$

```
graph TD; CC[Complex conjugation] --> KC1[KCγ5 P]; KC1 --> D1[D]; D1 --> KC2[KCγ5 P]; CG[Conjugation of generators: Ta* = -PTaP⁻¹] --> TA1[Ta]; TA1 --> D2[D]; D2 --> CG; D1 <--> D2;
```

- ♦ Guarantees positivity of Dirac determinant even with nonzero baryon chemical potential and magnetic field.

LOW-ENERGY EFT IN STRONG B -FIELDS

- ♦ Symmetry-breaking pattern (both real and pseudoreal theories):

$$\underbrace{\text{SU}(2) \times \text{SU}(2)}_{\text{Not the chiral group!}} \times \text{U}(1)_Q \rightarrow \text{SU}(2)_{\text{diag}} \times \text{U}(1)_Q$$

- ♦ Leading-order effective Lagrangian:

$$\mathcal{L} = \frac{f_\pi^2}{4} \left[\underbrace{(g_{\parallel}^{\mu\nu} + v^2 g_{\perp}^{\mu\nu})}_{\text{Anisotropy due to } B\text{-field}} \text{tr}(D_\mu \Sigma^\dagger D_\nu \Sigma) + m_\pi^2 \text{tr}(\Sigma + \Sigma^\dagger) \right] + \mathcal{L}_{\text{WZW}}$$

- ♦ All the couplings f_π , m_π , v depend on the B -field!

LOW-ENERGY EFT IN STRONG B -FIELDS

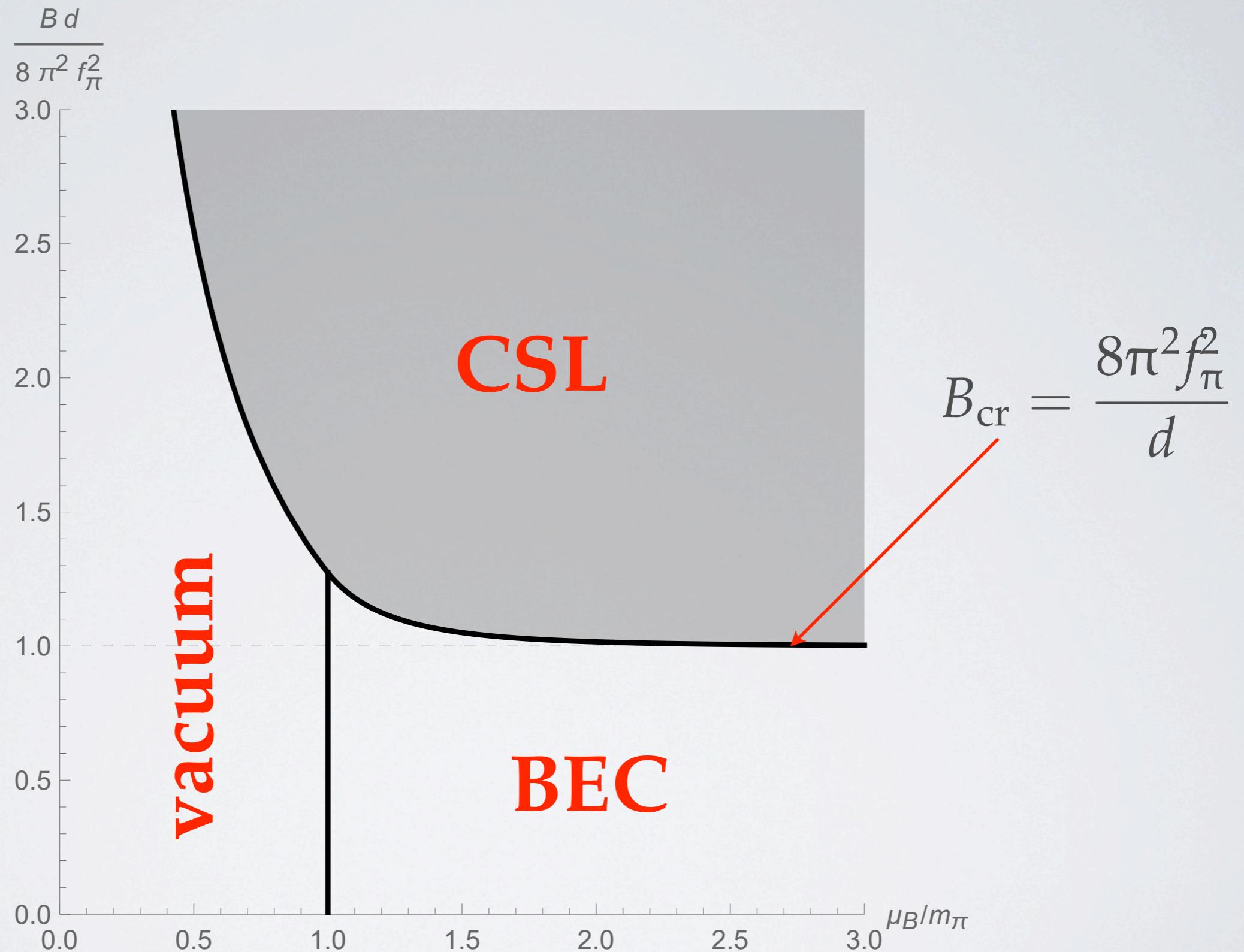
- ♦ Wess–Zumino–Witten term:

$$\begin{aligned}\mathcal{L}_{WZW} = & + \frac{i b C}{4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^Q A_\alpha^B \text{tr}[\tau_3 (\partial_\beta \Sigma \Sigma^\dagger - \partial_\beta \Sigma^\dagger \Sigma)] \\ & - \frac{C}{6} \epsilon^{\mu\nu\alpha\beta} A_\mu^Q \text{tr}(\partial_\nu \Sigma \partial_\alpha \Sigma^\dagger \partial_\beta \Sigma \Sigma^\dagger)\end{aligned}$$

$$C = \frac{d}{8\pi^2} (q_u - q_d)$$

of quark color d.o.f.

PHASE DIAGRAM



WHY IS THIS INTERESTING?

- ♦ Makes the presence of CSL testable on the lattice in a setting free of the sign problem.
- ♦ Until now, few attempts at lattice simulation of inhomogeneous phases.

de Forcrand, Wenger, PoS LAT**2006** (2006)

Yamamoto, PRL **112** (2014)

Pannullo, Lenz, Wagner, Welleghausen, Wipf, 1902.11066

Winstel, Stoll, Wagner, 1909.00064

- ♦ Disproves the conjecture that absence of sign problem implies absence of inhomogeneous order in the phase diagram.

Splittorff, Son, Stephanov, PRD **64** (2001)

CONCLUSIONS

CONCLUSIONS

- ♦ In sufficiently strong magnetic fields, QCD vacuum is unstable under formation of a soliton lattice of neutral pions.
- ♦ This can generate baryon densities relevant for neutron stars, although the required magnetic field seems too strong.
- ♦ The same inhomogeneous phase appears in other theories:
 - ♦ QCD with isospin chemical potential.
 - ♦ QCD-like theories with (pseudo)real quarks.
- ♦ There are many avenues for further investigation:
 - ♦ Effects of nonzero temperature?
 - ♦ Modification in finite volume?
 - ♦ Modification due to inhomogeneous magnetic fields?
 - ♦ Competition with nuclear matter?