Antiheavy-antiheavy-light-light ($\bar{Q}\bar{Q}qq$) tetraquarks from lattice QCD

"SIFT 2024: Strongly Interacting Field Theories"

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Why $\bar{Q}\bar{Q}qq$ tetraquarks?

- For almost 10 years several independent lattice QCD groups study the existence and properties of
  - $\bar{b}\bar{b}ud$ tetraquarks (not discussed in this talk),
  - $\bar{b}\bar{b}us$ tetraquarks (part 1 of this talk, straightforward [a deeply bound state]),
  - $\bar{b}\bar{c}ud$ tetraquarks (part 2 of this talk, difficult [scattering theory needed]).

  Theoretical motivation:

  - $\bar{Q}\bar{Q}qq$ systems are systems of at least four quarks, because quark-antiquark anihilation is not possible.
    $\rightarrow$ Simpler to study than e.g. $\bar{Q}Q(\bar{q}q)$ or $\bar{q}q(\bar{q}q)$ tetraquarks.
  - $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$ and $\bar{b}\bar{b}us$ with $J^P = 1^+$ are QCD-stable.
    $\rightarrow$ Very straightforward to study. (Just check, whether the ground state of the system is below the corresponding 2-meson threshold.)

  Experimental motivation:

  - Related $T_{cc}^+(\bar{c}\bar{c}ud)$ recently discovered by LHCb.
    [R. Aaij *et al.* [LHCb], Nature Commun. **13**, 3351 (2022) [arXiv:2109.01056]]
  - $\bar{b}\bar{c}qq$ might "soon" be within experimental reach.

Why lattice QCD?

- Lattice QCD = full QCD (numerically with high performance computers) ... i.e. no assumptions, no approximations, etc. needed.

- A lattice QCD result, if generated in a technically sound and solid way, is a full QCD result and can be confronted with experiment in a direct and meaningful way.

- However, lattice QCD is technically difficult, in particular, when studying exotic hadrons, e.g. $\bar{Q}\bar{Q}qq$ tetraquarks.

  $\rightarrow$ Often lattice QCD studies are not yet fully rigorous, i.e. certain assumptions are made, quark masses are unphysical, no continuum and/or infinite volume limit, no convincing separation and extraction of low-lying energy eigenstates, etc.

  $\rightarrow$ Important to read (at least some) technical details of lattice QCD papers, to be able to judge their quality.

Existing work and references (1)

- Summary of current status (only full lattice QCD results):
  - $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$:
    A **QCD-stable tetraquark** around $130\,\text{MeV}$ below the BB^* threshold.
  - $\bar{b}\bar{b}us$ with $J^P = 1^+$: **(part 1 of this talk)**
    A **QCD-stable tetraquark** around $90\,\text{MeV}$ below the BB_s^* threshold.
    * Masses of stable hadrons correspond to energy eigenvalues at infinite volume (for shallow bound states, it might be difficult to study this limit).
  - $\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$ and with $I(J^P) = 0(1^+)$: **(part 2 of this talk)**
    Contradictory results. The technically most advanced study points towards **very shallow bound states**, i.e. QCD-stable tetraquarks slightly below the BD and B^*D thresholds.
    * Masses and decay widths of resonances/shallow bound states can be calculated from the volume dependence of the energy eigenvalues (difficult).
  - $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^-)$:
    No full lattice QCD investigation yet. Antistatic-antistatic lattice QCD potentials and the Born-Oppenheimer approximation suggest the existence of a **tetraquark resonance** close to the B^*B^* threshold (which is not the lowest meson-meson threshold).

Existing work and references (2)

- This talk is mainly a summary of our recent works

  [S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. D **106**, 034507 (2022) [arXiv:2205.13982]] ($\bar{b}\bar{b}us$, $\bar{b}\bar{c}ud$)

  [C. Alexandrou, J. Finkenrath, T. Leontiou, S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. Lett. **132**, 151902 (2024) [arXiv:2312.02925]] ($\bar{b}\bar{c}ud$)

- Related lattice QCD works on $\bar{Q}\bar{Q}qq$ tetraquarks:

  [A. Francis, R. J. Hudspith, R. Lewis, K. Maltman, Phys. Rev. Lett. **118**, 142001 (2017) [arXiv:1607.05214]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)

  [A. Francis, R. J. Hudspith, R. Lewis, K. Maltman, Phys. Rev. D **99**, 054505 (2019) [arXiv:1810.10550]] ($\bar{b}\bar{c}ud$)

  [P. Junnarkar, N. Mathur, M. Padmanath, Phys. Rev. D **99**, 034507 (2019) [arXiv:1810.12285]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)

  [L. Leskovec, S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. D **100**, 014503 (2019) [arXiv:1904.04197]] ($\bar{b}\bar{b}ud$)

  [R. J. Hudspith, B. Colquhoun, A. Francis, R. Lewis, K. Maltman, Phys. Rev. D **102**, 114506 (2020) [arXiv:2006.14294]] ($\bar{b}\bar{c}ud$)

  [P. Mohanta, S. Basak, Phys. Rev. D **102**, 094516 (2020) [arXiv:2008.11146]] ($\bar{b}\bar{b}ud$)

  [R. J. Hudspith, D. Mohler, Phys. Rev. D **107**, 114510 (2023) [arXiv:2303.17295]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)

  [T. Aoki, S. Aoki, T. Inoue, Phys. Rev. D **108**, 054502 (2023) [arXiv:2306.03565]] ($\bar{b}\bar{b}ud$)

  [M. Padmanath, A. Radhakrishnan and N. Mathur, Phys. Rev. Lett. **132**, 20 (2024) [arXiv:2307.14128]] ($\bar{b}\bar{c}ud$)

  [C. Alexandrou, J. Finkenrath, T. Leontiou, S. Meinel, M. Pflaumer and M. Wagner, Phys. Rev. D **110**, 054510 (2024) [arXiv:2404.03588]] ($\bar{b}\bar{b}ud$, $\bar{b}\bar{b}us$)

Part 1:
Masses of QCD-stable $\bar{Q}\bar{Q}qq$ tetraquarks: eigenvalues of the QCD Hamiltonian
(mainly $\bar{b}\bar{b}us$ with $J^P = 0^+$)

[S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. D **106**, 034507 (2022) [arXiv:2205.13982]]

Basics of lattice hadron spectroscopy (1)

- Masses of QCD-stable hadrons (e.g. the mass of a $\bar{b}\bar{b}us$ tetraquark) correspond to low-lying energy eigenvalues E_n with matching quantum numbers (typically the ground state energy E_0) and are determined from the exponential decays of temporal correlation functions $C_{jk}(t)$ of (hadron creation) operators O_j:

$$C_{jk}(t) = \langle\Omega|O_j^\dagger(t)O_k(0)|\Omega\rangle = \langle\Omega|O_j^\dagger|0\rangle\langle 0|O_k|\Omega\rangle e^{-E_0 t} + \langle\Omega|O_j^\dagger|1\rangle\langle 1|O_k|\Omega\rangle e^{-E_1 t} + \ldots$$

  - $C_{jk}(t)$ can be computed with lattice QCD.
  - The analytical expression on the right hand side is used to determine E_0, E_1, ...

Basics of lattice hadron spectroscopy (2)

- $C_{jk}(t) = \langle \Omega | O_j^\dagger(t) O_k(0) | \Omega \rangle = \langle \Omega | O_j^\dagger | 0 \rangle \langle 0 | O_k | \Omega \rangle e^{-E_0 t} + \langle \Omega | O_j^\dagger | 1 \rangle \langle 1 | O_k | \Omega \rangle e^{-E_1 t} + \ldots$
- In principle one can use any operator O_j, which generates the same quantum numbers as the hadron of interest. (but then you have to compute $C_{jk}(t)$ precisely for very large t ...)
- In practice one needs operators with the following properties:
  - The operators have to generate large overlap to the low-lying energy eigenstates (not only the hadron of interest, but also multi-particle states of similar mass).
  - There must be at least one operator for each low-lying state.
  - The operators must not be too similar (ideally "they are almost orthogonal").

  Otherwise it is questionable, whether an analysis correctly extracts E_0, E_1, ... from the correlation function $C_{jk}(t)$.
  A major problem is that such analyses always provide numbers, but these might be wrong ... e.g. one could obtain $\approx (E_0 + E_1)/2$ instead of E_0, if one does not use both bound state and scattering operators.
- **We improve on existing lattice QCD studies by considering both local and scattering operators for $\bar{Q}\bar{Q}qq$ systems. This allows a more trustworthy and precise extraction of energy eigenvalues as well as to carry out scattering analyses.**

Lattice setup

- Five ensembles of gauge link configurations generated with 2+1 quark flavors by the **RBC and UKQCD collaboration**. These have different volumes, different lattice spacings and different light quark masses.

ensemble	$N_s^3 \times N_t$	a [fm]	m_π [MeV]
C00078	$48^3 \times 96$	$0.1141(3)$	$139(1)$
C005	$24^3 \times 64$	$0.1106(3)$	$340(1)$
C01	$24^3 \times 64$	$0.1106(3)$	$431(1)$
F004	$32^3 \times 64$	$0.0828(3)$	$303(1)$
F006	$32^3 \times 64$	$0.0828(3)$	$360(1)$

[Y. Aoki *et al.* [RBC and UKQCD], Phys. Rev. D **83**, 074508 (2011) [arXiv:1011.0892]]
[T. Blum *et al.* [RBC and UKQCD], Phys. Rev. D **93**, 074505 (2016) [arXiv:1411.7017]]

- Domain-wall action for u, d and s quarks.

- NRQCD action for valence b quarks, anisotropic clover action for valence c quarks.

- Local operators (representing bound states) and scattering operators (representing meson-meson states).

- Scattering operators only at one end of the correlation functions, because we were using existing point-to-all-operators. (for scattering operators at both ends see 2404.03588)

$\bar{b}\bar{b}us$ with $J^P = 1^+$: operators

- Local operators (at the source and at the sink):

$$O_1 = O_{[BB_s^*](0)} = \sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\, \bar{b}\gamma_j s(\mathbf{x}) \quad (BB_s^* \text{ bound state})$$

$$O_2 = O_{[B^*B_s](0)} = \sum_{\mathbf{x}} \bar{b}\gamma_j u(\mathbf{x})\, \bar{b}\gamma_5 s(\mathbf{x}) \quad (B^*B_s \text{ bound state})$$

$$O_3 = O_{[B^*B_s^*](0)} = \epsilon_{jkl} \sum_{\mathbf{x}} \bar{b}\gamma_k u(\mathbf{x})\, \bar{b}\gamma_l s(\mathbf{x}) \quad (B^*B_s^* \text{ bound state})$$

$$O_4 = O_{[Dd](0)} = \sum_{\mathbf{x}} \bar{b}^a \gamma_j \mathcal{C} \bar{b}^{b,T}(\mathbf{x})\, u^{a,T} \mathcal{C}\gamma_5 s^b(\mathbf{x}) \quad (\text{diquark-antidiquark}).$$

- Scattering operators (only at the sink):

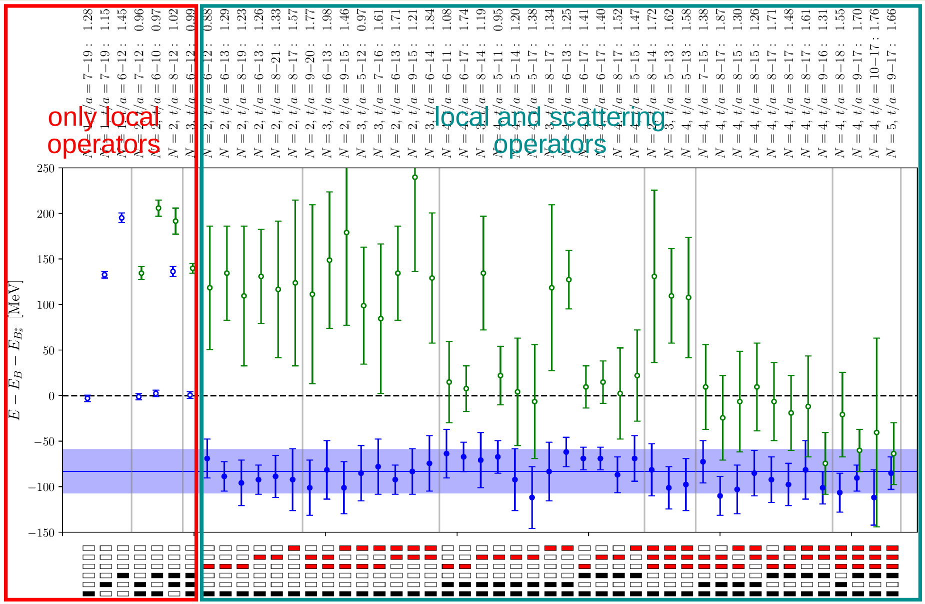
$$O_5 = O_{B(0)B_s^*(0)} = \Big(\sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\Big)\Big(\sum_{\mathbf{y}} \bar{b}\gamma_j s(\mathbf{y})\Big) \quad (BB_s^* \text{ 2-particle state})$$

$$O_6 = O_{B^*(0)B_s(0)} = \Big(\sum_{\mathbf{x}} \bar{b}\gamma_j u(\mathbf{x})\Big)\Big(\sum_{\mathbf{y}} \bar{b}\gamma_5 s(\mathbf{y})\Big) \quad (B^*B_s \text{ 2-particle state})$$

$$O_7 = O_{B^*(0)B_s^*(0)} = \epsilon_{jkl}\Big(\sum_{\mathbf{x}} \bar{b}\gamma_k u(\mathbf{x})\Big)\Big(\sum_{\mathbf{y}} \bar{b}\gamma_l s(\mathbf{y})\Big) \quad (B^*B_s^* \text{ 2-particle state}).$$

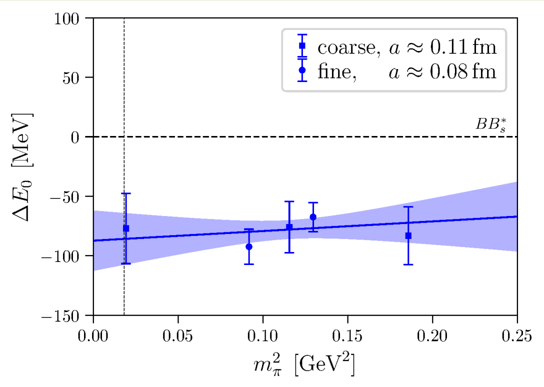
$\bar{b}\bar{b}us$ with $J^P = 1^+$: energy levels

- Plot: Energy levels $\Delta E_n = E_n - E_B - E_{B_s^*}$ for ensemble C01 obtained with various operator subsets and temporal fitting ranges.
- Only local operators $\rightarrow$ $\Delta E_0 \approx 0\,$MeV.
- Local and scattering operators $\rightarrow$ $\Delta E_0 \approx -100\,$MeV, $\Delta E_1 \approx 0\,$MeV.
  $\rightarrow$ Ground state corresponds to a QCD-stable tetraquark.

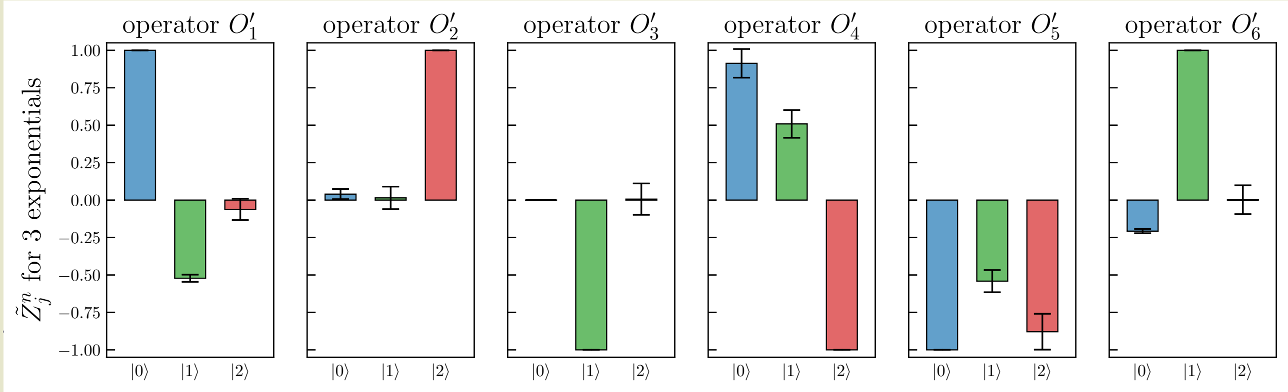


$\bar{b}\bar{b}us$ with $J^P = 1^+$: final results

- Bottom plot: Overlaps of each operator to the lowest three energy eigenstates (O'_1 to O'_3 are linear combinations of O_1 to O_4, O'_4 to O'_6 correspond to O_5 to O_7).
  - Roughly equal contributions to the ground state from a local BB^*_s / B^*B_s operator ("$I = 0$") ...
  - ... and a local $B^*B^*_s$ operator, ...
  - ... a smaller but still sizable contribution from a diquark-antidiquark operator.

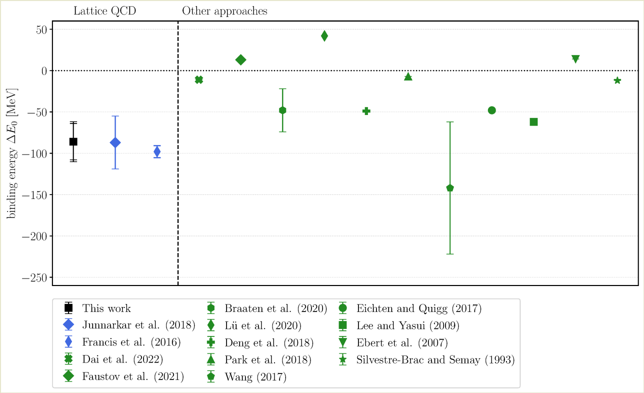


- Right plot: Almost no light quark mass dependence.
  $\rightarrow$ $\Delta E_0(m_{\pi,\text{phys}}) = (-86 \pm 22 \pm 10)$ MeV,
  $m_{\bar{b}\bar{b}us\ \text{tetraquark}}(m_{\pi,\text{phys}}) = (10609 \pm 22 \pm 10)$ MeV.



$\bar{b}\bar{b}us$ with $J^P = 1^+$: existing results

- Lattice QCD results from three independent groups (Francis et al., Junnarkar et al., our work) consistent within statistical errors.
- Strong discrepancies between non-lattice QCD results.



$\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$: operators

- Local operators (at the source and at the sink):

$$O_1 = O_{[BD](0)} = \sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\, \bar{c}\gamma_5 d(\mathbf{x}) - (u \leftrightarrow d) \quad (BD \text{ bound state})$$

$$O_2 = O_{[Dd](0)} = \sum_{\mathbf{x}} \bar{b}^a \gamma_5 \mathcal{C} \bar{c}^{b,T}(\mathbf{x})\, u^{a,T} \mathcal{C} \gamma_5 d^b(\mathbf{x}) - (u \leftrightarrow d) \quad (\text{diquark-antidiquark}).$$

- Scattering operators (only at the sink):

$$O_3 = O_{B(0)D(0)} = \Big(\sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\Big) \Big(\sum_{\mathbf{y}} \bar{c}\gamma_5 d(\mathbf{y})\Big) - (u \leftrightarrow d) \quad (BD \text{ 2-particle state}).$$

$\bar{b}\bar{c}ud$ with $I(J^P) = 0(1^+)$: operators

- Local operators (at the source and at the sink):

$$O_1 = O_{[B^* D](0)} = \sum_{\mathbf{x}} \bar{b}\gamma_j u(\mathbf{x})\, \bar{c}\gamma_5 d(\mathbf{x}) - (u \leftrightarrow d) \quad (B^* D \text{ bound state})$$

$$O_2 = O_{[B D^*](0)} = \sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\, \bar{c}\gamma_j d(\mathbf{x}) - (u \leftrightarrow d) \quad (B D^* \text{ bound state})$$

$$O_3 = O_{[Dd](0)} = \sum_{\mathbf{x}} \bar{b}^a \gamma_j \mathcal{C} \bar{c}^{b,T}(\mathbf{x})\, u^{a,T} \mathcal{C}\gamma_5 d^b(\mathbf{x}) - (u \leftrightarrow d) \quad (\text{diquark-antidiquark}),$$

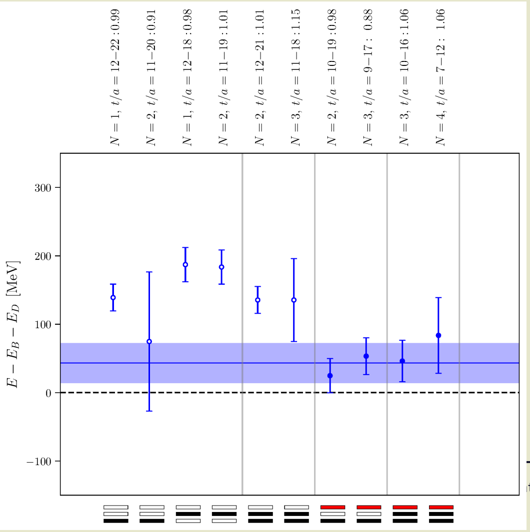
- Scattering operators (only at the sink):

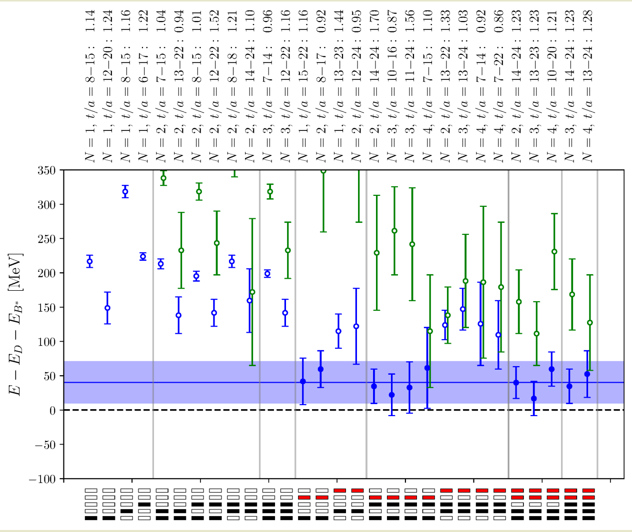
$$O_4 = O_{B^*(0)D(0)} = \bigg(\sum_{\mathbf{x}} \bar{b}\gamma_j u(\mathbf{x})\bigg) \bigg(\sum_{\mathbf{y}} \bar{c}\gamma_5 d(\mathbf{y})\bigg) - (u \leftrightarrow d) \quad (B^* D \text{ 2-particle state})$$

$$O_5 = O_{B(0)D^*(0)} = \bigg(\sum_{\mathbf{x}} \bar{b}\gamma_5 u(\mathbf{x})\bigg) \bigg(\sum_{\mathbf{y}} \bar{c}\gamma_j d(\mathbf{y})\bigg) - (u \leftrightarrow d) \quad (B D^* \text{ 2-particle state}).$$

$\bar{b}\bar{c}ud$: energy levels

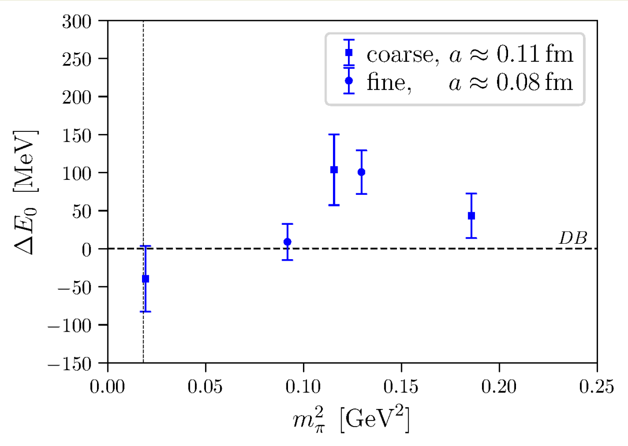
- Left plot: $I(J^P) = 0(0^+)$, energy levels $\Delta E_j = E_j - E_B - E_D$ for ensemble C01 obtained with various operator subsets and temporal fitting ranges.
- Right plot: $I(J^P) = 0(1^+)$, energy levels $\Delta E_j = E_j - E_{B^*} - E_D$ for ensemble C01 obtained with various operator subsets and temporal fitting ranges.
- Ground states always consistent with or above the lowest meson-meson thresholds.
  $\rightarrow$ No indication for the existence of a QCD-stable tetraquark.
  $\rightarrow$ Operator overlaps support this, i.e. suggest that the ground states are meson-meson scattering states.

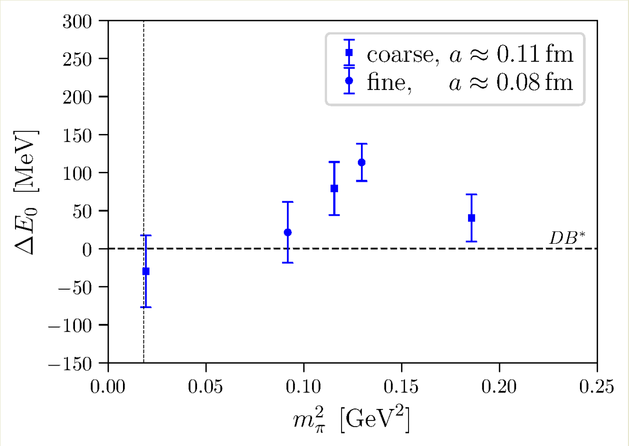




$\bar{b}\bar{c}ud$: final results

- Left plot: $I(J^P) = 0(0^+)$, ensemble dependence of ground state energy.
- Right plot: $I(J^P) = 0(1^+)$, ensemble dependence of ground state energy.
- To exclude the existence of a shallow bound state with binding energy of only a few MeV, more precise data and an infinite volume extrapolation is needed.





Part 2:

Finite volume scattering analysis for $\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$ and $I(J^P) = 0(1^+)$

[C. Alexandrou, J. Finkenrath, T. Leontiou, S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. Lett. **132**, 151902 (2024) [arXiv:2312.02925]]

$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (1)

- T_{cc} ($\bar{c}\bar{c}ud$ with $I(J^P) = 0(1^+)$): slightly below the DD^* threshold, almost QCD-stable. (experiment)
- $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$: ≈ 100 MeV below the DD^* threshold, QCD-stable. (lattice QCD)
- What about $\bar{b}\bar{c}ud$ with $I(J^P) = 0(1^+)$ (and also $I(J^P) = 0(0^+)$)?
  - Physics might be somewhat different, because of non-identical heavy quark flavors.
  - Existing lattice studies contradictory or inconclusive.
    [A. Francis, R. J. Hudspith, R. Lewis, K. Maltman, Phys. Rev. D **99**, 054505 (2019) [arXiv:1810.10550]] (hints for a bound state)
    [R. J. Hudspith, B. Colquhoun, A. Francis, R. Lewis, K. Maltman, Phys. Rev. D **102**, 114506 (2020) [arXiv:2006.14294]] (previous hints disappeared)
    [S. Meinel, M. Pflaumer, M.W., Phys. Rev. D **106**, 034507 (2022) [arXiv:2205.13982]] (no evidence for a bound state, a shallow bound state could not be ruled out [**part 1 of this talk**])
    [M. Padmanath, A. Radhakrishnan, N. Mathur, Phys. Rev. Lett. **132**, 20 (2024) [arXiv:2307.14128]] (bound state ≈ 43 MeV below the BD^* threshold via Lüscher's method)
  - Expected to be close to the B^*D threshold.
    $\rightarrow$ Lattice QCD studies technically difficult.

$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (2)

- In the following a summary of
  [C. Alexandrou, J. Finkenrath, T. Leontiou, S. Meinel, M. Pflaumer, M. Wagner, Phys. Rev. Lett. **132**, 151902 (2024) [arXiv:2312.02925]]
  - $\bar{b}\bar{c}ud$ systems with $I(J^P) = 0(1^+)$ and $I(J^P) = 0(0^+)$.
  - Different lattice setup and substantially more advanced methods compared to previous work.
    $\rightarrow$ Local and scattering operators at the source and at the sink of correlation functions.
    $\rightarrow$ Application of Lüscher's finite-volume method to multiple excited states.
    $\rightarrow$ Reliable determination of the energy dependence of B-D and B^*-D S-wave scattering amplitudes.

Lattice setup

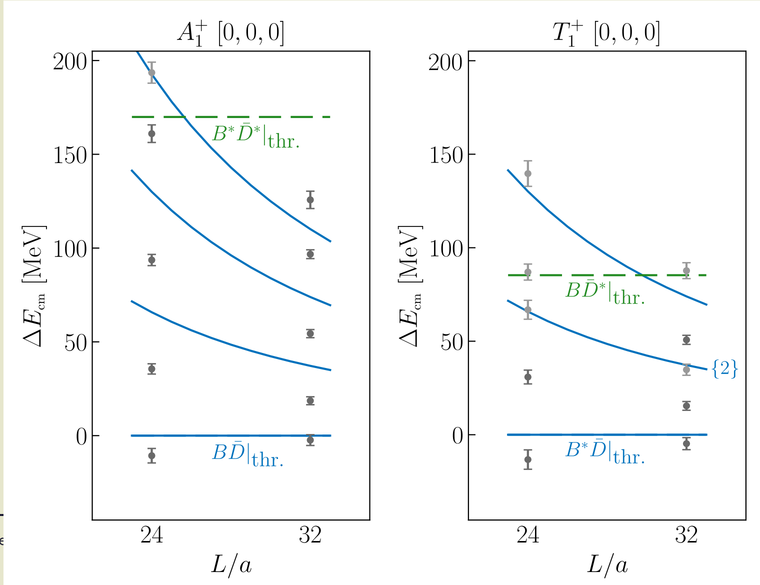
- Gauge link configurations generated with $N_f = 2 + 1 + 1$ flavors of highly improved staggered (HISQ) quarks by the MILC collaboration.
  [A. Bazavov *et al.* [MILC], Phys. Rev. D **87**, 054505 (2013) [arXiv:1212.4768]]
  - Two ensembles, which differ in the spatial volume:
    * $a \approx 0.12\,$fm.
    * $24^3 \times 64$, i.e. spatial lattice extent $\approx 2.9\,$fm,
      $32^3 \times 64$, i.e. spatial lattice extent $\approx 3.8\,$fm.
    * Pion mass $m_\pi \approx 220\,$MeV.
- Mixed-action setup tested and used by the PNDME collaboration for nucleon-structure computations.
  [T. Bhattacharya *et al.* [PNDME], Phys. Rev. D **92**, 094511 (2015) [arXiv:1506.06411]]
  [R. Gupta, Y. C. Jang, B. Yoon, H. W. Lin, V. Cirigliano, T. Bhattacharya, Phys. Rev. D **98**, 034503 (2018) [arXiv:1806.09006]]
  - Clover-improved Wilson action with HYP-smeared gauge links for the valence light and charm quarks.

$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (3)

- Black and gray data points:
  Lowest five finite-volume energy levels as functions of the spatial lattice extent L.
  - First lattice QCD study of $\bar{b}\bar{c}ud$ using both local operators ("tetraquark structure") and scattering operators ("meson-meson structure") at the source and at the sink.
  - Such a set of operators seem to be necessary to get correct and precise results for the low-lying finite-volume energy levels.

- Blue curves:
  Noninteracting $B^{(*)}$-D energy levels, $E = E_{B^{(*)}}(\mathbf{p}^2) + E_D(\mathbf{p}^2)$ with momenta $\mathbf{p}$ satisfying periodic boundary conditions.

- Significant downward shift of finite-volume energy levels compared to noninteracting energy levels ("a larger number of energy levels").
  $\rightarrow$ A hint for the existence of a pole in the scattering amplitude, i.e. a shallow bound state or a resonance.



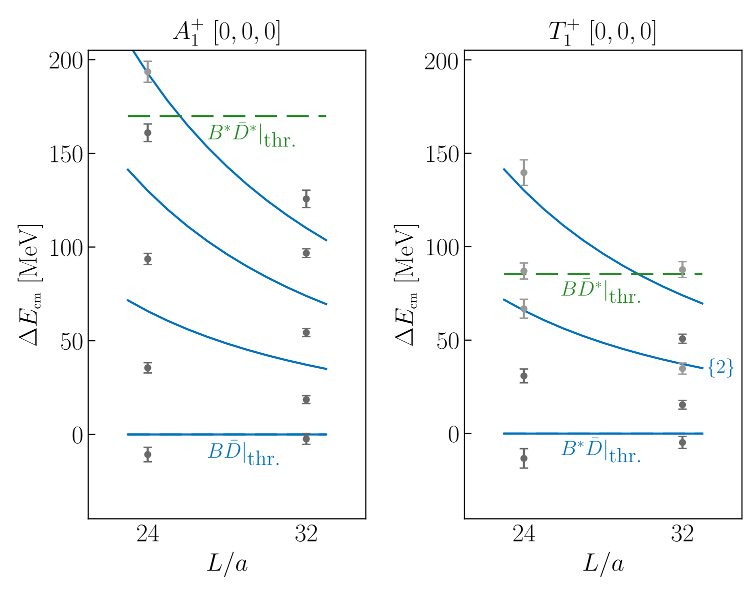
$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (4)

- Rigorously investigate the existence of bound states or resonances by mapping the finite-volume energy levels E_n to infinite-volume S-wave $B^{(*)}$-D scattering phase shifts,

$$\cot \delta_0(k_n) \quad = \quad \frac{2 Z_{00}(1; (k_n L / 2\pi)^2)}{\pi^{1/2} k_n L}$$

  (Lüscher's method).

  - Z_{00}: generalized zeta function.
  - k_n: scattering momenta associated with energy levels E_n, calculated via $E_n = E_{B^{(*)}}(k_n^2) + E_D(k_n^2)$.
  - Single-channel, single-partial-wave approach:
    $\rightarrow$ Only extract the phase shifts for energy levels below the B^*-D^* ($J = 0$) and B-D^* ($J = 1$) thresholds.
    $\rightarrow$ For $J = 1$ exclude the second excitation, because it is strongly D-wave dominated.
    (use black points, exclude gray points)



$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (5)

- Blue data points:
  Infinite-volume S-wave $B^{(*)}$-D scattering phase shifts.
  $\rightarrow$ Data points / Lüscher's method valid above the left-hand cut associated with two-pion exchange and below the next threshold (B^*-D^* for $J = 0$ and B-D^* for $J = 1$).

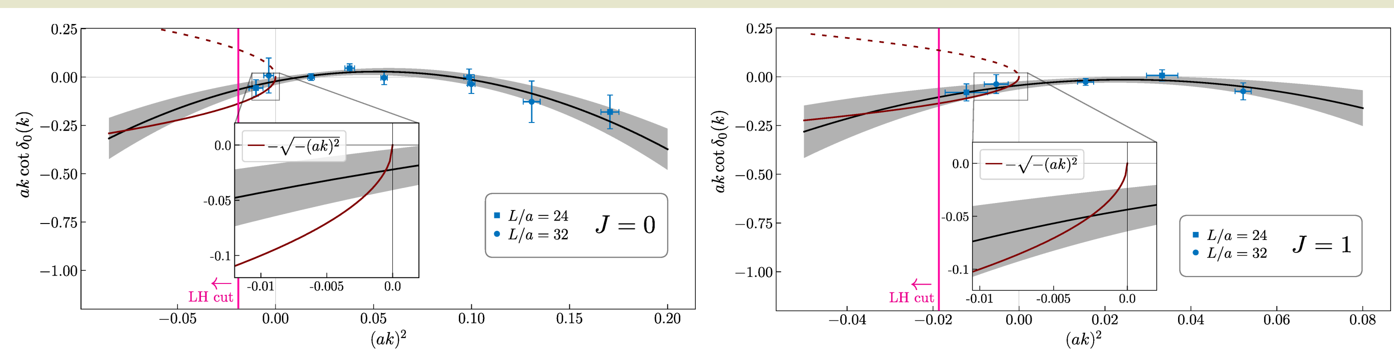
- Black curve:
  Effective-range expansion (ERE) fit,

$$k \cot \delta_0(k) \quad = \quad \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + b_0 k^4 .$$

- S-wave scattering amplitude:

$$T_0(k) \quad = \quad \frac{1}{\cot \delta_0(k) - i} ,$$

  i.e. poles for $k \cot \delta_0(k) = ik$.



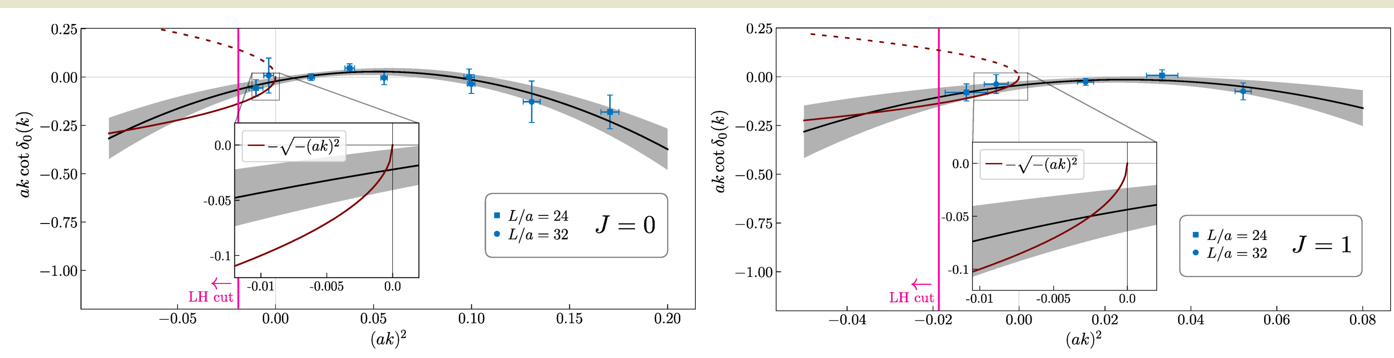
$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ **and** $0(1^+)$ **(6)**

Bound states (1)

- Condition for poles in the scattering amplitude $k \cot \delta_0(k) = ik = \pm\sqrt{-k^2}$.
- For real energies, i.e. real k^2, the right-hand-side ik is real for $k^2 \leq 0$ (plotted in red); intersections with $k \cot \delta_0(k)$ correspond to poles below threshold, i.e. indicate bound states.

$\rightarrow$ A bound state for $J = 0$ at $-0.5^{+0.4}_{-1.5}$ MeV (88.5% bound state, 11.5% virtual bound state).

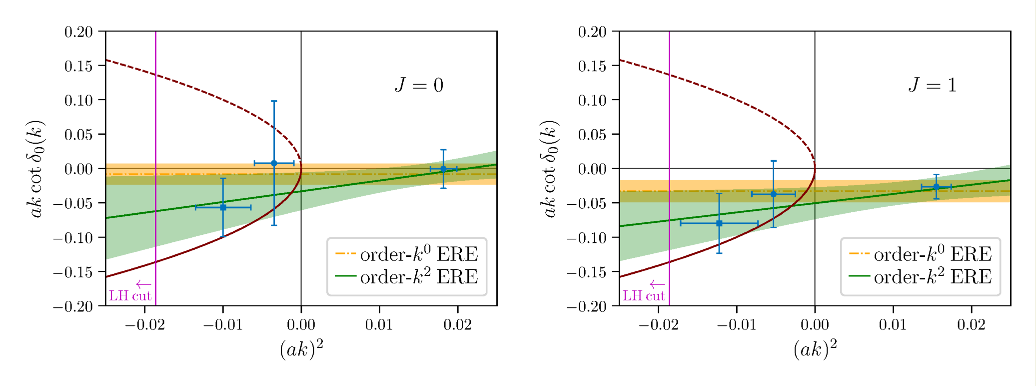
$\rightarrow$ A bound state for $J = 1$ at $-2.4^{+2.0}_{-0.7}$ MeV (97.7% bound state, 2.3% virtual bound state).



$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (7)

Bound states (2)

- Additional test of our prediction of shallow bound states:
  - ERE fits of order k^0 and order k^2 using only the three data points closest to threshold.
  - $\rightarrow$ Consistent results on the existence of shallow bound states and their masses.



$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ and $0(1^+)$ (8)

Resonances

- Poles in the scattering amplitude with real part of the energy above threshold, i.e. $\text{Re}(k^2) > 0$, and negative imaginary part indicate resonances.

$\rightarrow$ A resonance for $J = 0$ at $138(13)$ MeV, decay width $229(35)$ MeV.

$\rightarrow$ A resonance for $J = 1$ at $67(24)$ MeV, decay width $132(32)$ MeV.

- **These results on resonances should be treated with caution:**
  The resonance poles lie outside the radius of convergence of the ERE, which is limited by the presence of a left-hand cut associated with two-pion exchange (position of the cut ≈ 18 MeV below threshold for both $J = 0$ and $J = 1$).

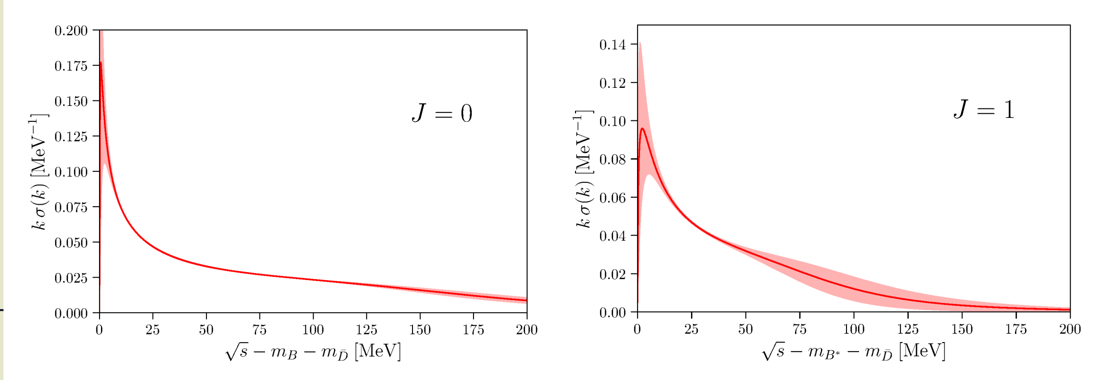
$\bar{b}\bar{c}ud$, $I(J^P) = 0(0^+)$ **and** $0(1^+)$ **(9)**

- S-wave cross section,

$$\sigma(k) = \frac{4\pi}{k^2}|T_0(k)|^2 \quad , \quad T_0(k) = \frac{1}{\cot\delta_0(k) - i}$$

  with the ERE fit $k\cot\delta_0(k) = 1/a_0 + (r_0/2)k^2 + b_0 k^4$.

- scattering rate $=$ flux $\times\ \sigma(k) \propto k\sigma(k)$ (for nonrelativistic k).
- Sharp enhancements in the scattering rates close to the thresholds, because of the shallow bound states.
- At higher energies still enhanced, because of the broad resonances.



Existing work ... again

- Summary of current status (only full lattice QCD results):
  - $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^+)$:
    A **QCD-stable tetraquark** around 130 MeV below the BB^* threshold.
  - $\bar{b}\bar{b}us$ with $J^P = 1^+$: **(part 1 of this talk)**
    A **QCD-stable tetraquark** around 90 MeV below the BB_s^* threshold.
    * Masses of stable hadrons correspond to energy eigenvalues at infinite volume (for shallow bound states, it might be difficult to study this limit).
  - $\bar{b}\bar{c}ud$ with $I(J^P) = 0(0^+)$ and with $I(J^P) = 0(1^+)$: **(part 2 of this talk)**
    Contradictory results. The technically most advanced study points towards **very shallow bound states**, i.e. QCD-stable tetraquarks slightly below the BD and B^*D thresholds.
    * Masses and decay widths of resonances (or shallow bound states) can be calculated from the volume dependence of the energy eigenvalues (difficult).
  - $\bar{b}\bar{b}ud$ with $I(J^P) = 0(1^-)$:
    No full lattice QCD investigation yet. Antistatic-antistatic lattice QCD potentials and the Born-Oppenheimer approximation suggest the existence of a **tetraquark resonance** close to the B^*B^* threshold (which is not the lowest meson-meson threshold).