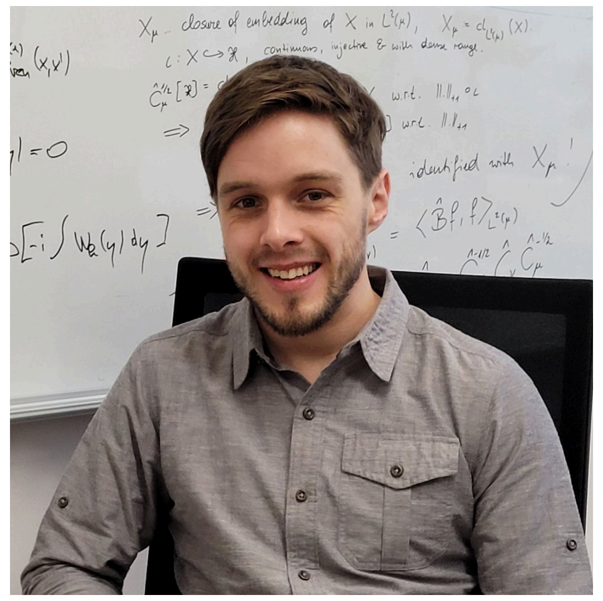
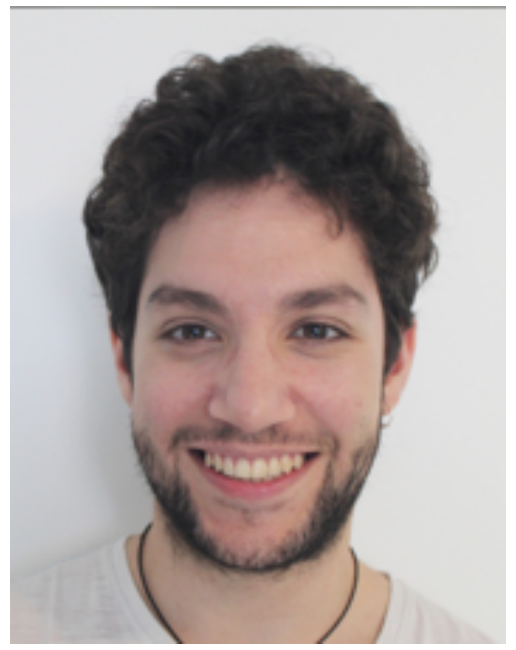
Cosmological Scattering Potentials and their Quantum Simulation

theory


Christian Schmidt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RTG Spring Combo 2024

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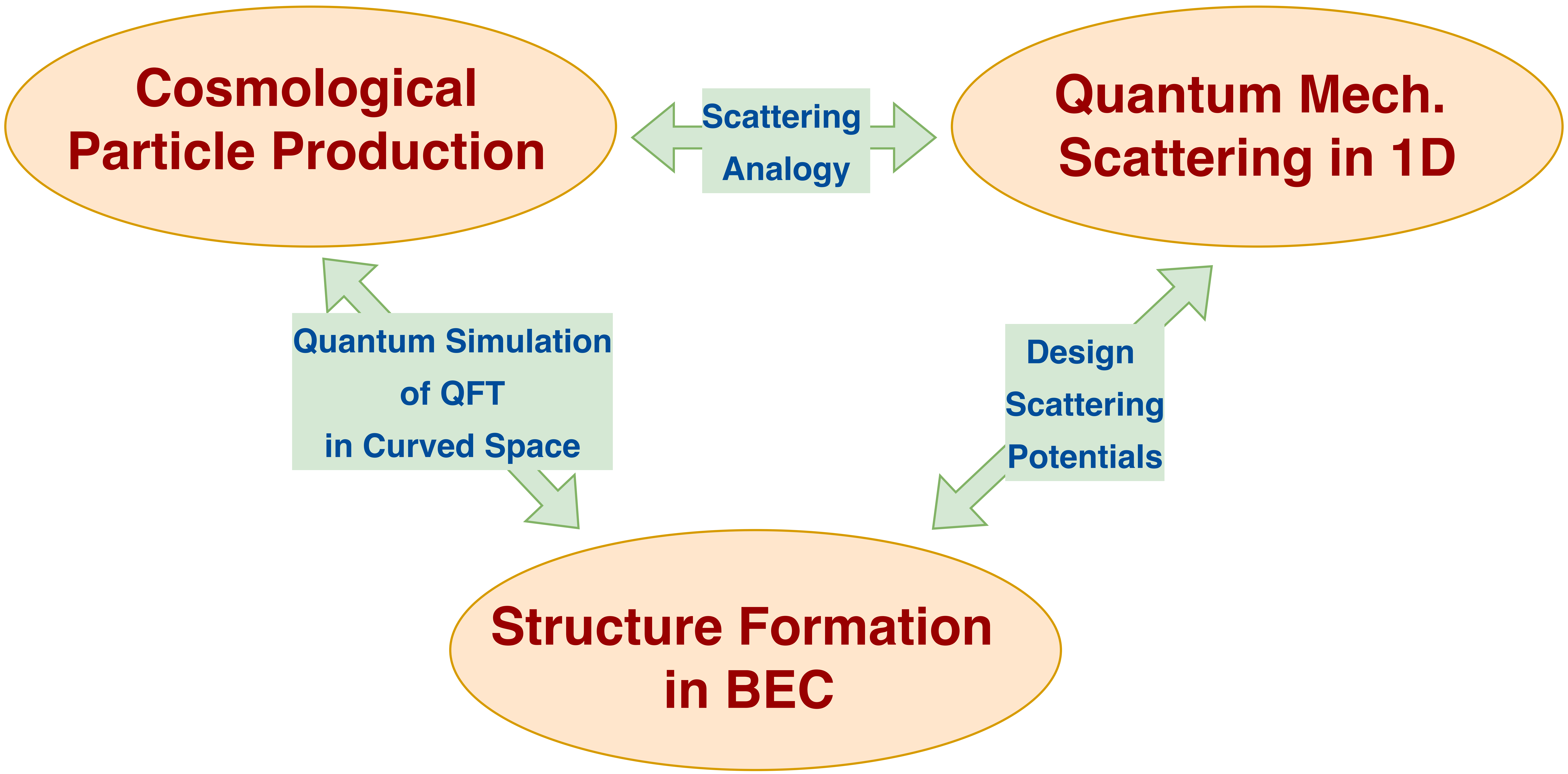


Cosmological Particle Production
Scattering Analogy
Quantum Mech. Scattering in 1D
Quantum Simulation of QFT in Curved Space
Design Scattering Potentials
Structure Formation in BEC

Analogue Quantum Simulation of Kinematics in Curved Spacetime

Black Hole Spacetimes

Horizons Hawking Radiation

Class. Fluids
Weinfurtner et al. (2011)

Bose-Einstein-Condensate (BEC)
Lahav et al. (2010) | Steinhauer (2016)| de Nova et al. (2019)

Optical Fibre
Philbin et al. (2008) | Choudhary et al. (2012) | Jaquet (2018)

Microcavity Polaritons
Nguyen et al. (2015) | Jacquet et.al (2022)

Superfluid 3-He
Človečko et al. (2019)

Rotating Spacetimes Black Hole Superradiance

Class. Fluids
Torres et al. (2017) I Cromb et.al (2020)

Photon Superfluid
Vocke et al. (2017)

Microcavity Polaritons
Falque et al. (2023) | Delhom et al. (2023)

Superfluid 4-He
Svancara et al. (2023)

New avenues for fermionic fields:
Simeón et.al (2023)
Haller, Meng et.al (2023)

Cosmological Particle Production

BEC
Hung et al. (2012) I Eckel et al. (2018) I Tajik et al. (2023)

2+1 dim: Viermann et al. (2022), Simeón et al. (2022)

Trapped Ions
Wittemer et al. (2018)

Laser Pulse
Steinhauer et al. (2022)

False Vacuum Decay

BEC
Berti et al.(2023), Cominoti et al. (2023) | Jenkins et.al(2024)

Dynamical Casimir effect

BEC
Jaskula et.al (2011)

Superconducting Circuit
Wilson et.al (2011)

Quantum Simulation of Scalar Field in Curved Spacetime

Effective action for weakly interacting BECs

$$\Gamma[\Phi] = \int \mathrm{d}t\,\mathrm{d}^D x \left\{ \hbar\Phi^*(t,\mathbf{x}) \left[\mathrm{i}\frac{\partial}{\partial t} - V(t,\mathbf{x}) \right] \Phi(t,\mathbf{x}) - \frac{\hbar^2}{2m} \boldsymbol{\nabla}\Phi^*(t,\mathbf{x}) \boldsymbol{\nabla}\Phi(t,\mathbf{x}) - \frac{\lambda(t)}{2} \left[\Phi^*(t,\mathbf{x})\Phi(t,\mathbf{x})\right]^2 \right\}$$

1. Expand to 2nd order in **fluctuations**

$$\Phi(t,\mathbf{x}) = e^{\mathrm{i}S_0(t,\mathbf{x})} \left(\sqrt{n_0(t,\mathbf{x})} + \frac{1}{\sqrt{2}} [\phi_1(t,\mathbf{x}) + \mathrm{i}\phi_2(t,\mathbf{x})] \right)$$

[Simeón et al.(2022)]

2. Evaluate background on **Gross-Pitaevskii equation** (in hydrodynamic form)

$$\partial_t n_0 + \nabla(n_0 \boldsymbol{v}) = 0$$

$$\hbar\partial_t S_0 + V + \lambda n_0 + \frac{\hbar^2}{2m} \left[(\nabla S_0)^2 - \frac{\nabla^2 \sqrt{n_0}}{n_0} \right] = 0$$

3. Employ acoustic approximation

$$k \ll \frac{\sqrt{2}}{\xi}$$

→ Neglect quantum pressure

$$q = \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n_0}}{n_0}$$

$$\Gamma_2[\phi] = -\frac{\hbar^2}{2} \int \mathrm{d}t\,\mathrm{d}^2 r\, \sqrt{g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Acoustic FLRW-Spacetime

BEC as stationary background

$$\boldsymbol{v}(t,\boldsymbol{x}) = (\hbar/m)\nabla S_0(t,x) = 0$$

Isotropic density profile

$$n_0(\mathbf{x}) = \bar{n}_0 \left[1 + \frac{\kappa}{4} r^2\right]^2$$

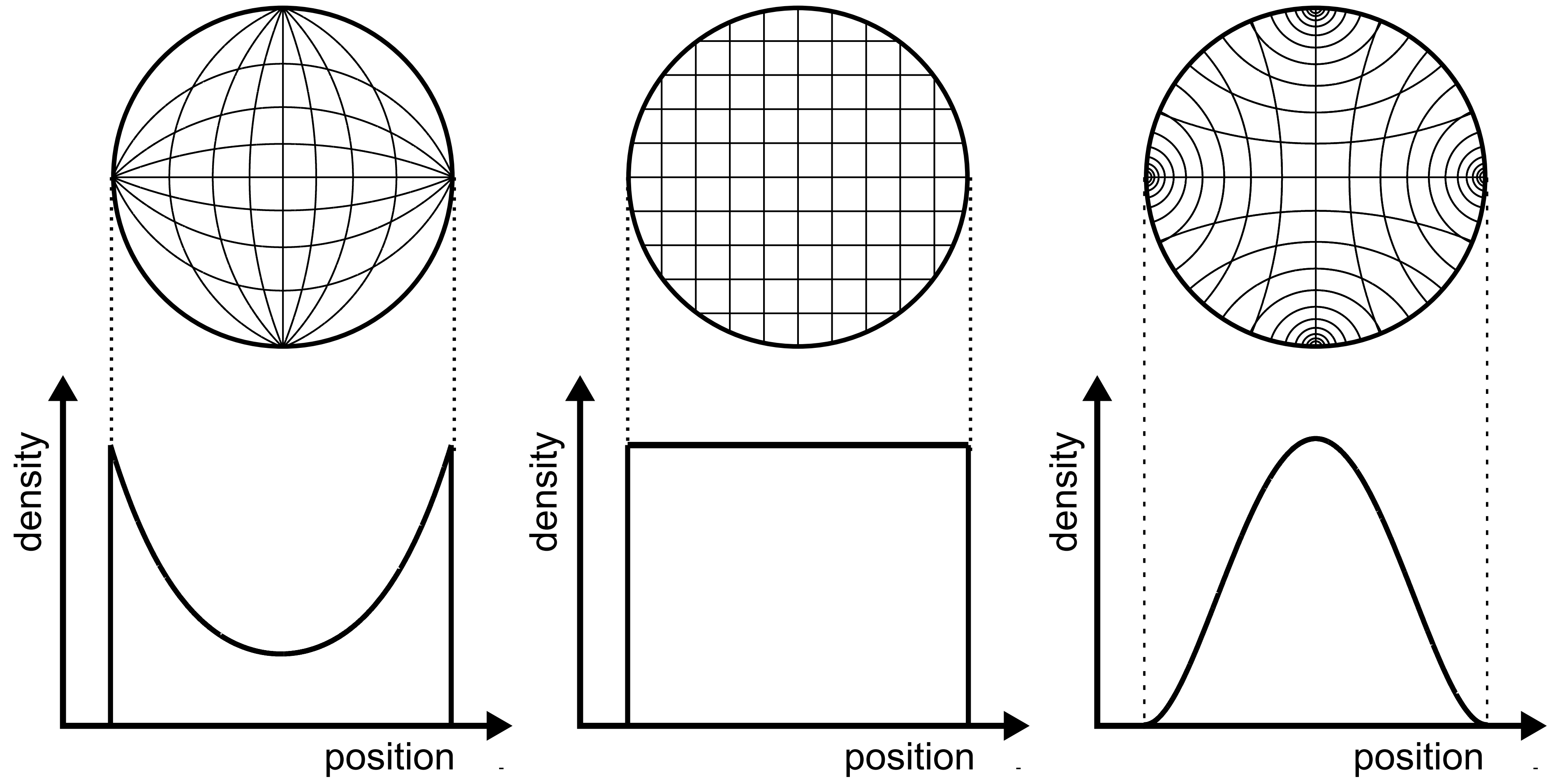
[Simeón et al.(2022), Viermann et al.(2022)]

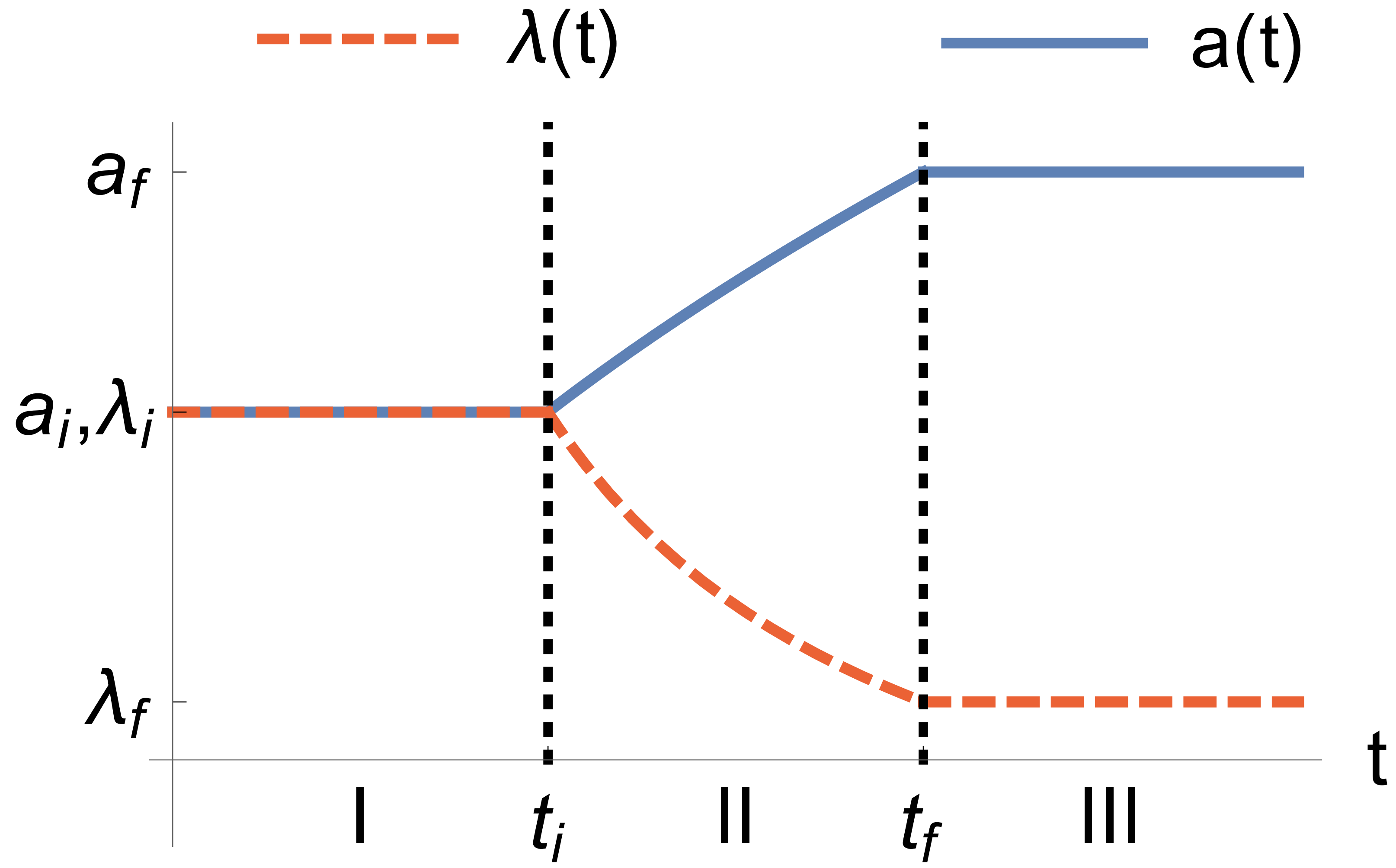
FLRW line-element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}u^2}{1-\kappa u^2} + u^2 \mathrm{d}\varphi^2\right)$$

Analogue scale-factor

$$a^2(t) = \frac{1}{c(t,\boldsymbol{x}=0)^2} = \frac{m}{\bar{n}_0}\frac{1}{\lambda(t)}$$



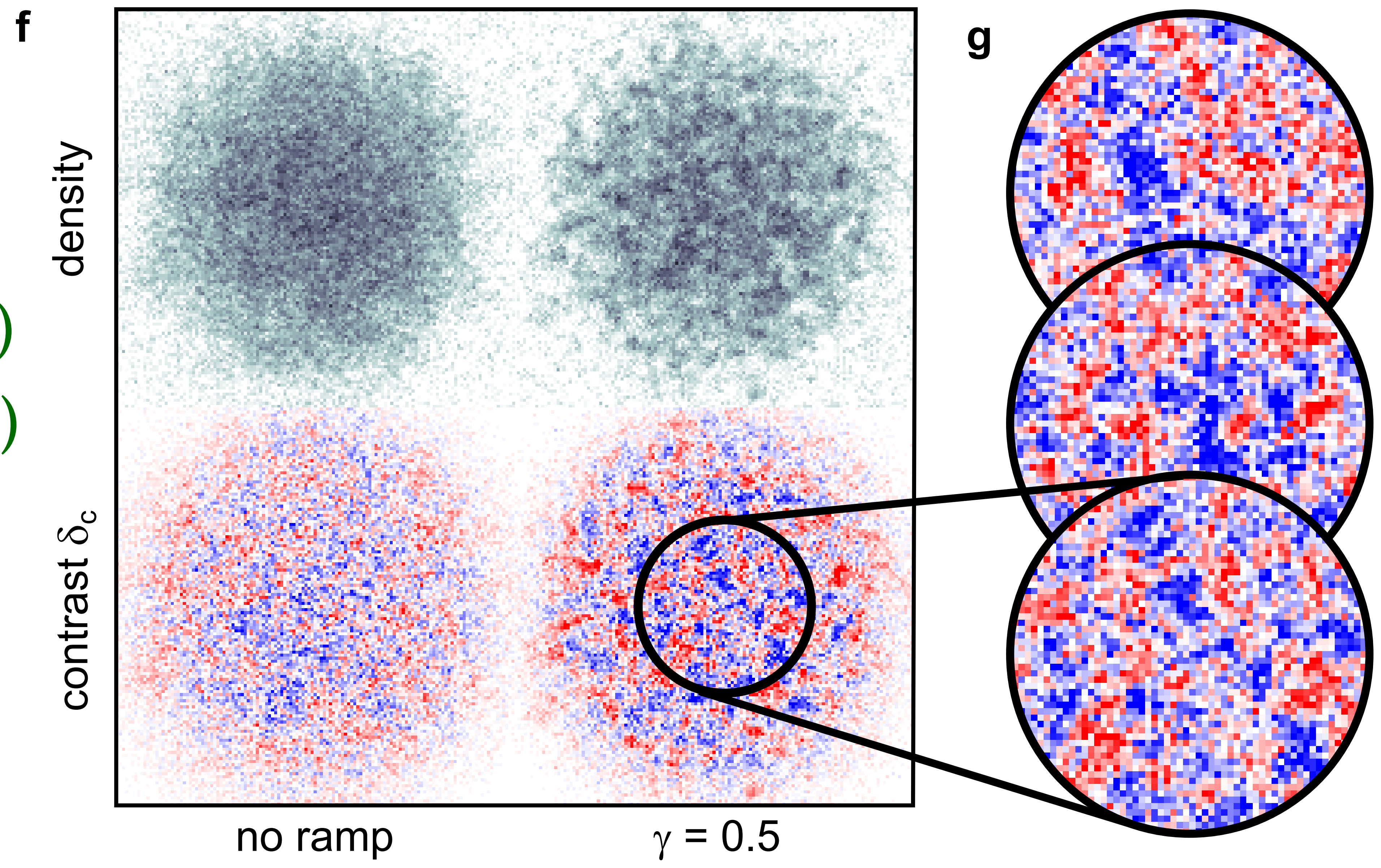


Analogue Cosmological Particle Production

Density-Contrast

$$\delta_c(t,u,\varphi) = \sqrt{\frac{n_0(u)}{\bar{n}_0^3}}\,[n(t,u,\varphi) - n_0(u)] \quad \sim \partial_t \phi + \mathcal{O}(\phi^2)$$

Viermann et al.(2022)
Simeón et al.(2022)

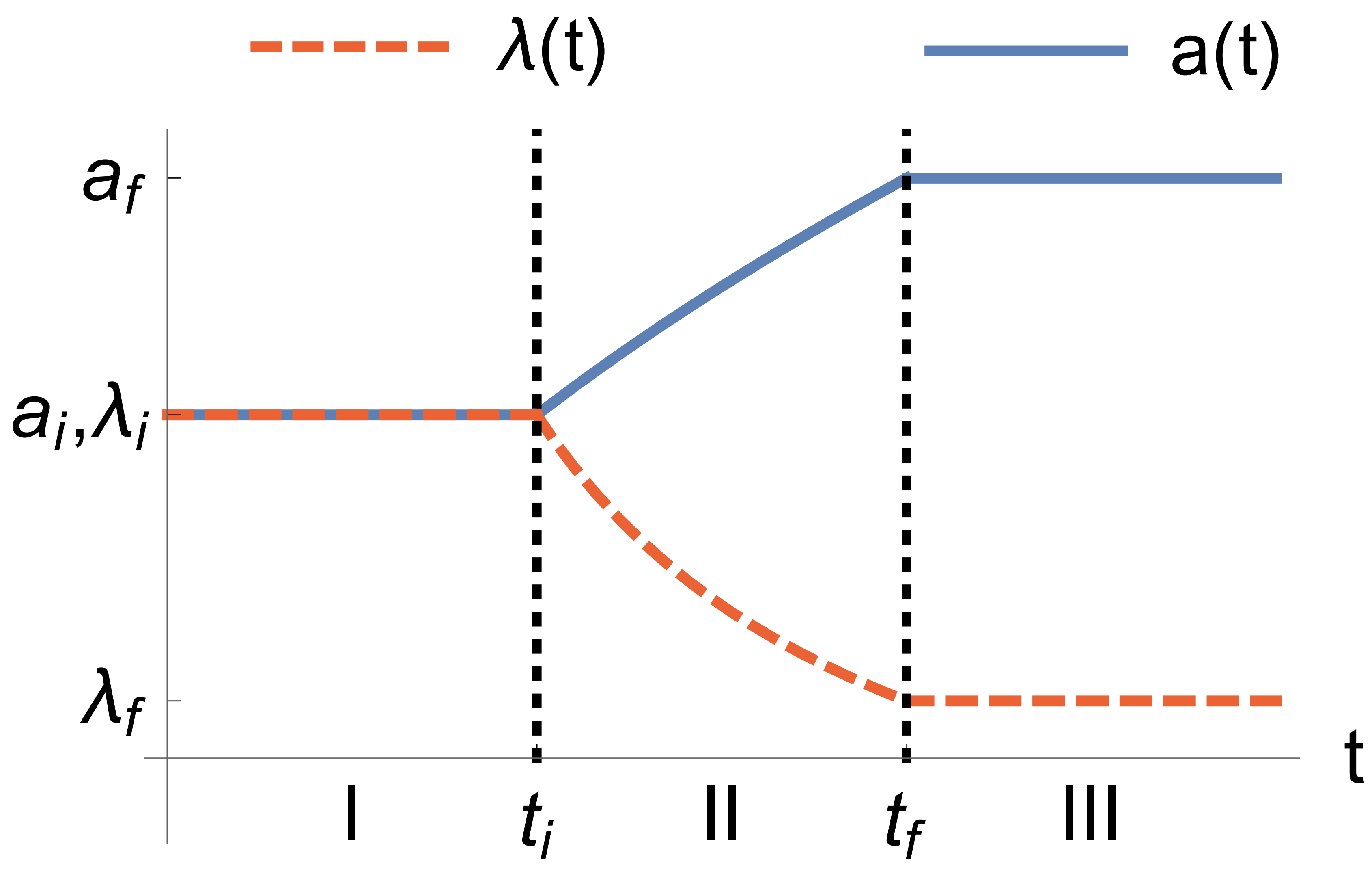


$$\langle \delta_c(t,u,\varphi)\delta_c(t,u',\varphi')\rangle = \frac{\hbar^2 m}{\lambda_f^2 \bar{n}_0^3}\langle \dot{\phi}(t,u,\varphi)\dot{\phi}(t,u',\varphi')\rangle$$

Quasi-particle occupancies measurable through full density-contrast correlations

Power Spectrum

$$\frac{1}{2}\langle 0|\,\{\dot{\phi}(t,x),\dot{\phi}(t,x')\}\,|0\rangle_c = \int_k \mathcal{F}(k,L)\frac{\sqrt{-h(k)}}{a_f^3} S_k(t)$$



(Analogue) Cosmological Particle Production as a Scattering Problem

Mode equation

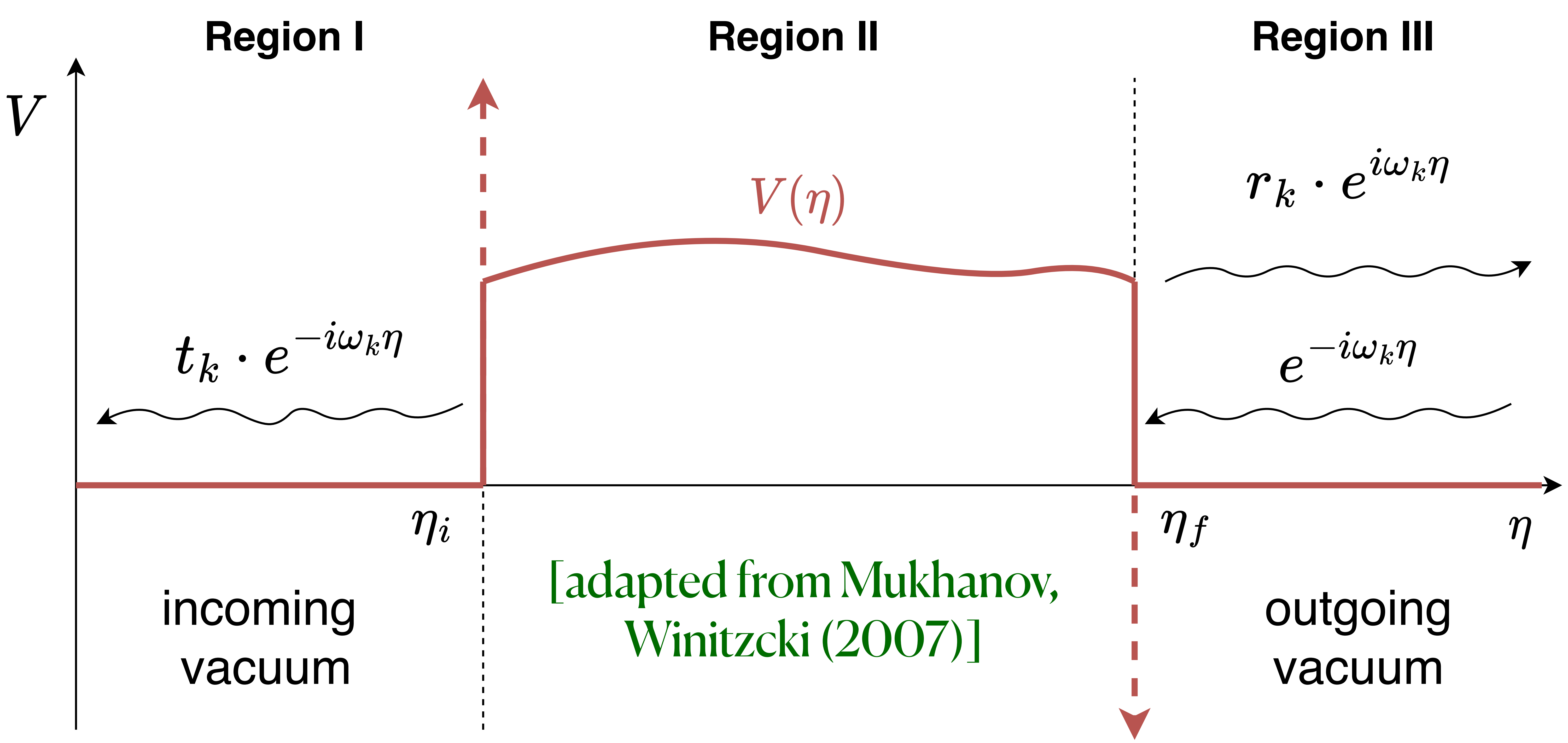
$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + V(\eta)\right]\psi_k(\eta) = -h(k)\psi_k(\eta)$$

$$\eta = \int \frac{\mathrm{d}t}{a(t)} = \int c_s(t)\,\mathrm{d}t$$

$$V(\eta) = -a^2(\eta)\left[m^2 + \xi R(\eta)\right] + \frac{D-1}{2}\left[\frac{a''(\eta)}{a(\eta)} - \frac{3-D}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2\right]$$

Eigenenergies

$$h(k) = \begin{cases} -k\left[k + (D-1)\sqrt{|\kappa|}\right] & \text{for} \quad \kappa > 0 \\ -k^2 & \text{for} \quad \kappa = 0 \\ -\left[k^2 + \left(\frac{D-1}{2}\right)^2 |\kappa|\right] & \text{for} \quad \kappa < 0 \end{cases}$$



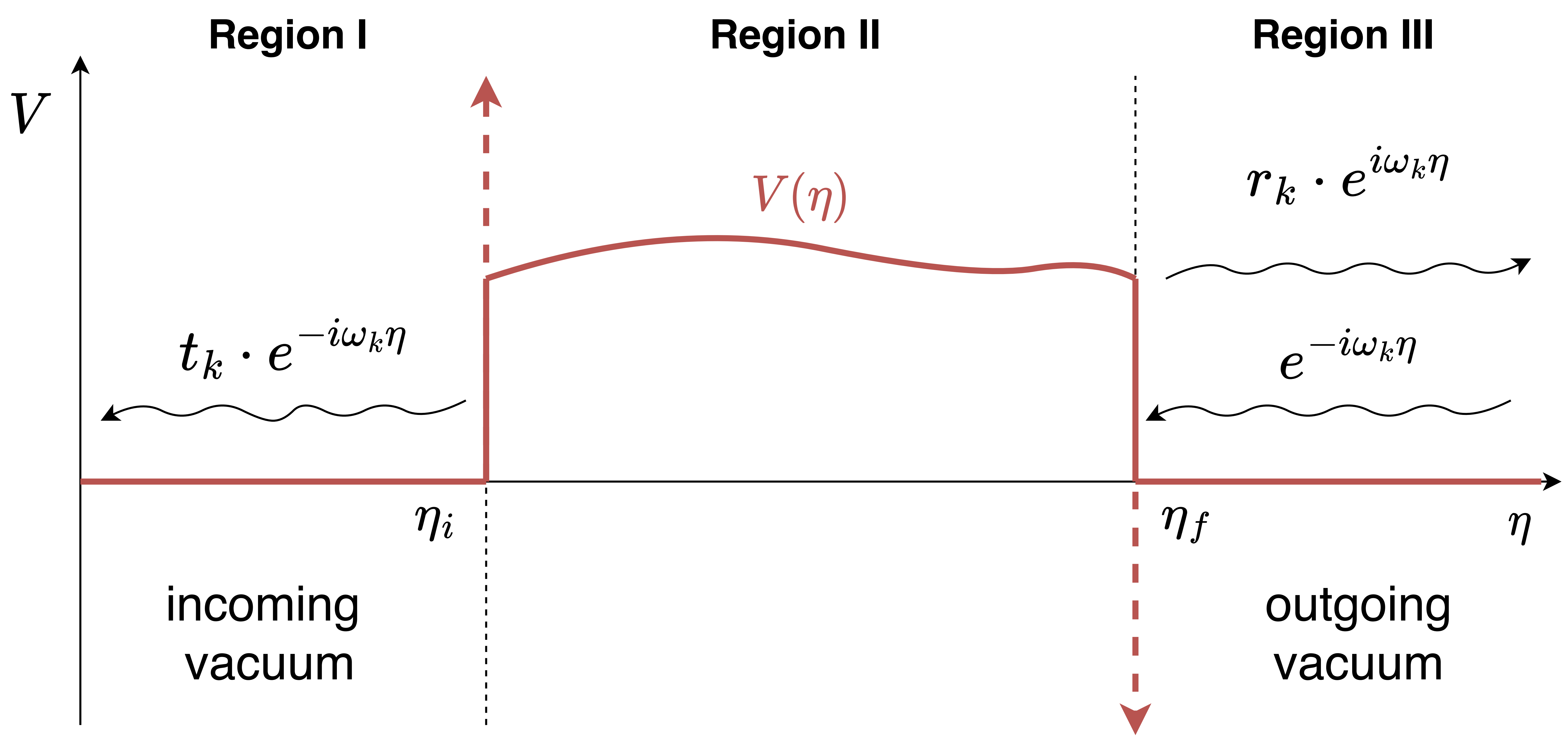
[adapted from Mukhanov, Winitzcki (2007)]

[CFS, Lopez, Simeón, Flörchinger, Oberthaler group (in preparation)]

(Analogue) Cosmological Particle Production as a Scattering Problem

Power Spectrum

$$S_k(t) = \frac{1}{2} + N_k + \Delta N_k^0 \cos(2\omega_k \eta(t) + \vartheta_k)$$



Offset

$$N_k = \left|\frac{r_k}{t_k}\right|^2$$

Amplitude

$$\Delta N_k^0 = \left|\frac{r_k}{t_k^2}\right|$$

Phase

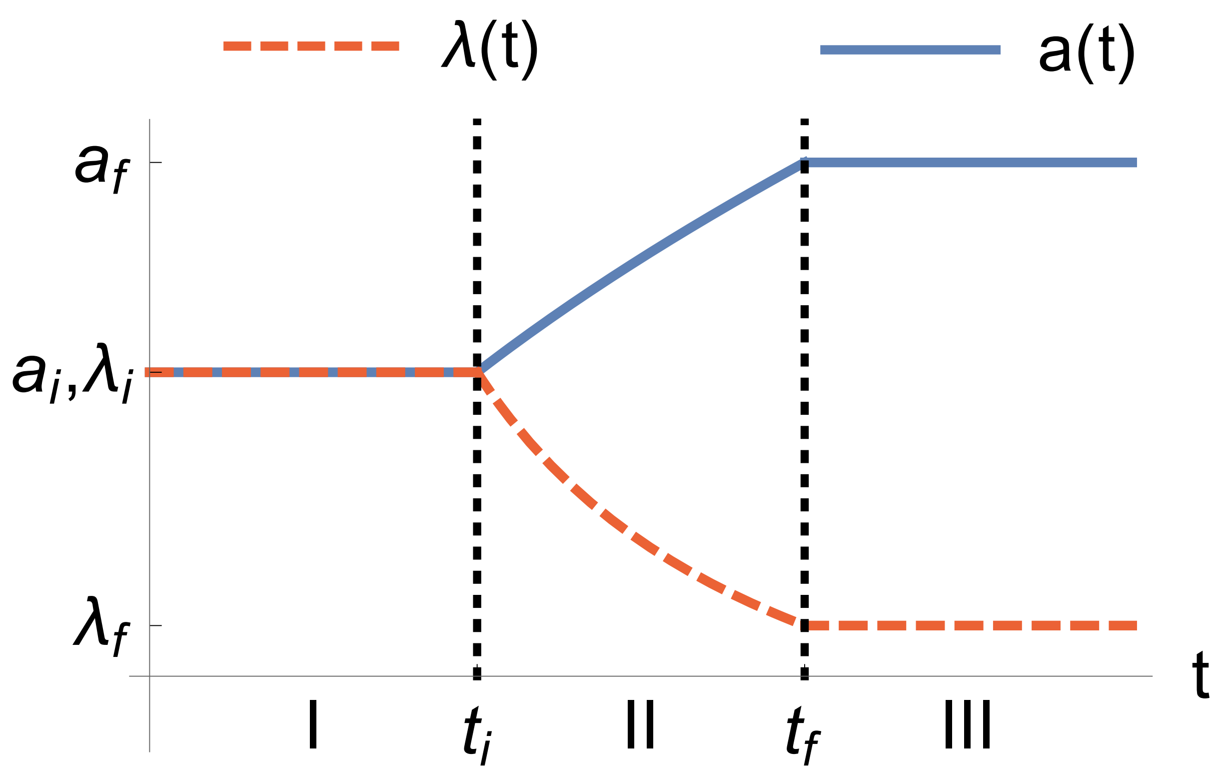
$$\vartheta_k = \arg(-r_k e^{-2i\omega_k \eta_f})$$

[CFS, Lopez, Simeón, Flörchinger, Oberthaler group (in preparation)]

Quantum Simulation of Cosmological Scattering Potentials

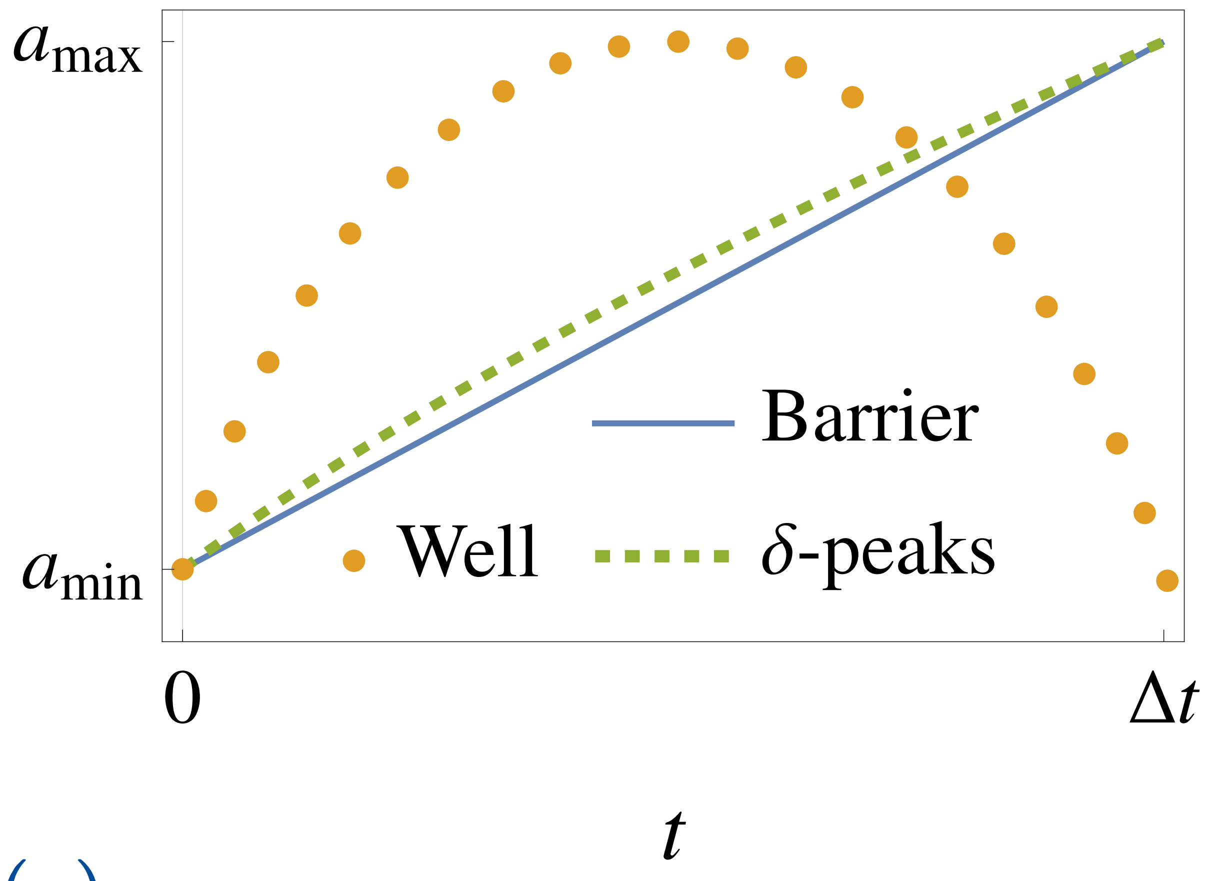
Focus on minimally coupled, massless fields in $D = 2$ spatial dimensions

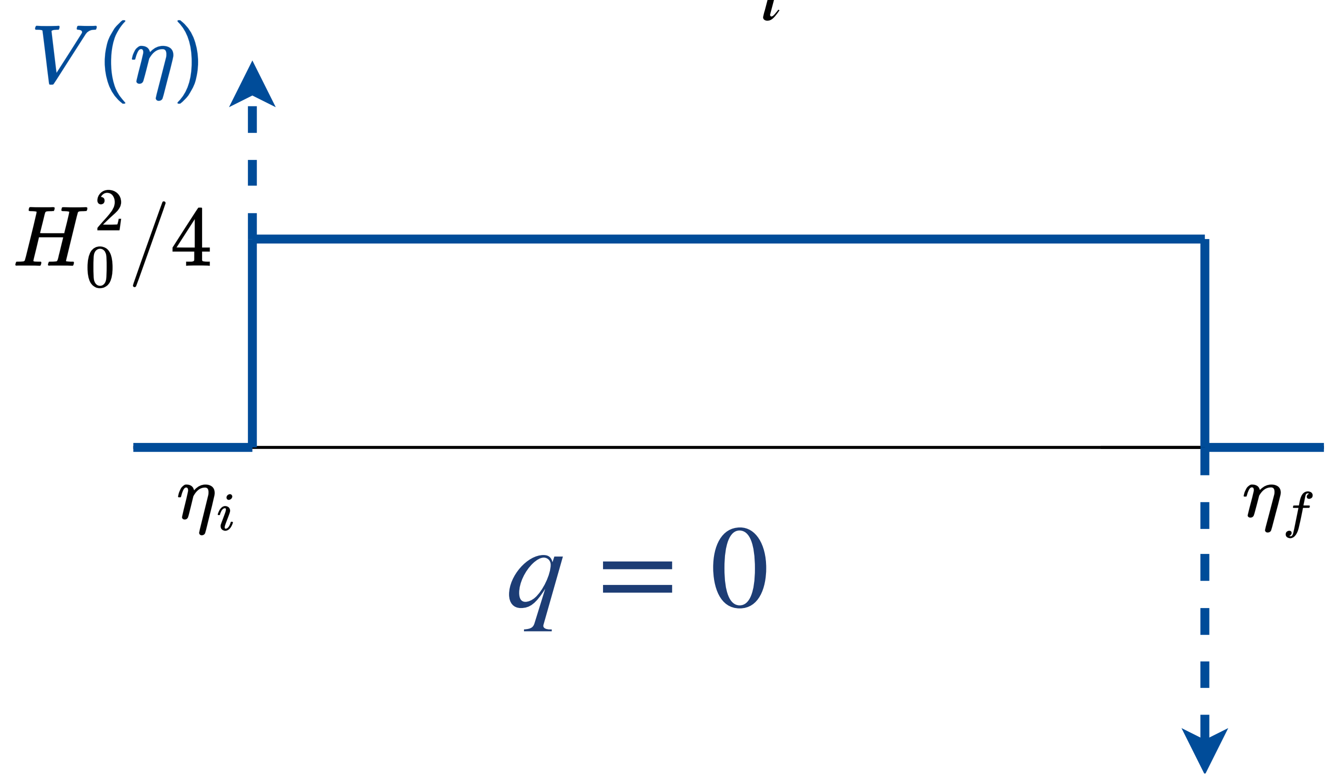
$$V(\eta) = -\frac{1}{4}\left(\frac{a'(\eta)}{a(\eta)}\right)^2 + \frac{1}{2}\frac{a''(\eta)}{a(\eta)} = \frac{m}{4\bar{n}_0}\left(\frac{7}{4}\frac{\dot{\lambda}(t)^2}{\lambda(t)^3} - \frac{\ddot{\lambda}(t)}{\lambda(t)^2}\right)$$

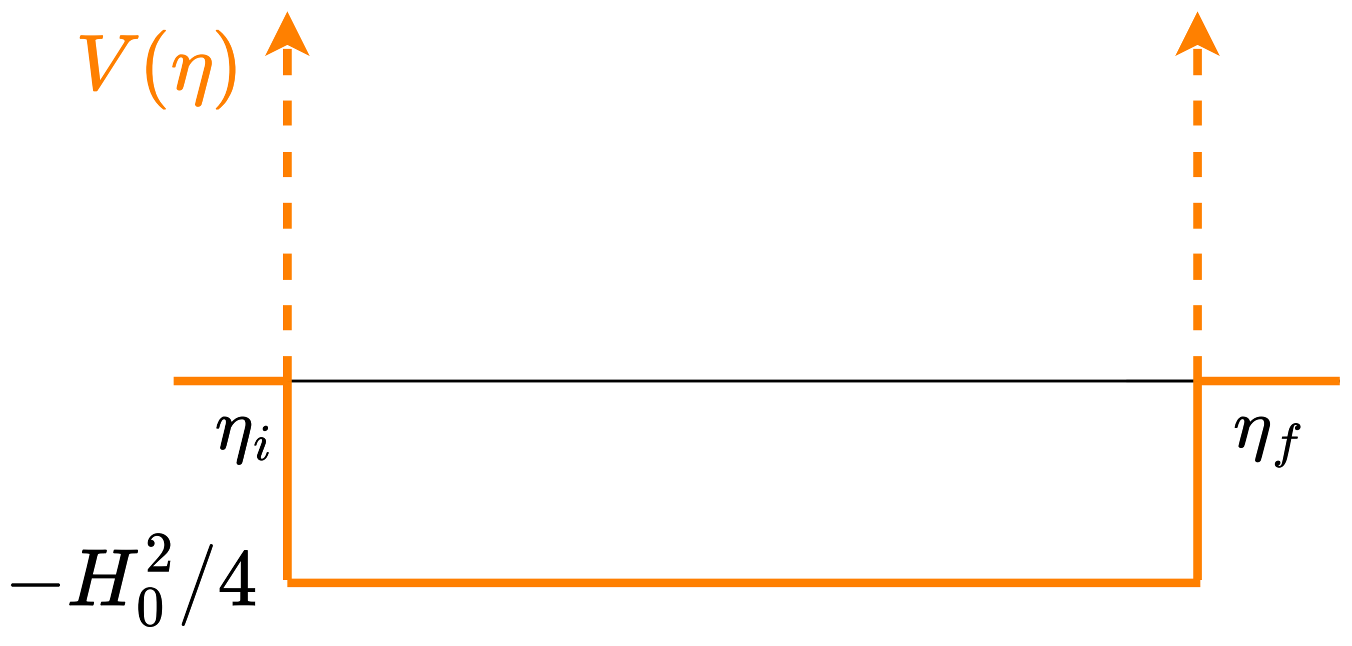


Discontinuous transitions imply singular contributions

$$V_s(\eta) = \frac{\dot{a}(t(\eta))}{2}\left[\delta(\eta - \eta_i) - \delta(\eta - \eta_f)\right]$$

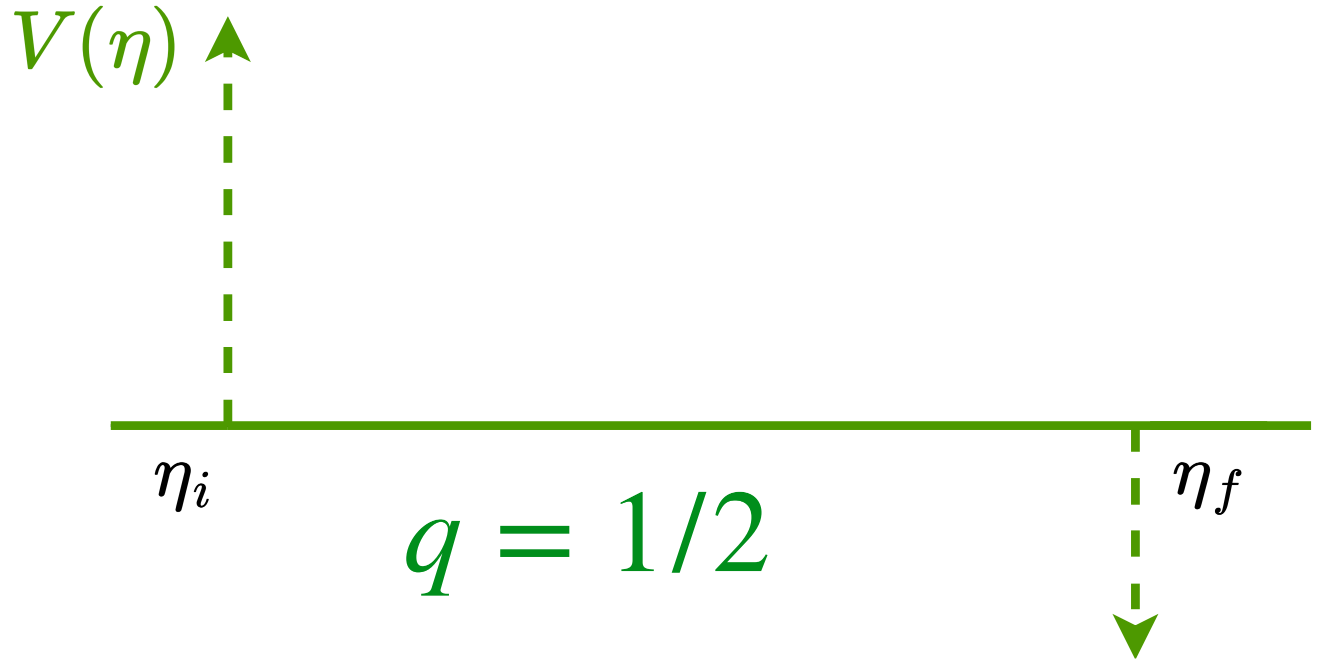






Power law

$$a(t) = [1 + (q+1)H_0 t]^{\frac{1}{q+1}}$$



Designed with

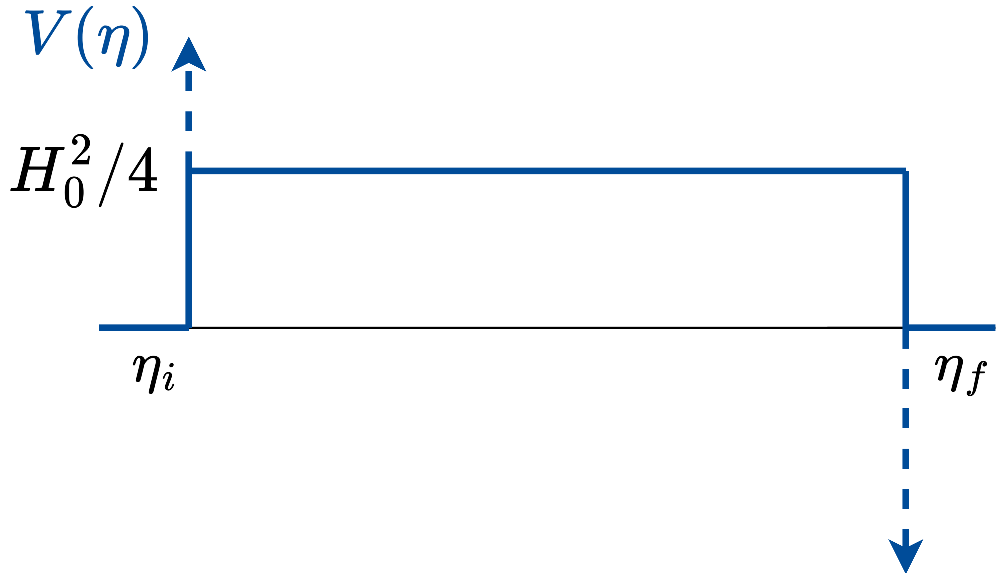
$$y''(\eta) - V(\eta)y(\eta) = 0$$

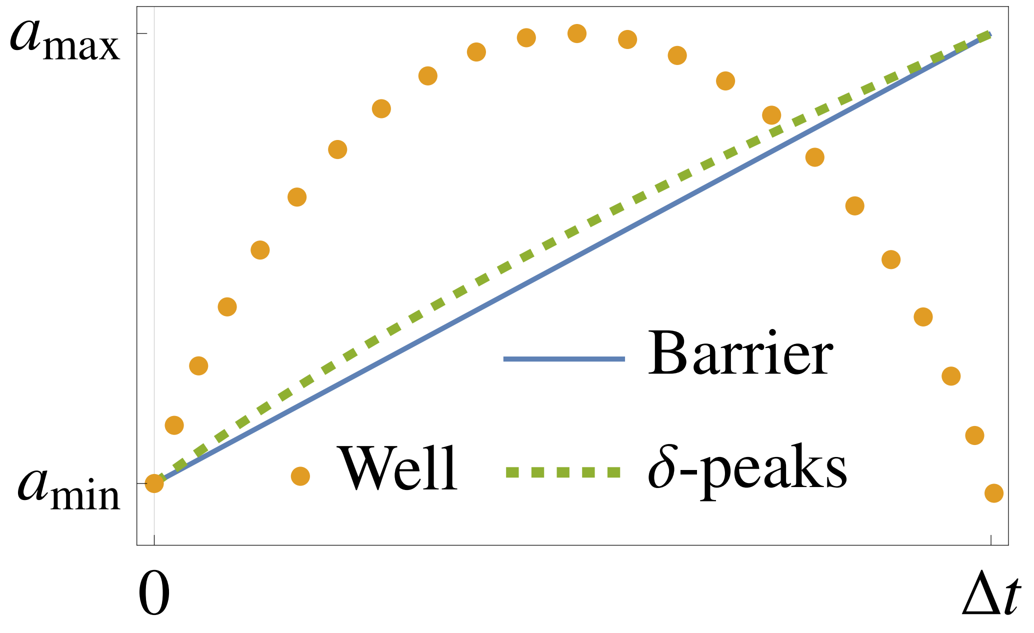
where

$$y(\eta) \propto a(\eta)^{1/2}$$

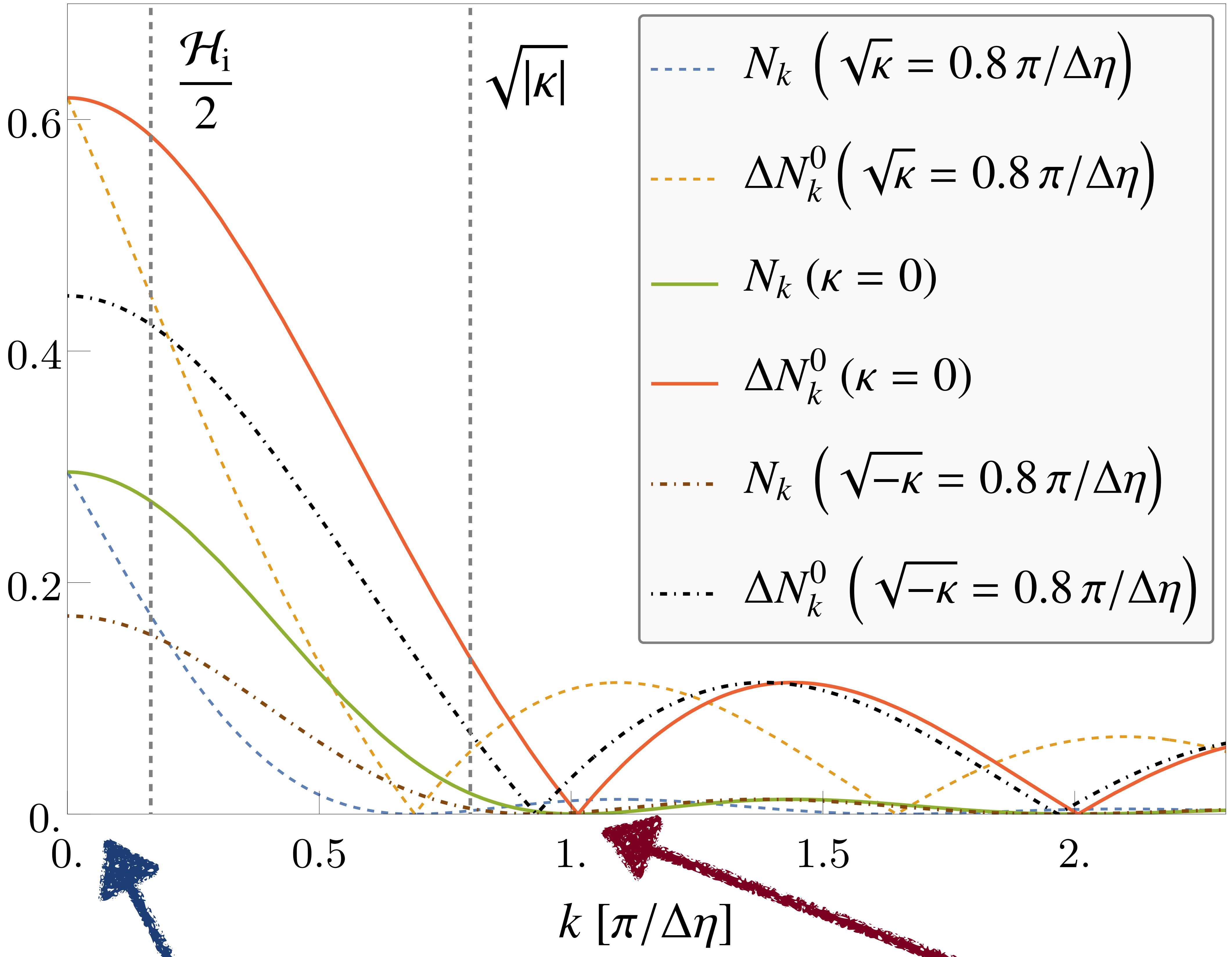
as zero-energy resonance

Results:

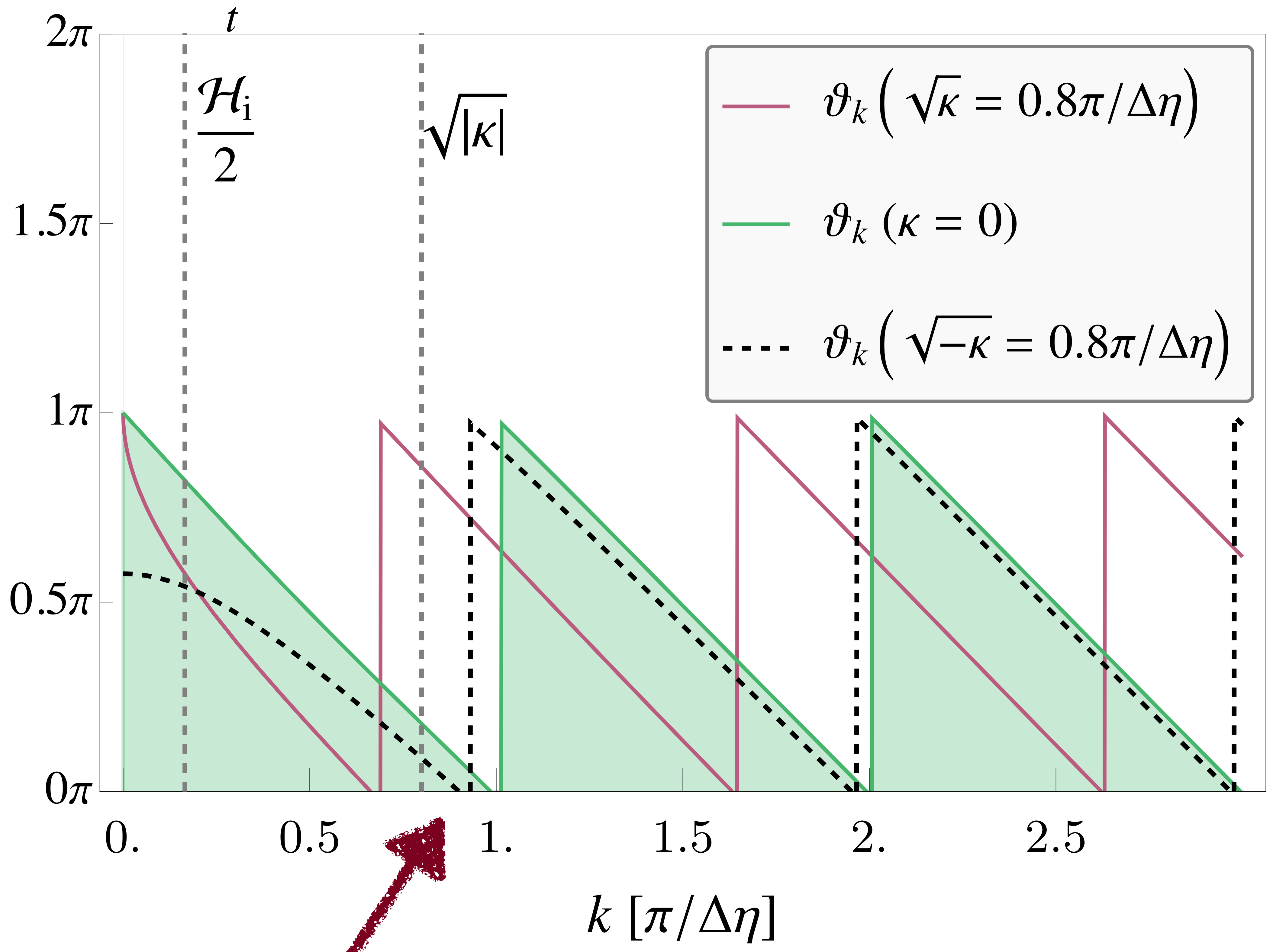




Offset, Amplitude



Phase



Resonance near continuum threshold

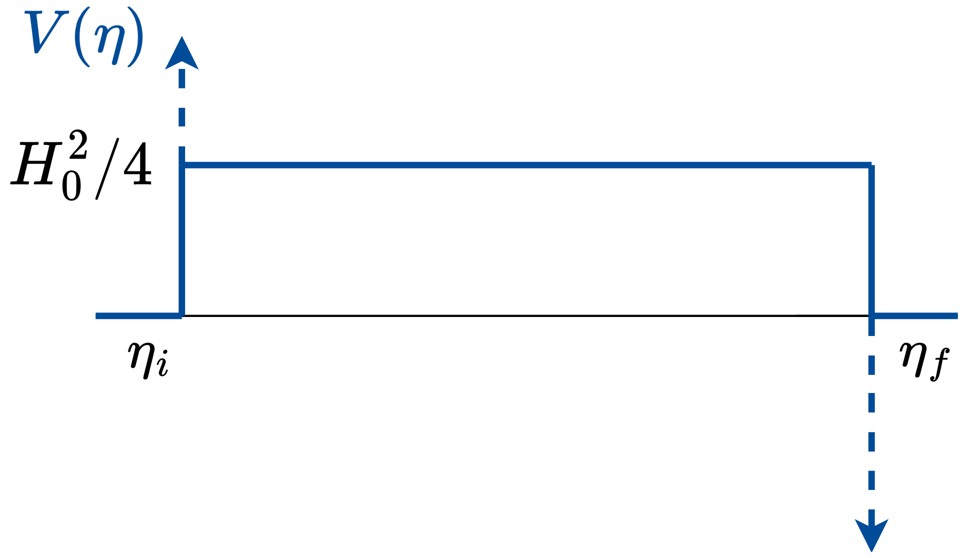
[Senn (1988), Boya (2008)]

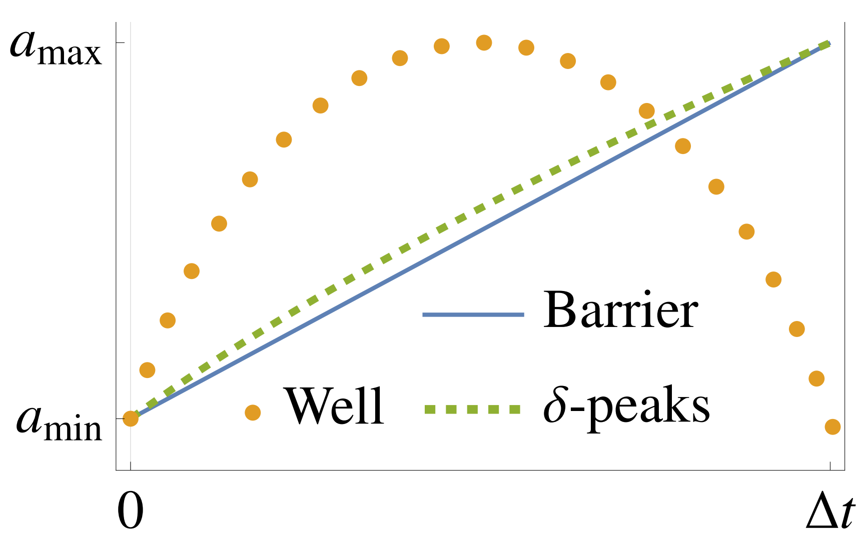
Resonant forward scattering

$$\sqrt{-h(k) - H_0^2/4}\;\Delta\eta = n\pi$$

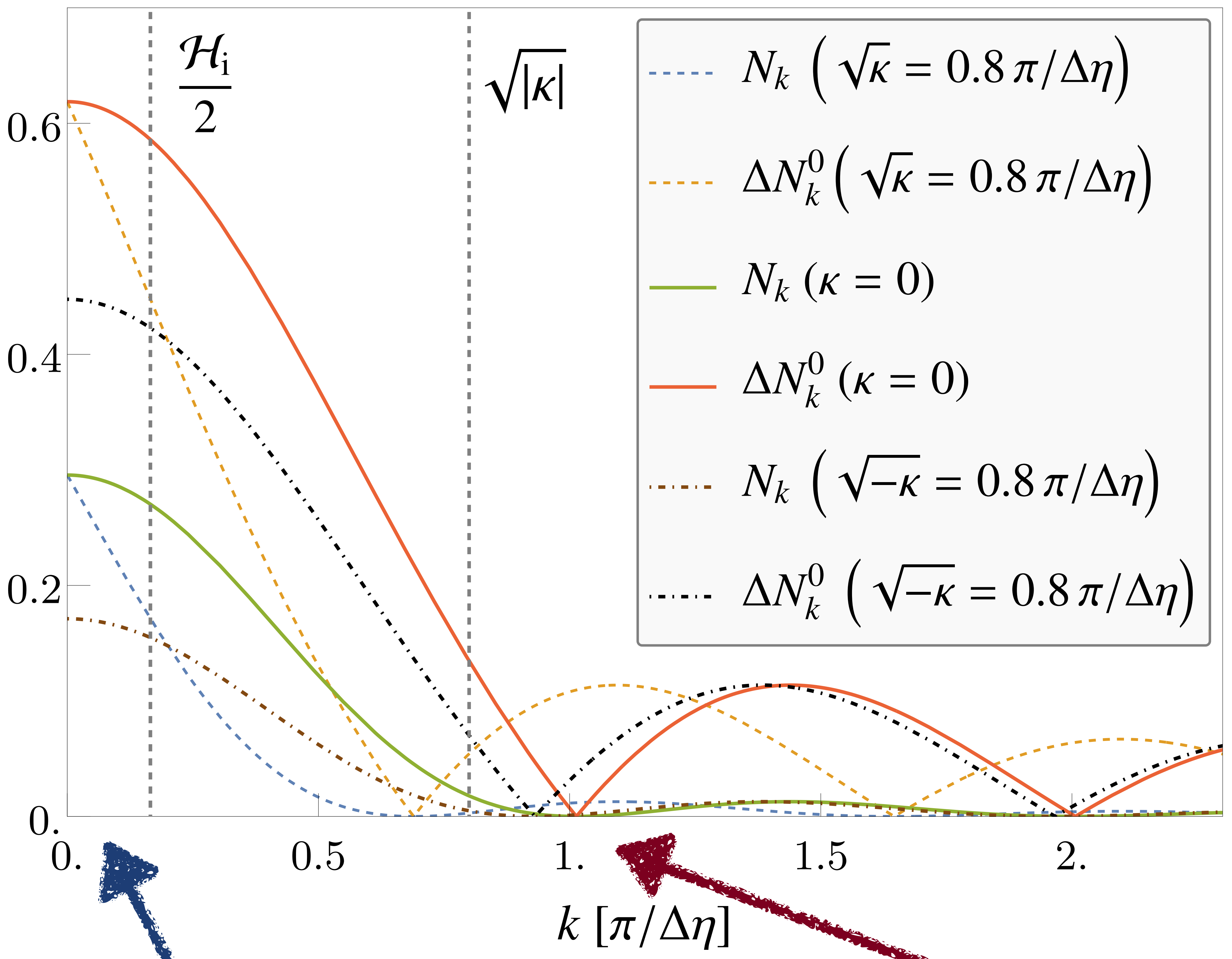
$$h(k) = \begin{cases} -k\left[k + (D-1)\sqrt{|\kappa|}\right] & \text{for} \quad \kappa > 0 \\ -k^2 & \text{for} \quad \kappa = 0 \\ -\left[k^2 + \left(\frac{D-1}{2}\right)^2 |\kappa|\right] & \text{for} \quad \kappa < 0 \end{cases}$$

Results:

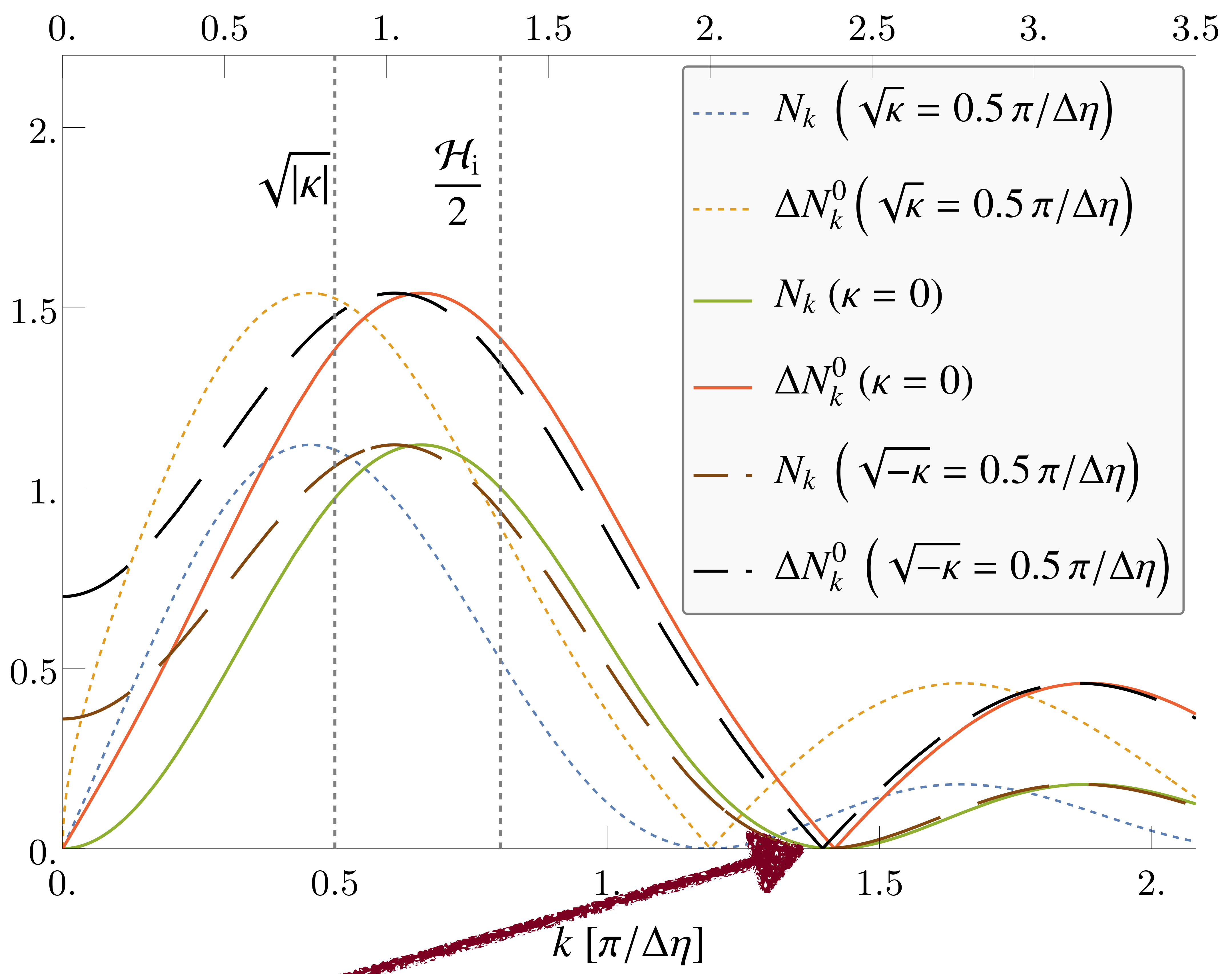




Offset, Amplitude (Barrier)



Offset, Amplitude (Well)



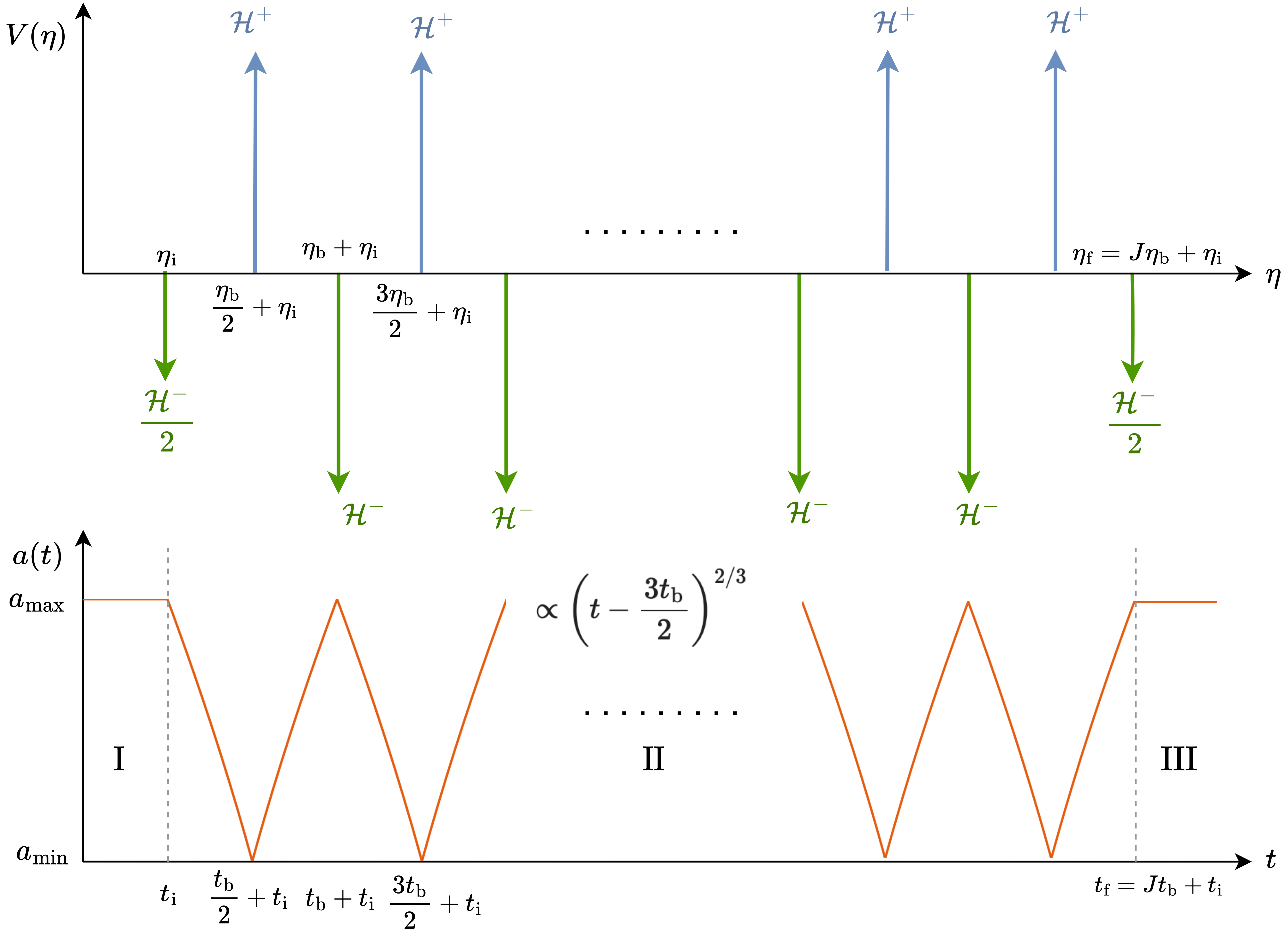
Resonance near continuum threshold

[Senn (1988), Boya (2008)]

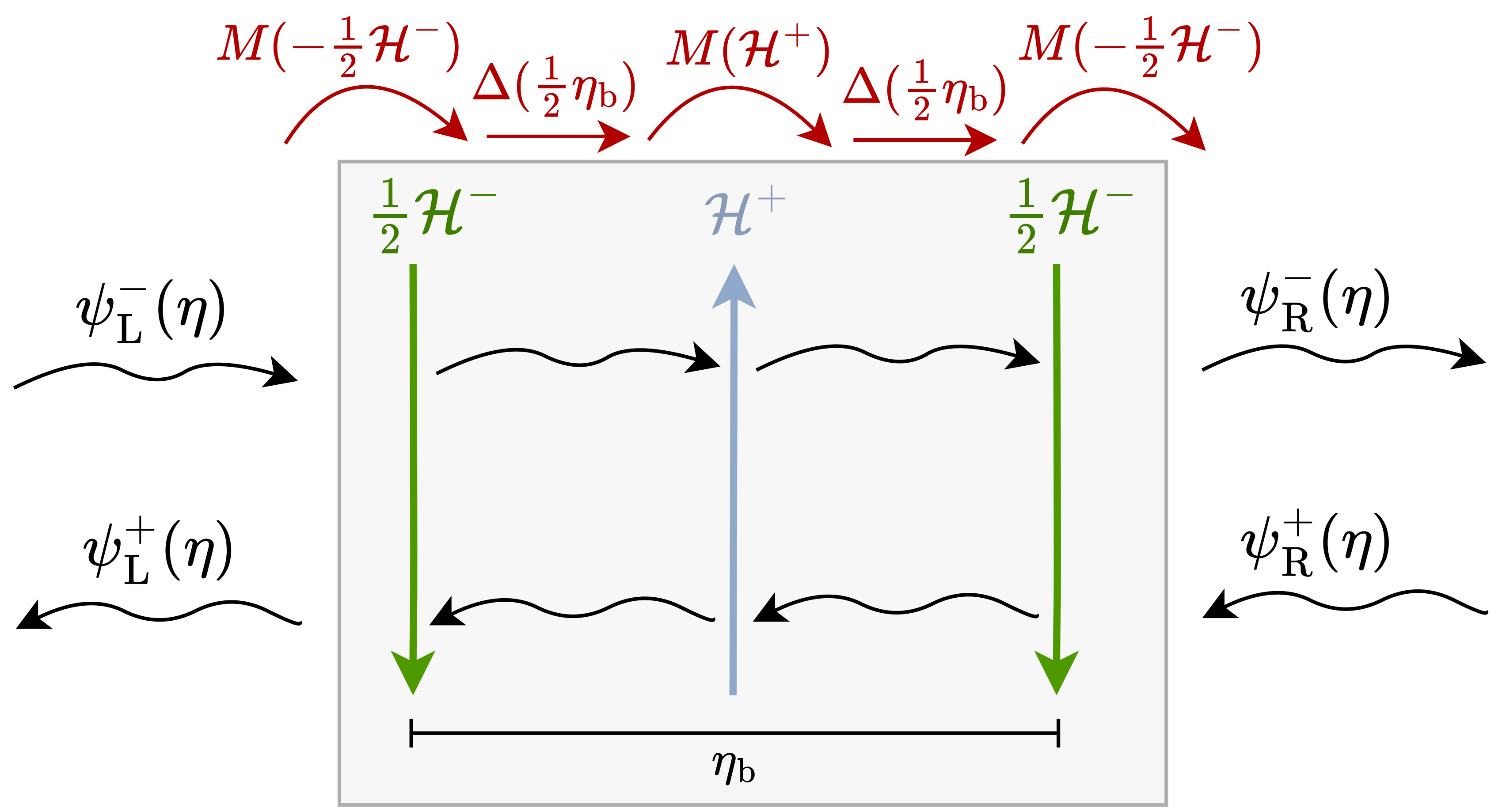
Resonant forward scattering

$$h(k) = \begin{cases} -k\left[k + (D-1)\sqrt{|\kappa|}\right] & \text{for} \quad \kappa > 0 \\ -k^2 & \text{for} \quad \kappa = 0 \\ -\left[k^2 + \left(\frac{D-1}{2}\right)^2 |\kappa|\right] & \text{for} \quad \kappa < 0 \end{cases}$$

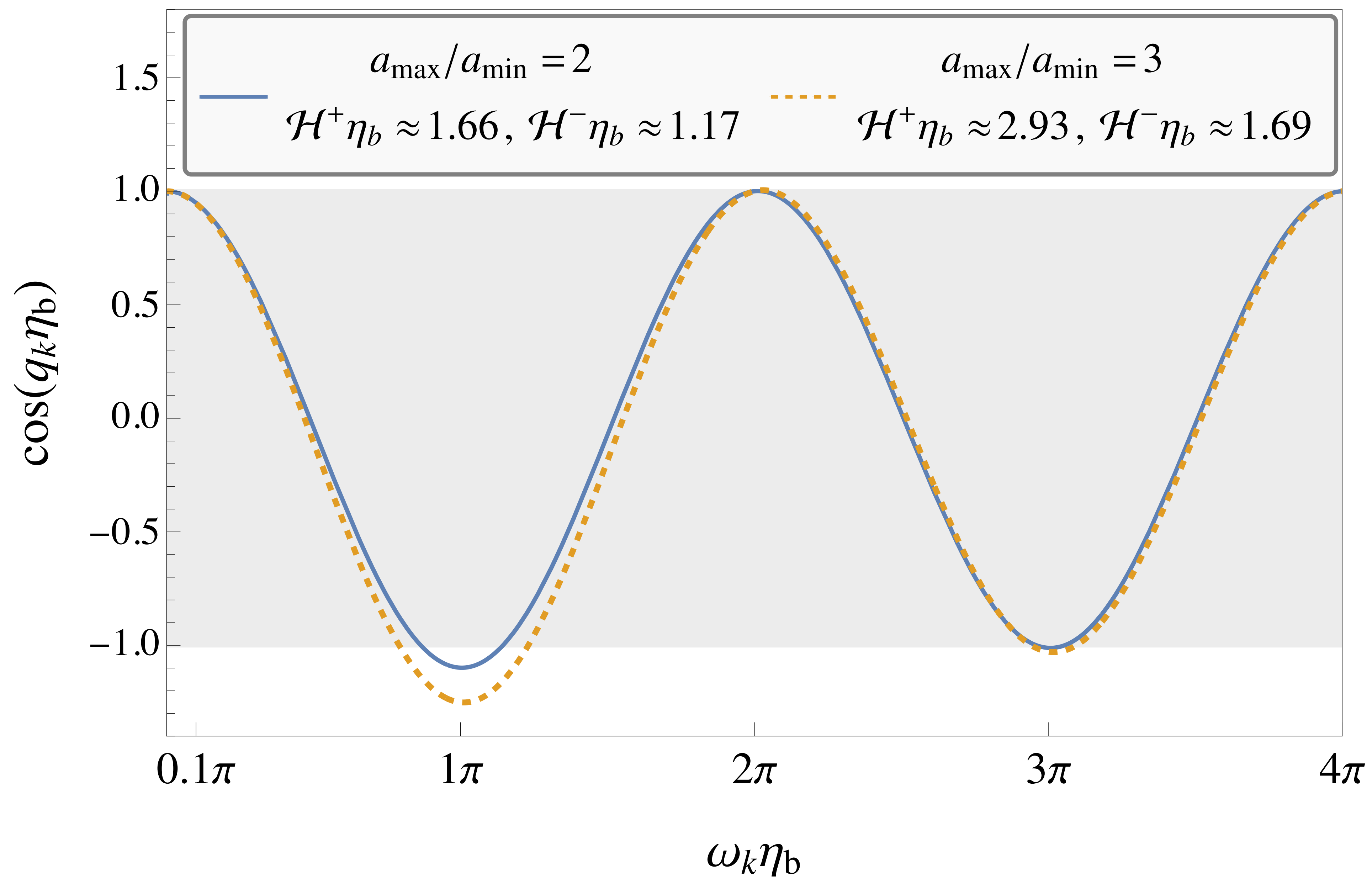
Periodical Cosmological Scattering Potentials



Transfer matrix method

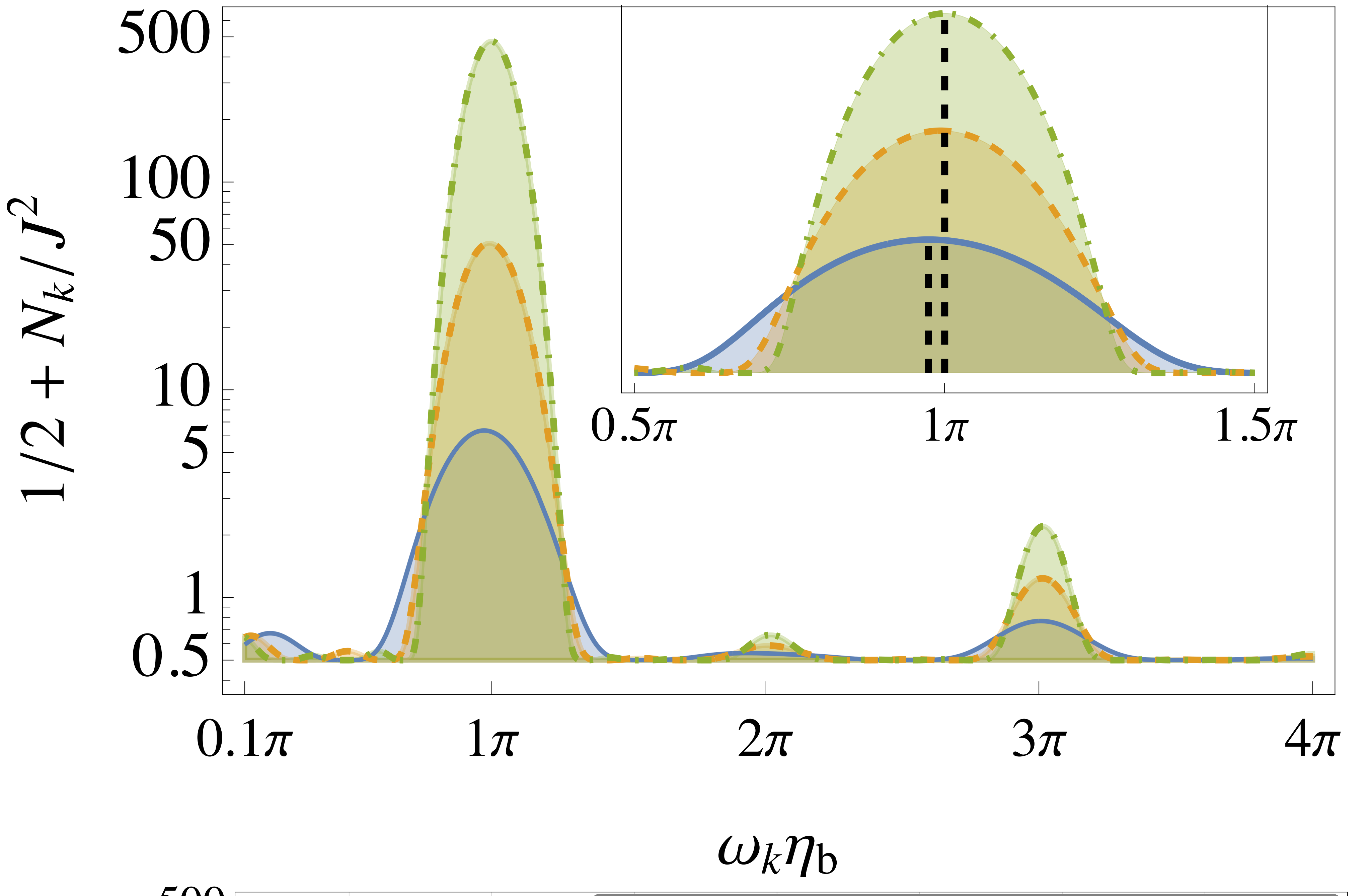


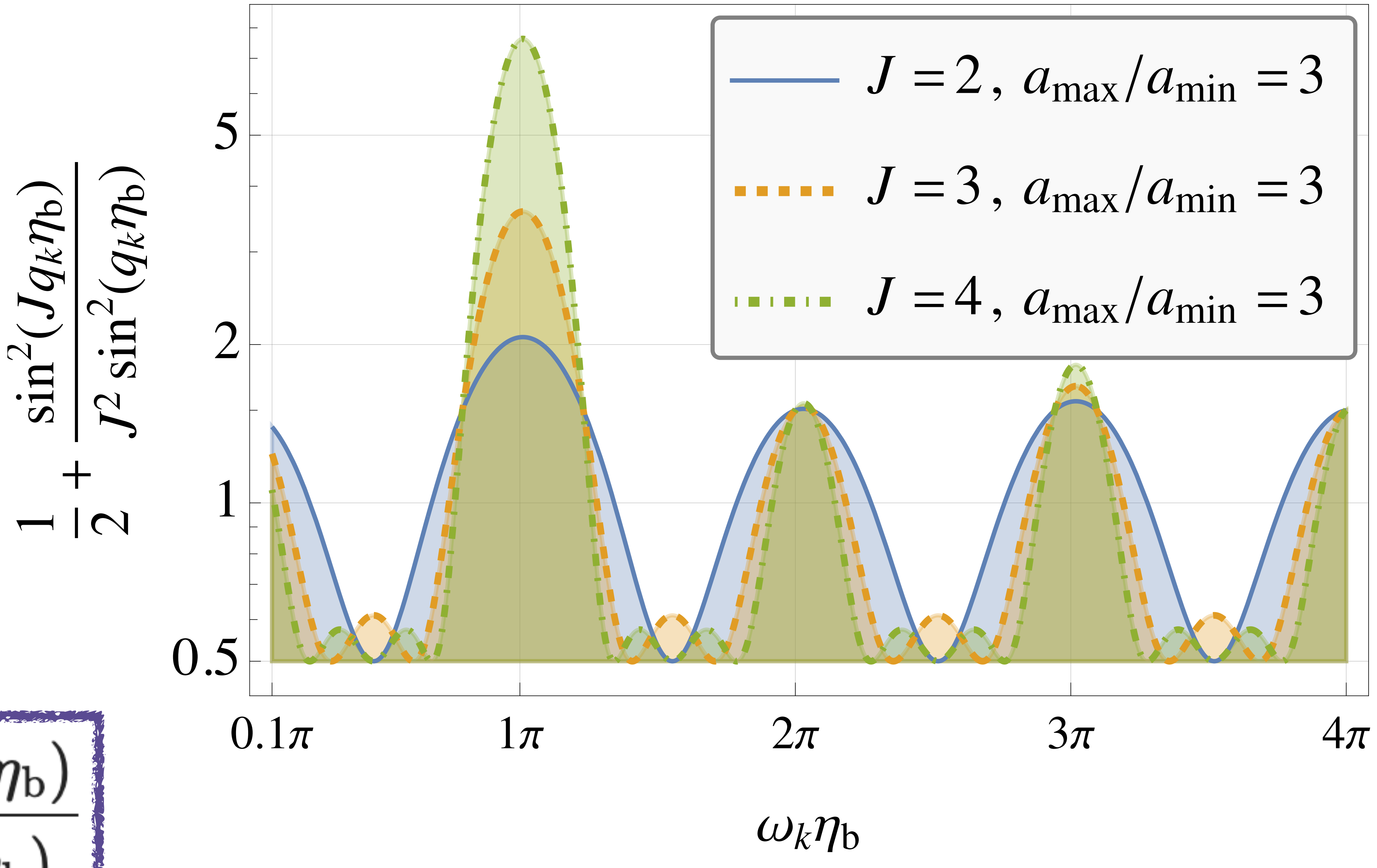
Gapped band structure



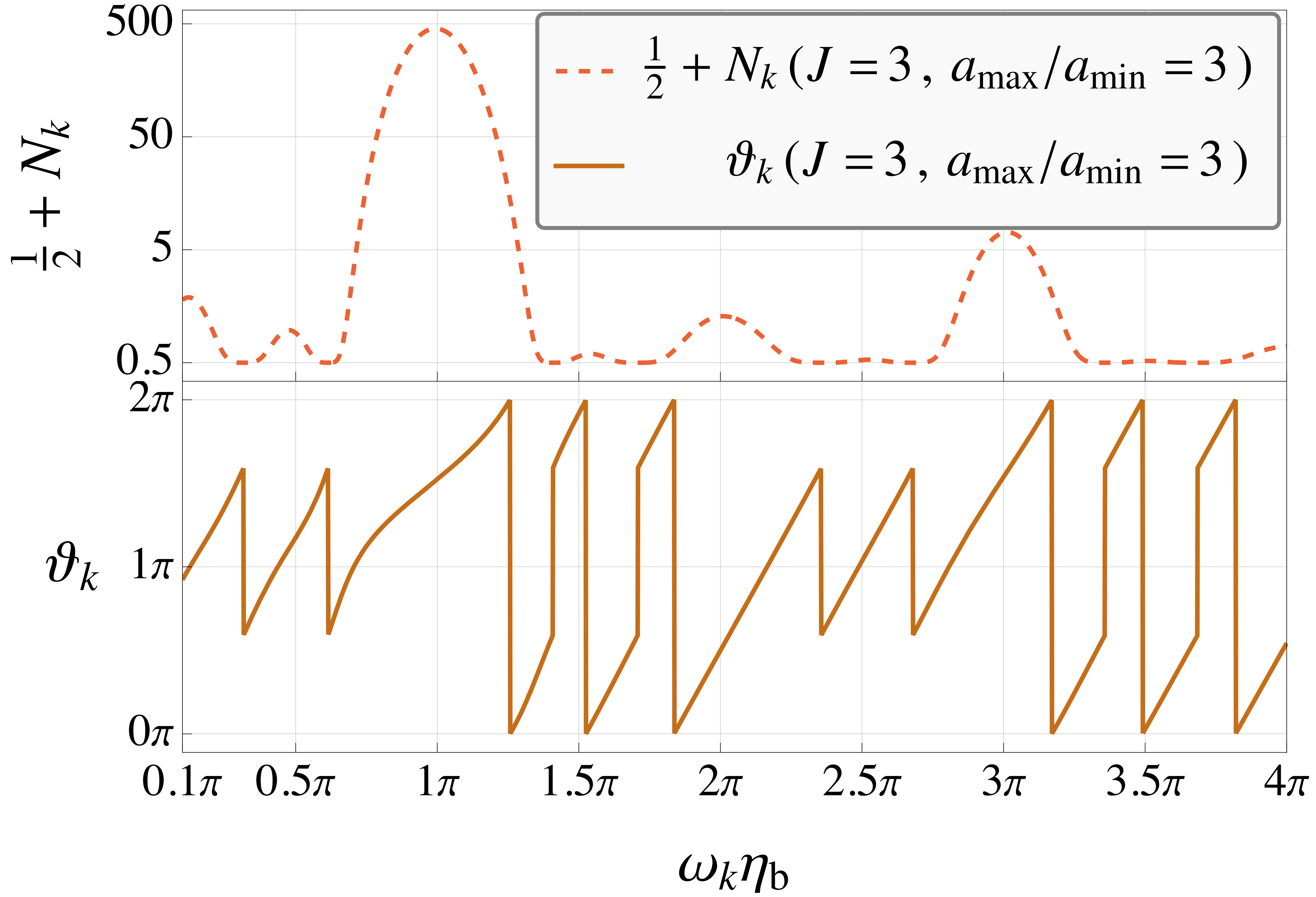
$$\cos(q_k\eta_b) = \cos(\omega_k\eta_b) + \left(\frac{\mathcal{H}^+}{2\omega_k} - \frac{\mathcal{H}^-}{2\omega_k}\right)\sin(\omega_k\eta_b) - \frac{\mathcal{H}^+\mathcal{H}^-}{2\omega_k^2}\sin^2\left(\frac{1}{2}\omega_k\eta_b\right)$$

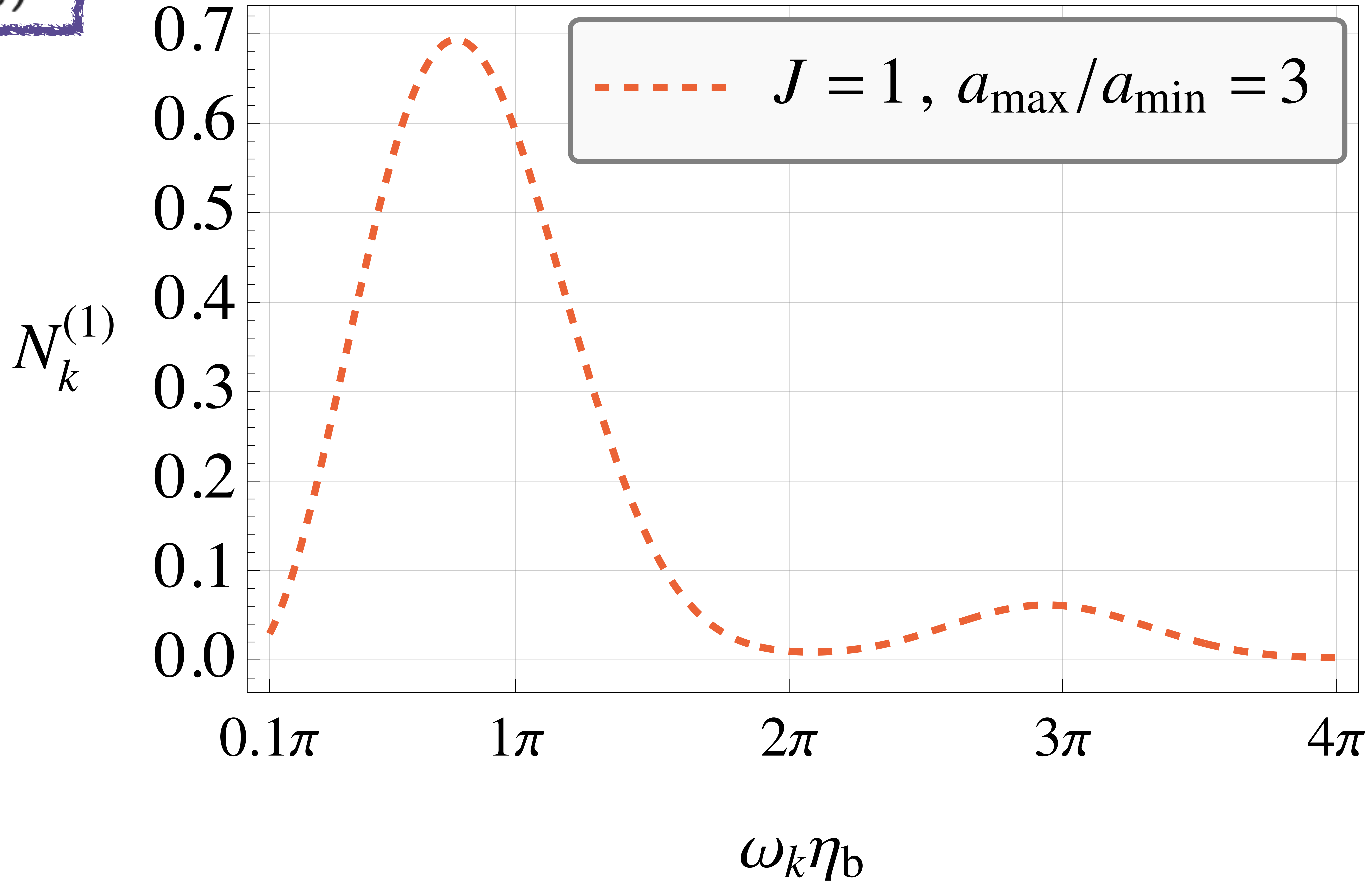
Periodical Cosmological Scattering Potentials





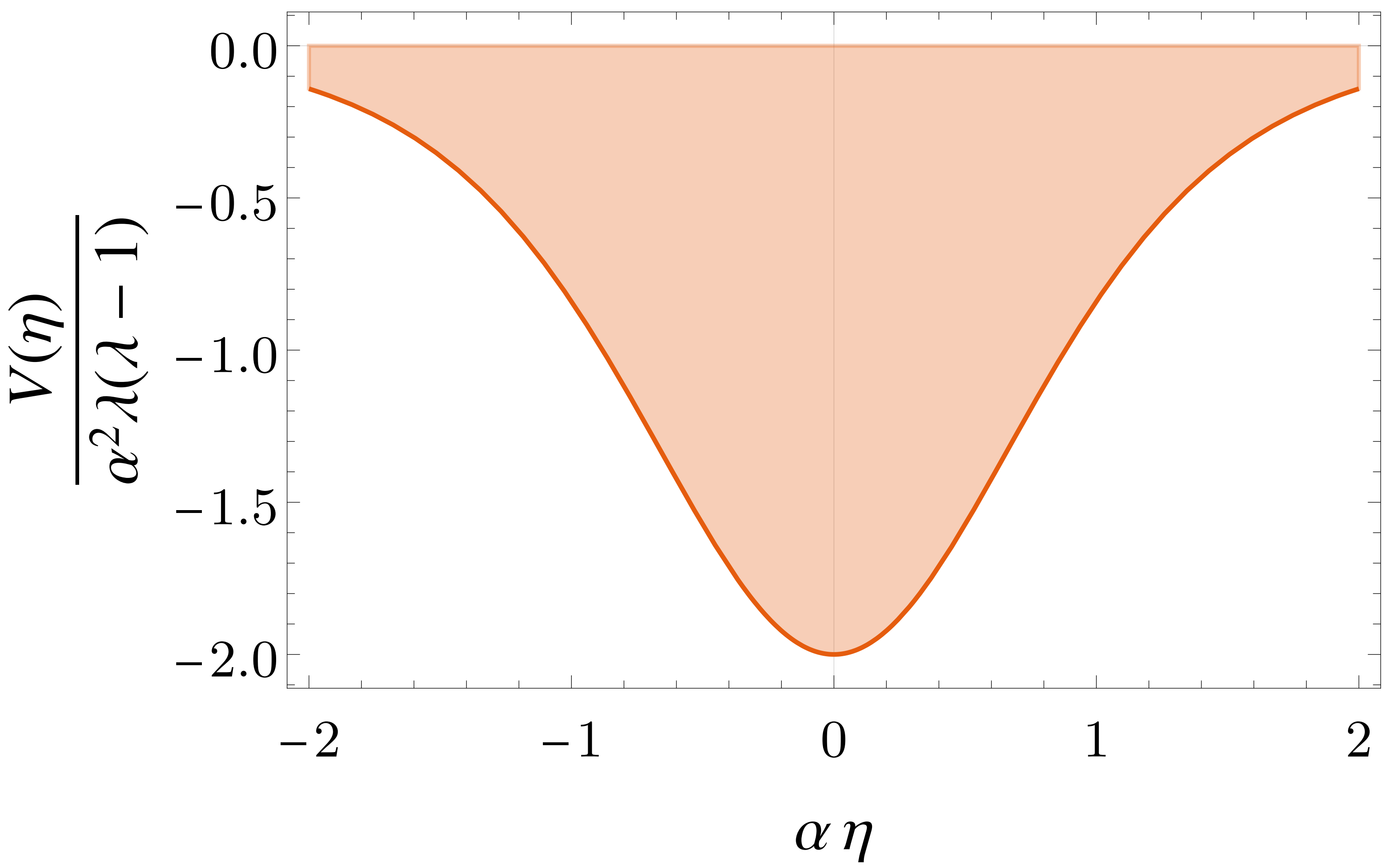
$$N_k = N_k^{(1)}\frac{\sin^2(Jq_k\eta_\mathrm{b})}{\sin^2(q_k\eta_\mathrm{b})}$$



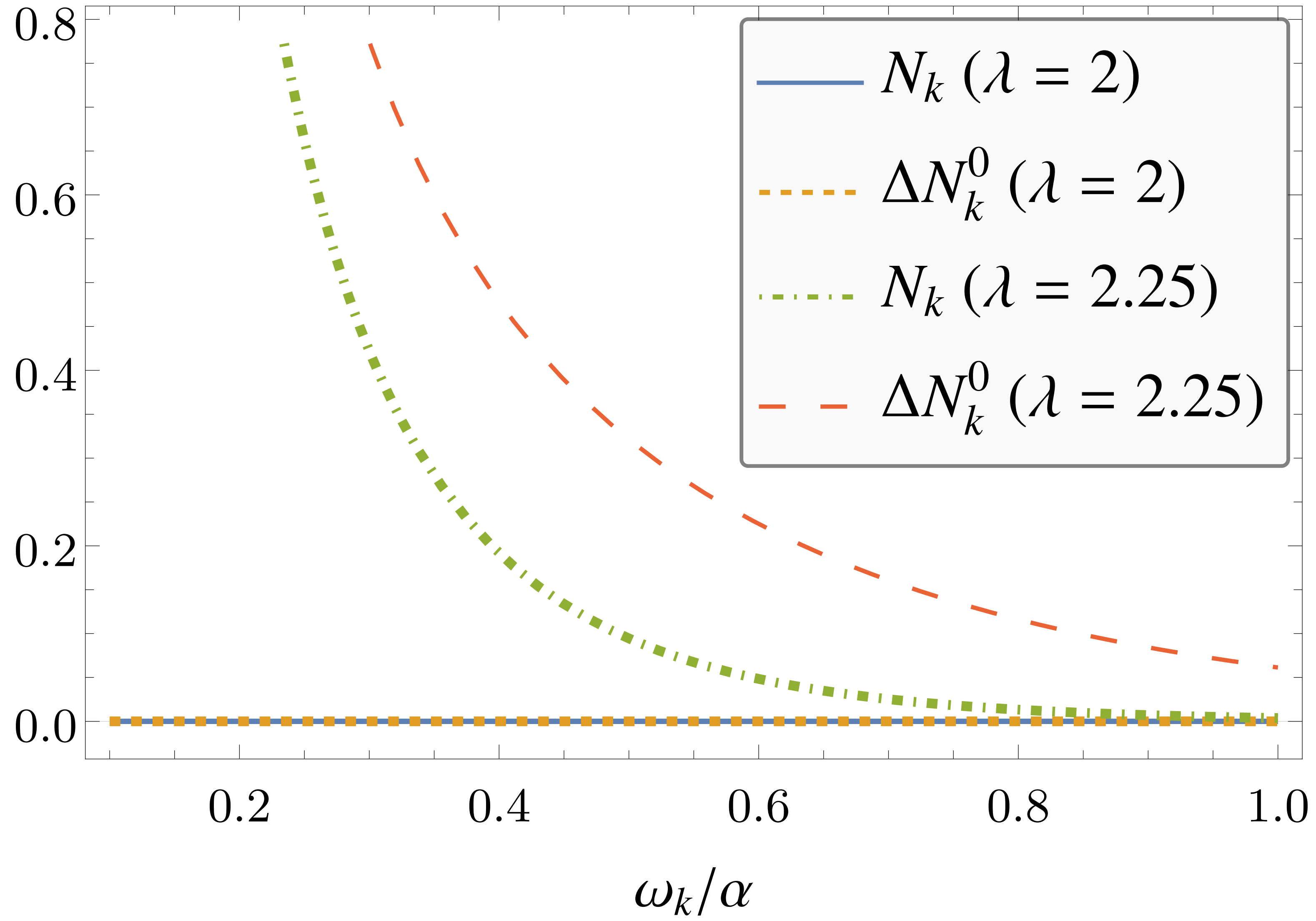


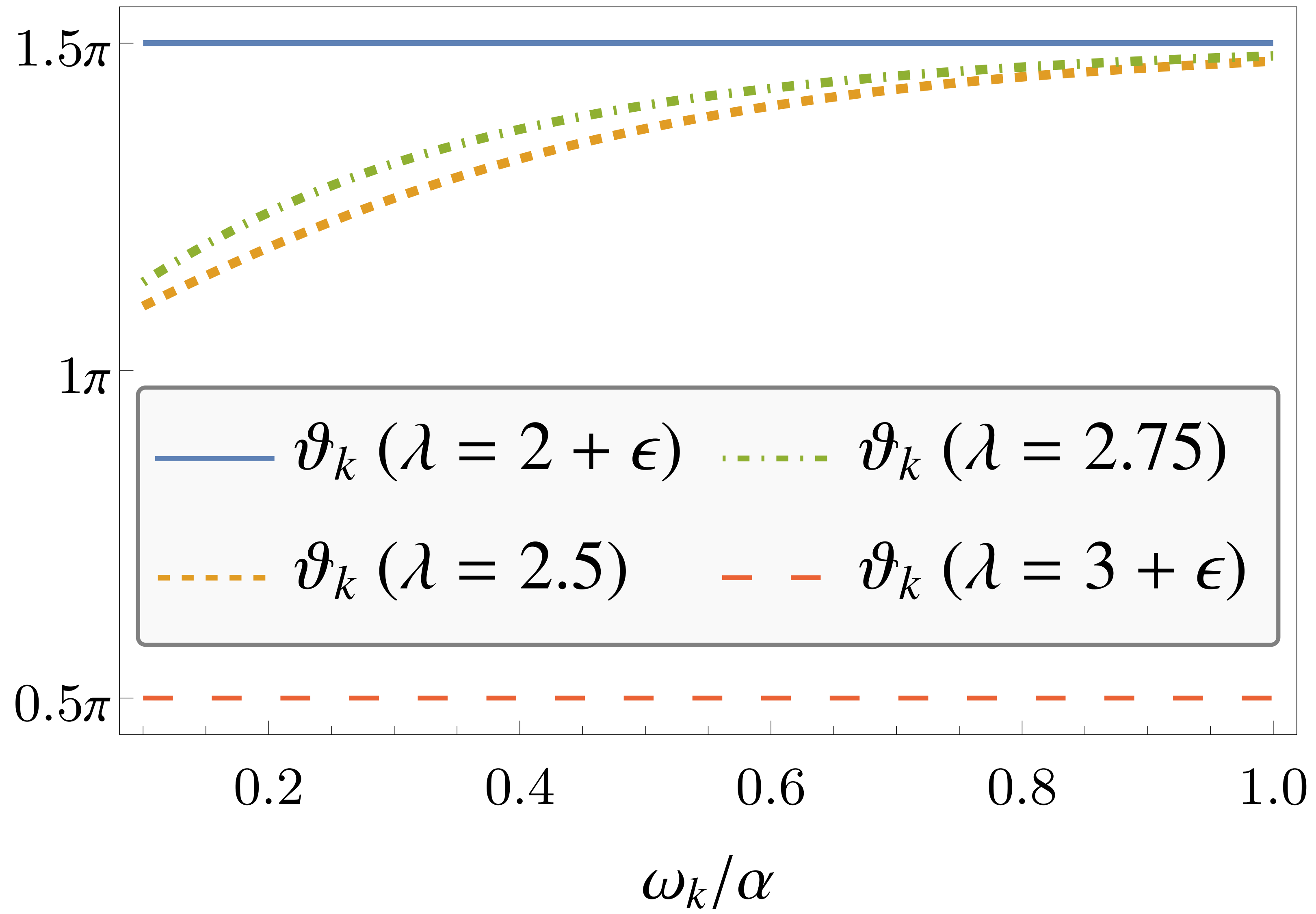
Invariant Cosmological Vacua through Transparant Potentials

Poeschl-Teller potential $V(\eta) = -\alpha^2\lambda(\lambda-1)\,\mathrm{sech}^2(\alpha\eta)$



Spectrum





$$a(\eta) = [c_1 P_{\lambda-1}(\tanh\alpha\eta) + c_2 Q_{\lambda-1}(\tanh\alpha\eta)]^{2/(D-1)}$$

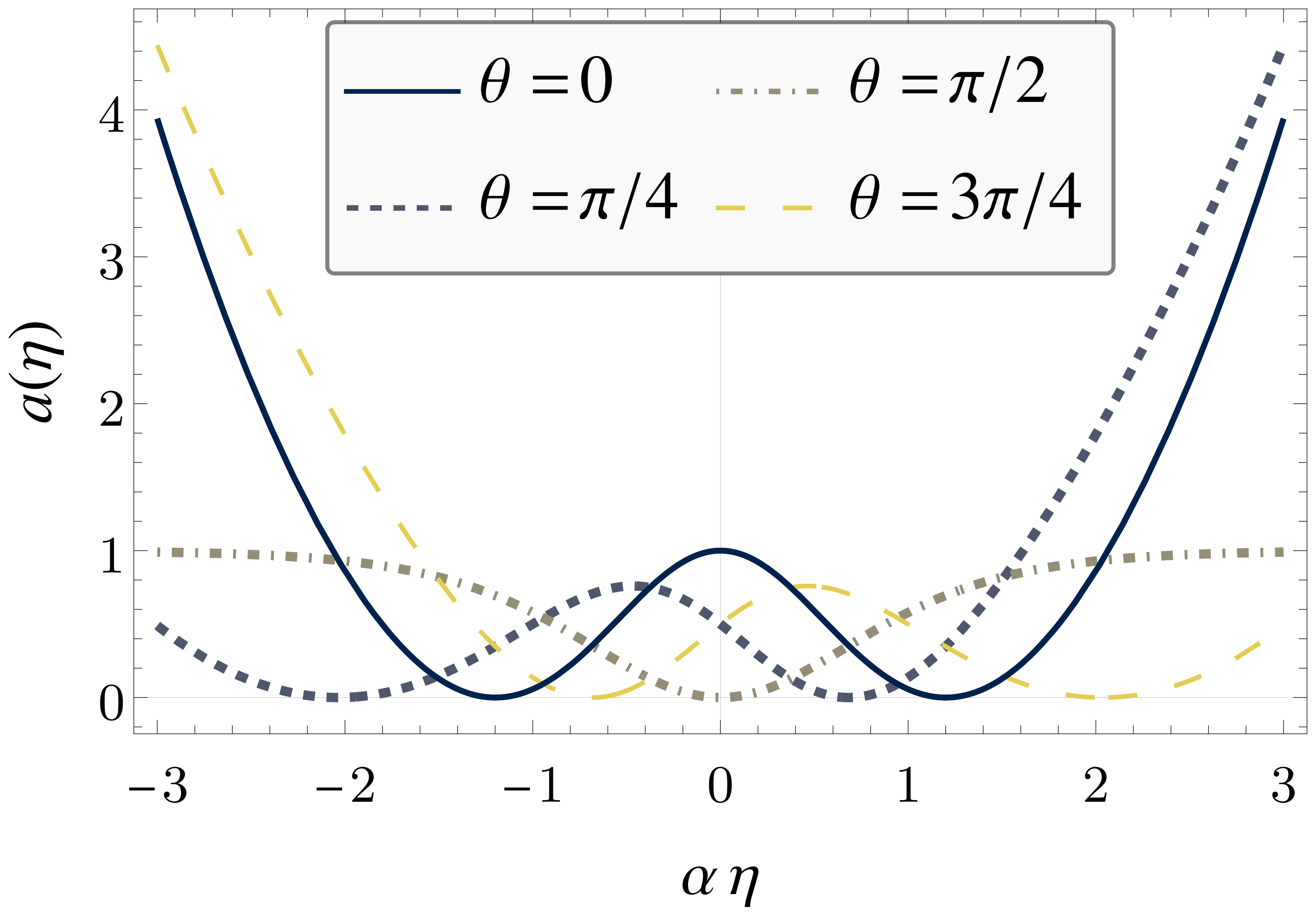
Scale factor

$D = 2$

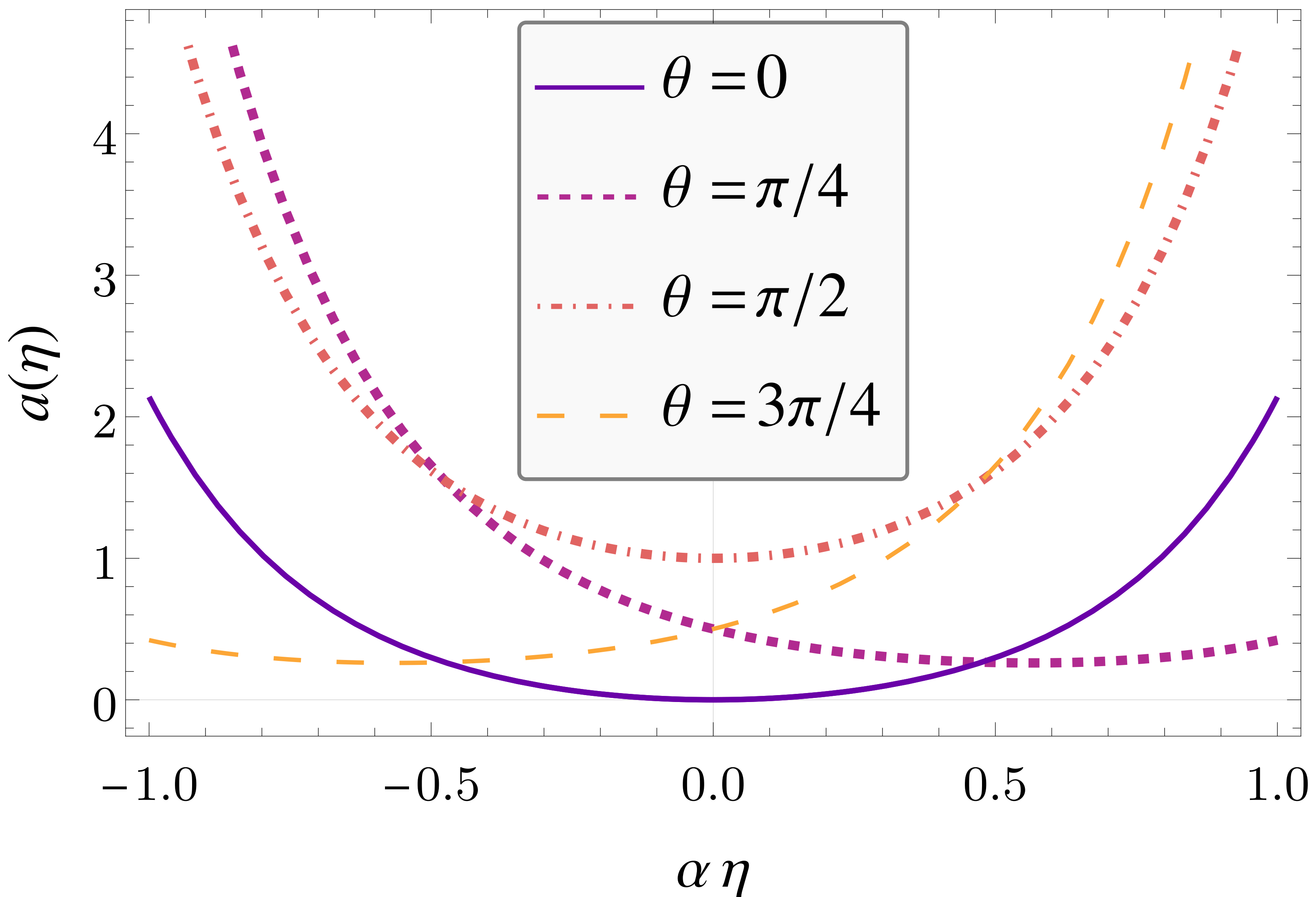
$\lambda = 2$

$c_1 = \cos\theta$

$c_2 = \sin\theta$



Continuum shift: $V(\eta) = \alpha^2\lambda^2 - \alpha^2\lambda(\lambda-1)\,\mathrm{sech}^2(\alpha\eta)$



Constant eff. mass (asympt.):

$$\omega_k^2 \to \omega_k^2 + (\alpha\lambda)^2$$

Invariant Cosmological Vacua through Transparent Potentials

Generalized transparent potentials [Kay,Moses (1956)]

$$V(\eta) = -2\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\log\det[\mathbb{1} + A(\eta)]$$
$$= -4\sum_{m=1}^{N}\kappa_m\psi_m^2(\eta)$$

$$\hat{A}_{nm} = \frac{\sqrt{A_n A_m}}{\kappa_n + \kappa_m}\exp\{(\kappa_n + \kappa_m)\eta\}$$

- Bound-state energies

$$-\kappa_m^2$$

Correspond to **Solitons of Korteweg-deVries-hierarchy** [e.g. Gardner et al. (1974)]

Transparent property related to integrability of
inverse scattering transform:

Scattering amplitudes
+
Bound state energies

→

Scattering potential

Gelfand, Levitan (1955); Marchenko (1955)

Future avenues

Inverse Scattering Theory and Isospectrality:

1. Infer cosmological evolution from power spectrum
2. Isospectral scattering potentials [Cooper,Khare,Sukhatme (1995), Dunne, Feinberg (1998)]

→ **Iso-spectral QFTCS (different coupling to background)**

Full Mode Dispersion:

Beyond healing-length: Bogoliubov dispersion [Bogoliubov (1946),Volovik (2009)]

Corresponds to Corley-Jacobson dispersion as UV-completion
[Corley, Jacobson (1996); Martin, Brandenberger (2002)]

Transplanckian Problem

Analogue rainbow metric

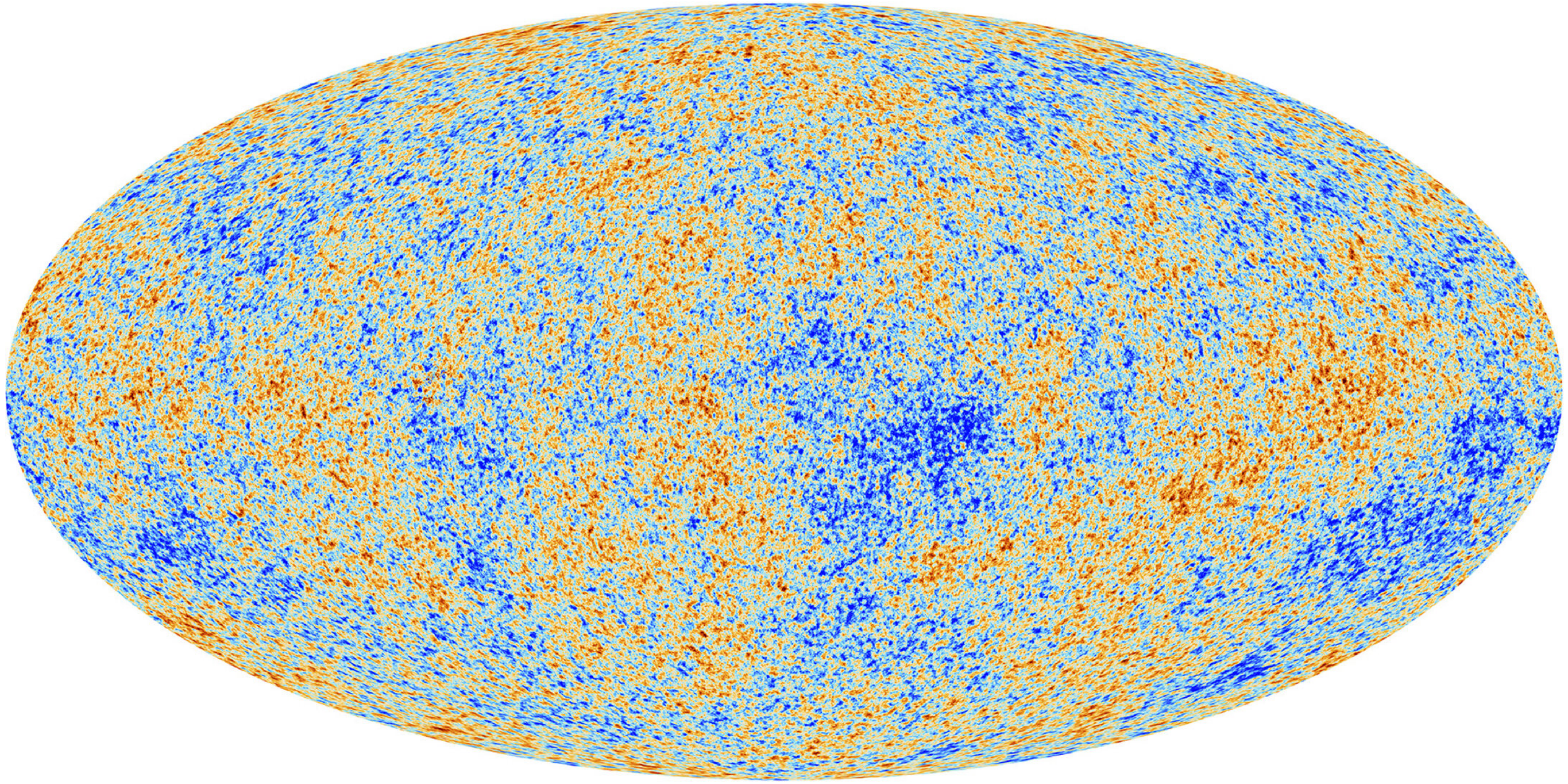
Cosmological Particle Production [e.g. Weinfurtner et al. (2008)]
Hawking radiation [e.g. Coutant, Weinfurtner (2017)]

Non-linear mode interaction in BEC:

Dissipation effects [e.g. Micheli,Robertson (2023)]

Quantum entanglement of two-mode-squeezed states

Primordial Cosmological Perturbations



Source: ESA/Planck Collaboration 2018

- Seeds for cosmic structure formation were generated through dynamic background Universe
- e.g Inflation: Miniscule vacuum fluctuations were amplified and stretched beyond cosmological horizon and ceased oscillating [Mukhanov, Feldman, Brandenberger (1992]
- Alternatives are possible, e.g Bouncing Cosmologies [Brandenberger,Peter (2016); Ijjas,Steinhardt (2018)]

- Generation of both scalar and tensor perturbations to metric is essentially captured by massless scalar on FLRW-spacetime [Martin,Brandenberger(2001); Mukhanov,Winitzki (2013); Brandenberger, Peter (2016)]

BackUp: Primordial Cosmological Perturbations

- Scalar and tensor perturbations to the metric obey (Martin (2008))

$$\frac{\mathrm{d}^2\mu_{S,T}(k,\eta)}{\mathrm{d}\eta^2} + \omega^2_{S,T}(k,\eta)\mu_{S,T}(k,\eta) = 0$$

with

$$\omega_S^2(k,\eta) = k^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \qquad \omega_T^2(k,\eta) = k^2 - \frac{a''}{a}$$

and

$$\gamma = 1 + \left(\frac{a}{a'}\right)'$$

↑

approximately constant for a wide class of inflation models
(Power-law-inflation)

Backup: Scattering Analogy of Cosmological Particle Production

Generalize theory

$$\Gamma[\phi] = -\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^D x\sqrt{g}\left[\partial_\mu\phi\partial^\mu\phi + \left(m^2 + \xi R\right)\phi^2\right]$$

Ricci scalar

Line-element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}u^2}{1-\kappa u^2} + u^2\mathrm{d}\Omega_D^2\right]$$

1. Conformal time

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}$$

2. Rescale

$$\chi(x) = a^{\frac{D-1}{2}}(\eta)\phi(x)$$

To wit

$$\Gamma[\chi] = -\frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^D x\sqrt{\gamma}\,\chi\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \Delta + m_{\mathrm{eff}}^2(\eta)\right]\chi$$

Mode expansion

$$\chi(\eta,\boldsymbol{x}) = \int_{\boldsymbol{k}}\left[a_{\boldsymbol{k}}\mathcal{H}_{\boldsymbol{k}}(\boldsymbol{x})\psi_k(\eta) + a_{\boldsymbol{k}}^*\mathcal{H}_{\boldsymbol{k}}^*(\boldsymbol{x})\psi_k^*(\eta)\right]$$

Eigenfunctions of Laplace-Beltrami

$$\Delta\cdot\mathcal{H}_k(\boldsymbol{x}) = -h(k)\mathcal{H}_k(\boldsymbol{x})$$

$$h(k) = \begin{cases} -k\left[k + (D-1)\sqrt{|\kappa|}\right] & \text{for} \quad \kappa > 0 \\ -k^2 & \text{for} \quad \kappa = 0 \\ -\left[k^2 + \left(\frac{D-1}{2}\right)^2|\kappa|\right] & \text{for} \quad \kappa < 0 \end{cases}$$

Mode equation has Schrödinger form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + V(\eta)\right]\psi_k(\eta) = E_k\psi_k(\eta)$$

Energy eigenvalue

$$E_k = \sqrt{-h(k)}$$

Scattering potential

$$\begin{aligned} V(\eta) &= -m_{\mathrm{eff}}^2(\eta) \\ &= -a^2(\eta)\left[m^2 + \xi R(\eta)\right] + \frac{D-1}{2}\left[\frac{a''(\eta)}{a(\eta)} - \frac{3-D}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2\right] \end{aligned}$$

Examples:

$$V(\eta) = -a_0^2 m^2 + \xi D(D-1)\kappa$$

Stationary space $a(t) = a_0$

$$V(\eta) = -\frac{(D-1)^2}{4}\kappa$$

Conformally coupled, massless $\xi = \frac{D-1}{4D}$, $m = 0$

<u>Focus on minimally coupled, massless fields from now on</u> $(\xi = 0,\ m = 0)$

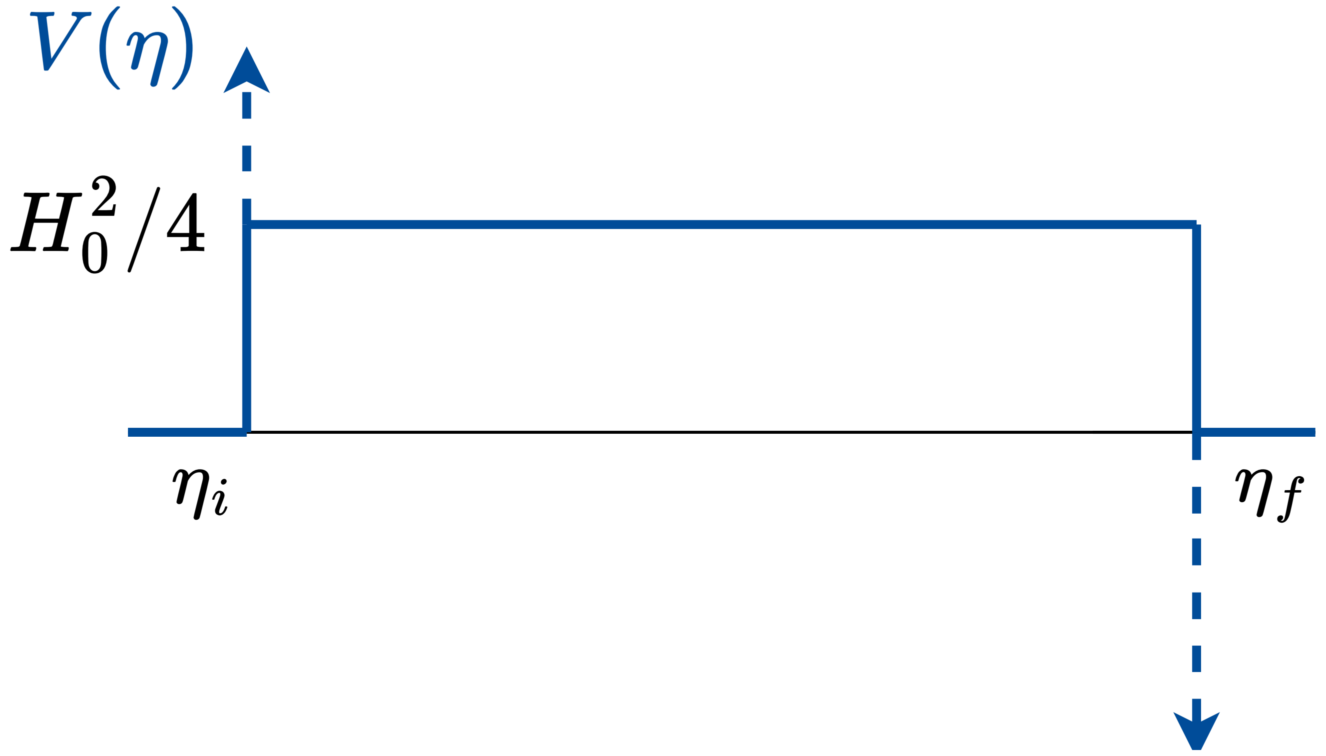
Vanishing potential: $V(\eta) = 0$

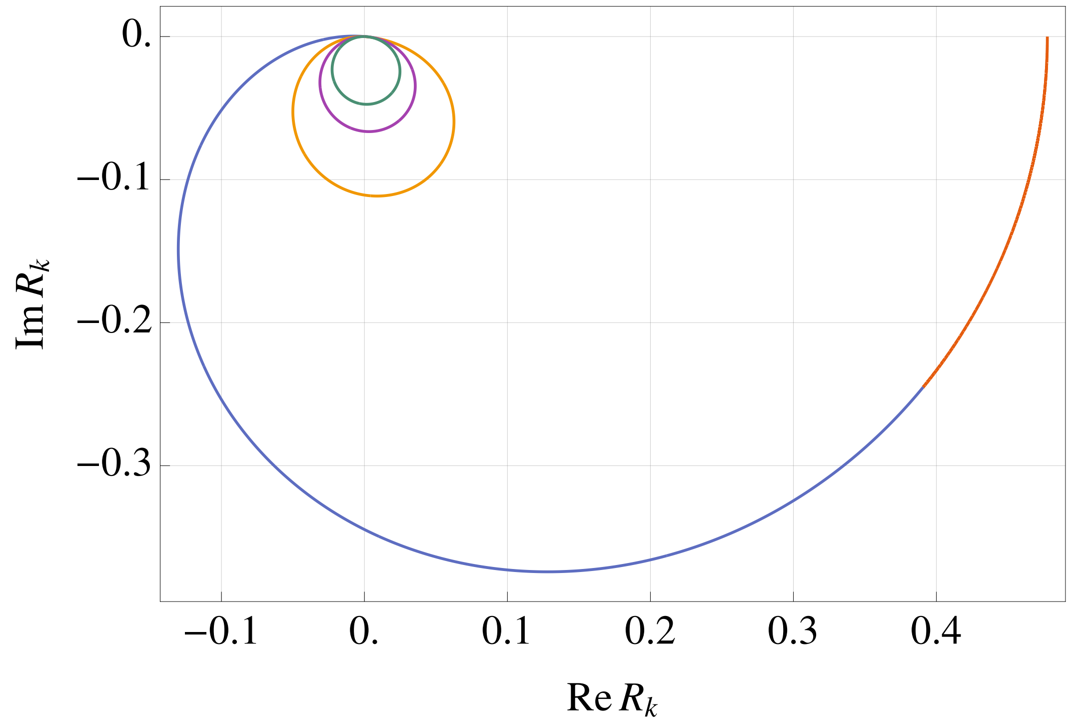
$$\left[\frac{D-1}{2} - q(t)\right]\dot{a}(t)^2 = 0$$

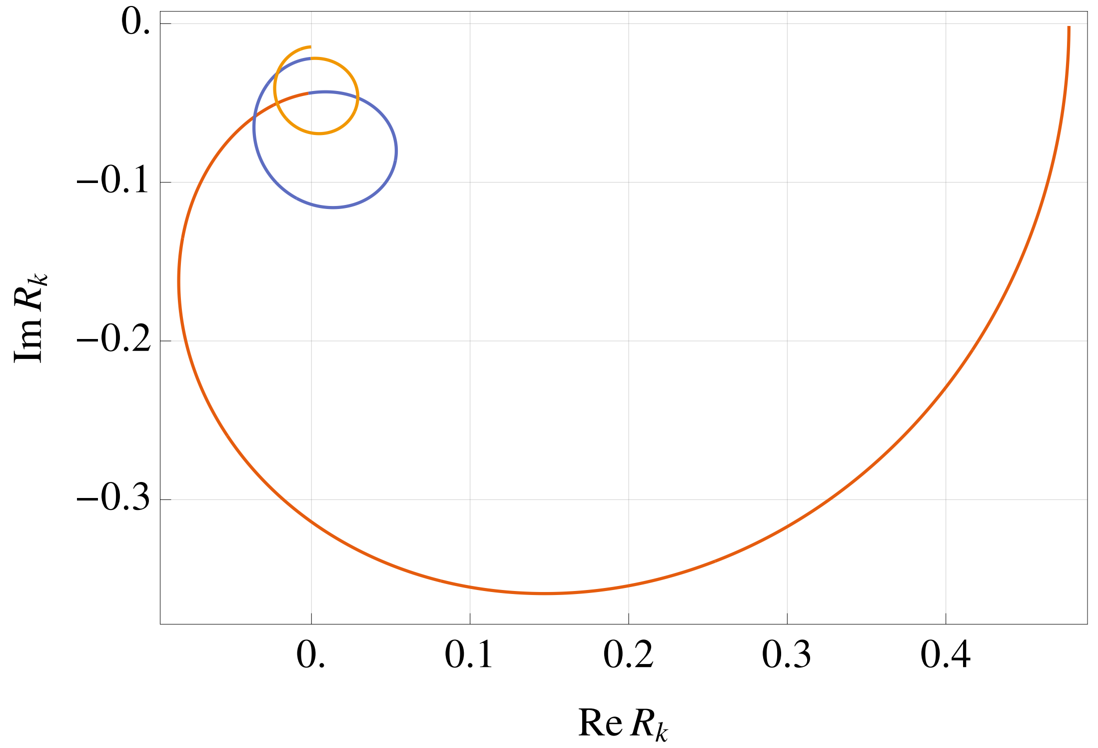
Deceleration parameter $q(t) = -\frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)}$

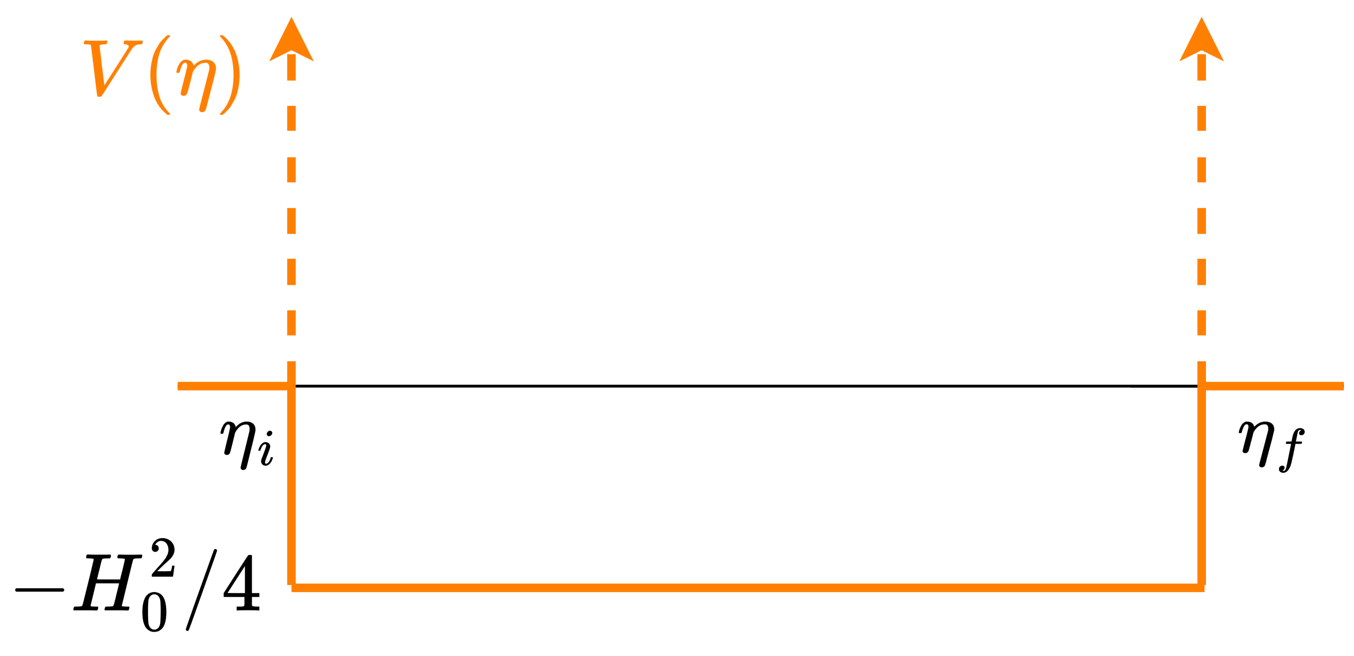
Implies either stasis $(\dot{a} = 0)$ **or radiation domination** $q = (D-1)/2$

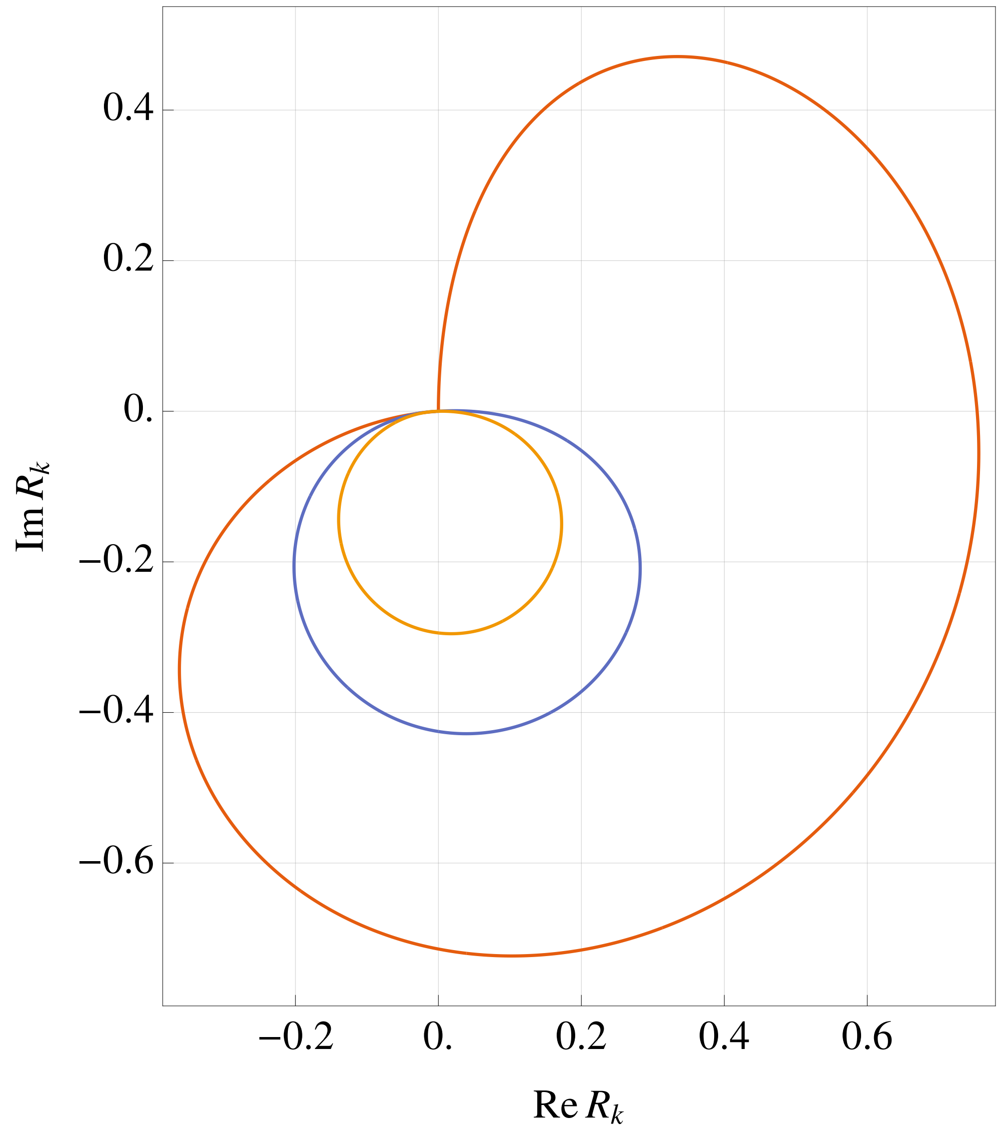
Backup: Resonant forward scattering



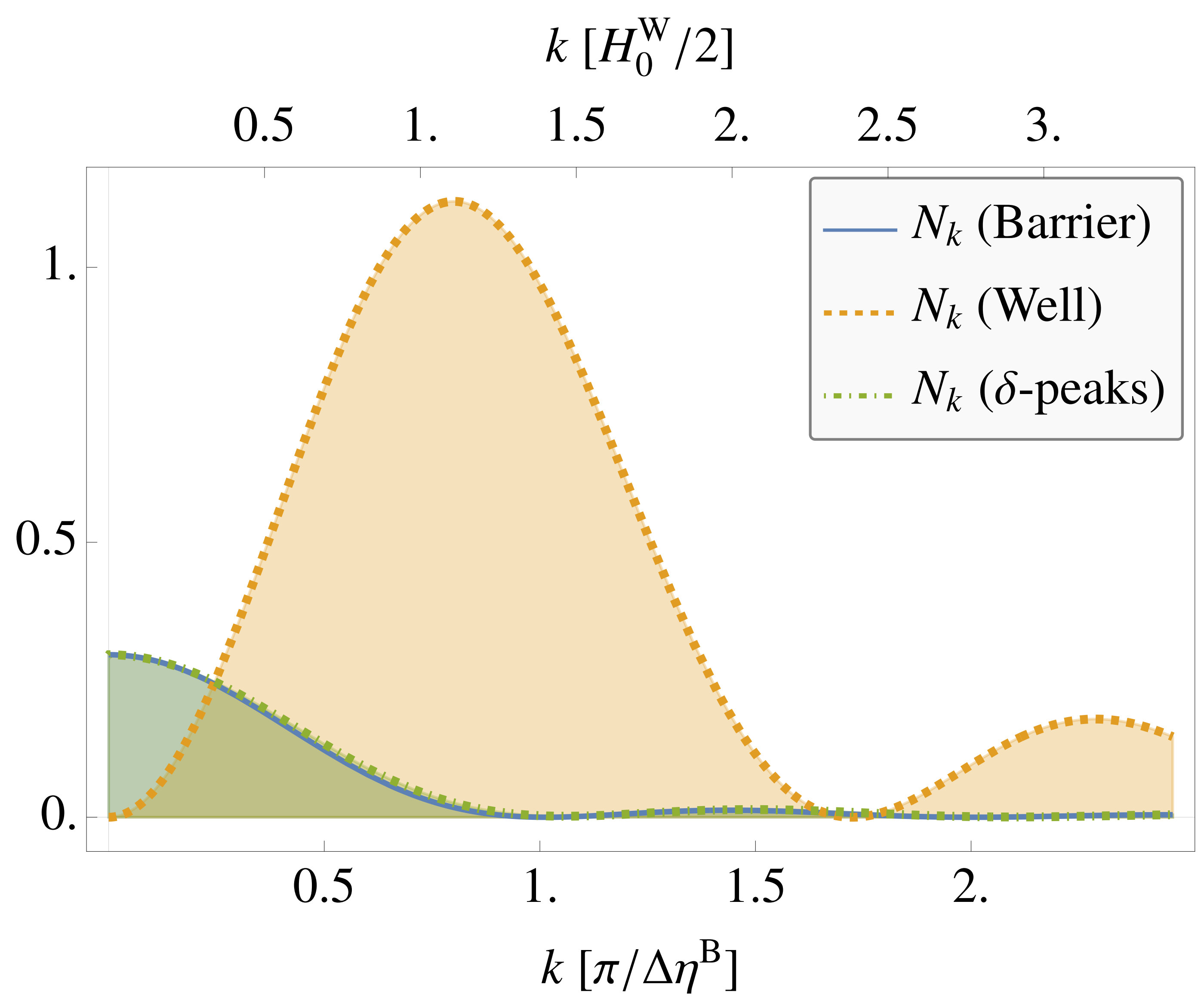


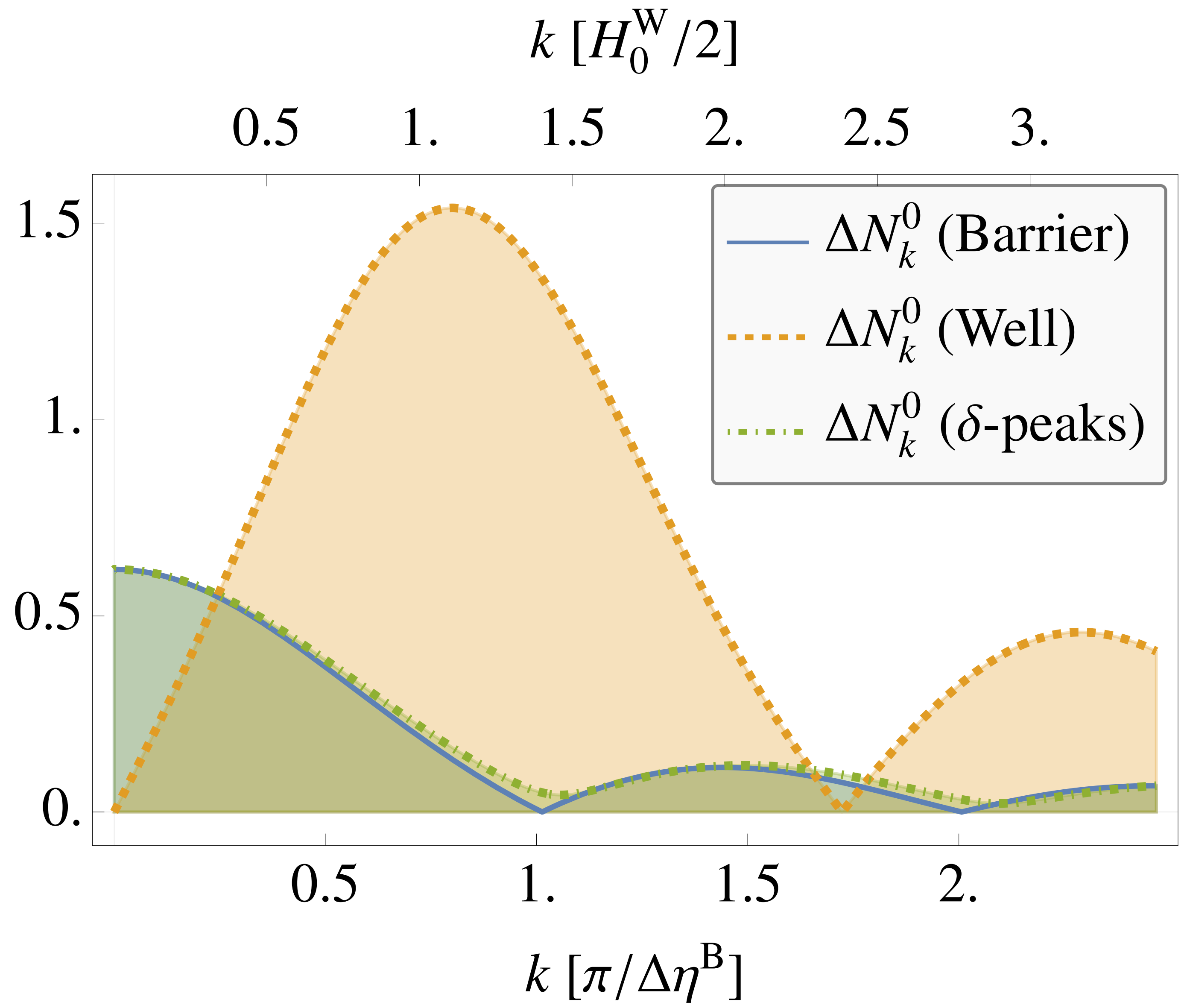


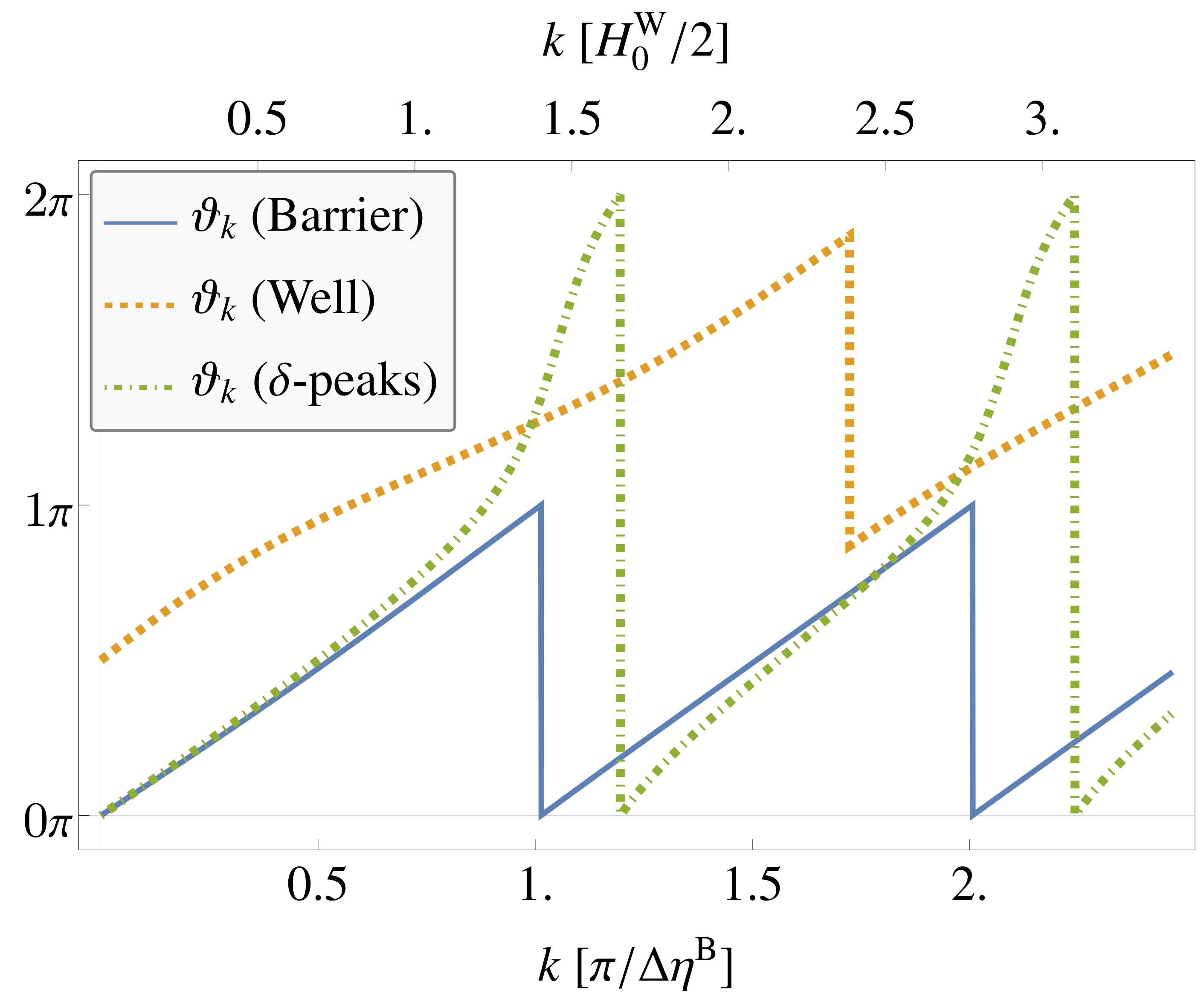


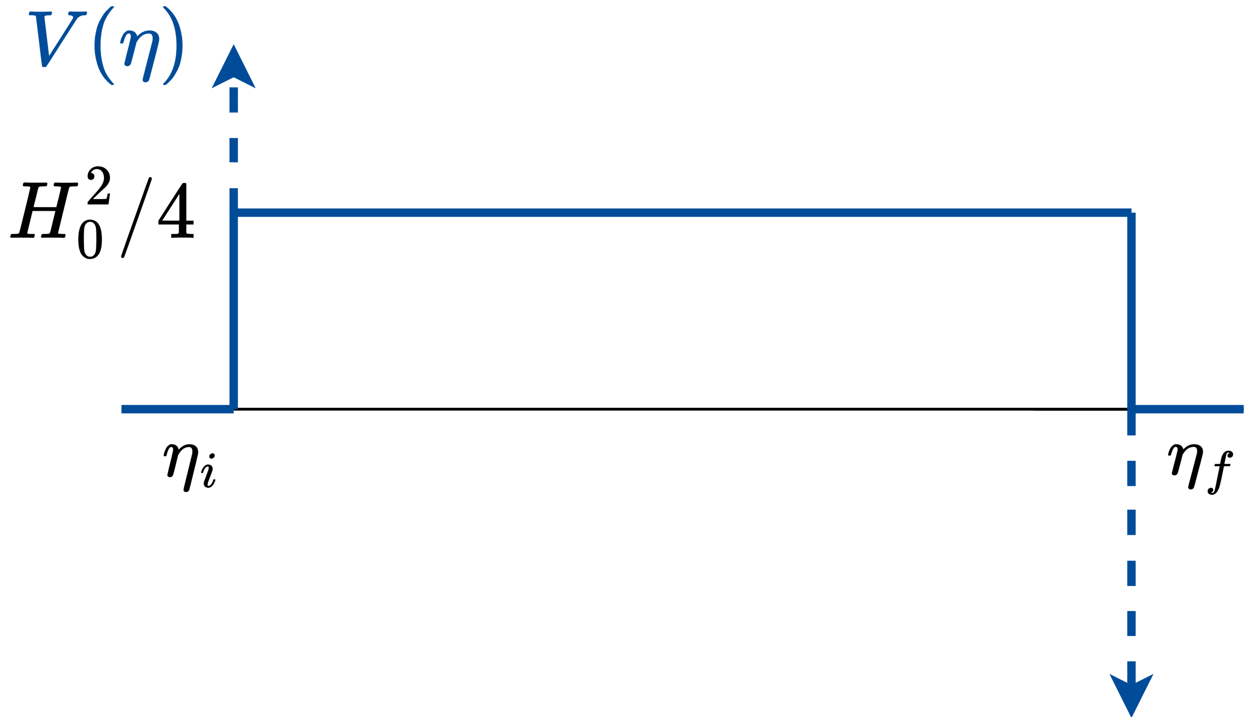


$$\Delta\eta = \int \frac{\mathrm{d}t}{a(t)} = \int c_\mathrm{S}\,\mathrm{d}t$$







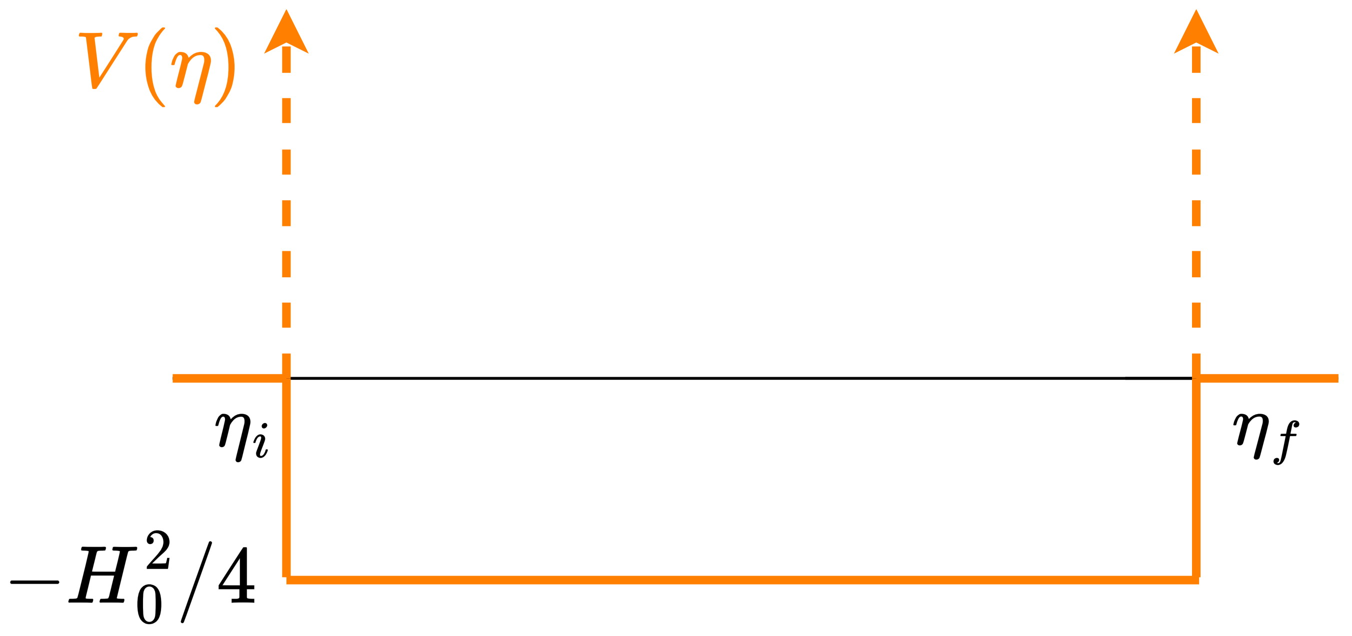


$$a(t) = 1 + H_0 t$$
$$H_0\left(\eta_{\rm f} - \eta_{\rm i}\right) = \ln\left(\frac{a_{\max}}{a_{\min}}\right)$$

$$V_{\rm s}(\eta) = \frac{H_0}{2}\left[\delta(\eta - \eta_{\rm i}) - \delta(\eta - \eta_{\rm f})\right]$$

$$a(t) = \left(1 + \frac{3}{2}H_0 t\right)^{\frac{2}{3}}$$
$$\frac{H_0}{2}(\eta_{\rm f} - \eta_{\rm i}) = \sqrt{a_{\max}} - \sqrt{a_{\min}}$$

$$V_{\rm s}(\eta) = \frac{H_0}{2\sqrt{a_{\min}}}\delta(\eta - \eta_{\rm i}) - \frac{H_0}{2\sqrt{a_{\max}}}\delta(\eta - \eta_{\rm f})$$



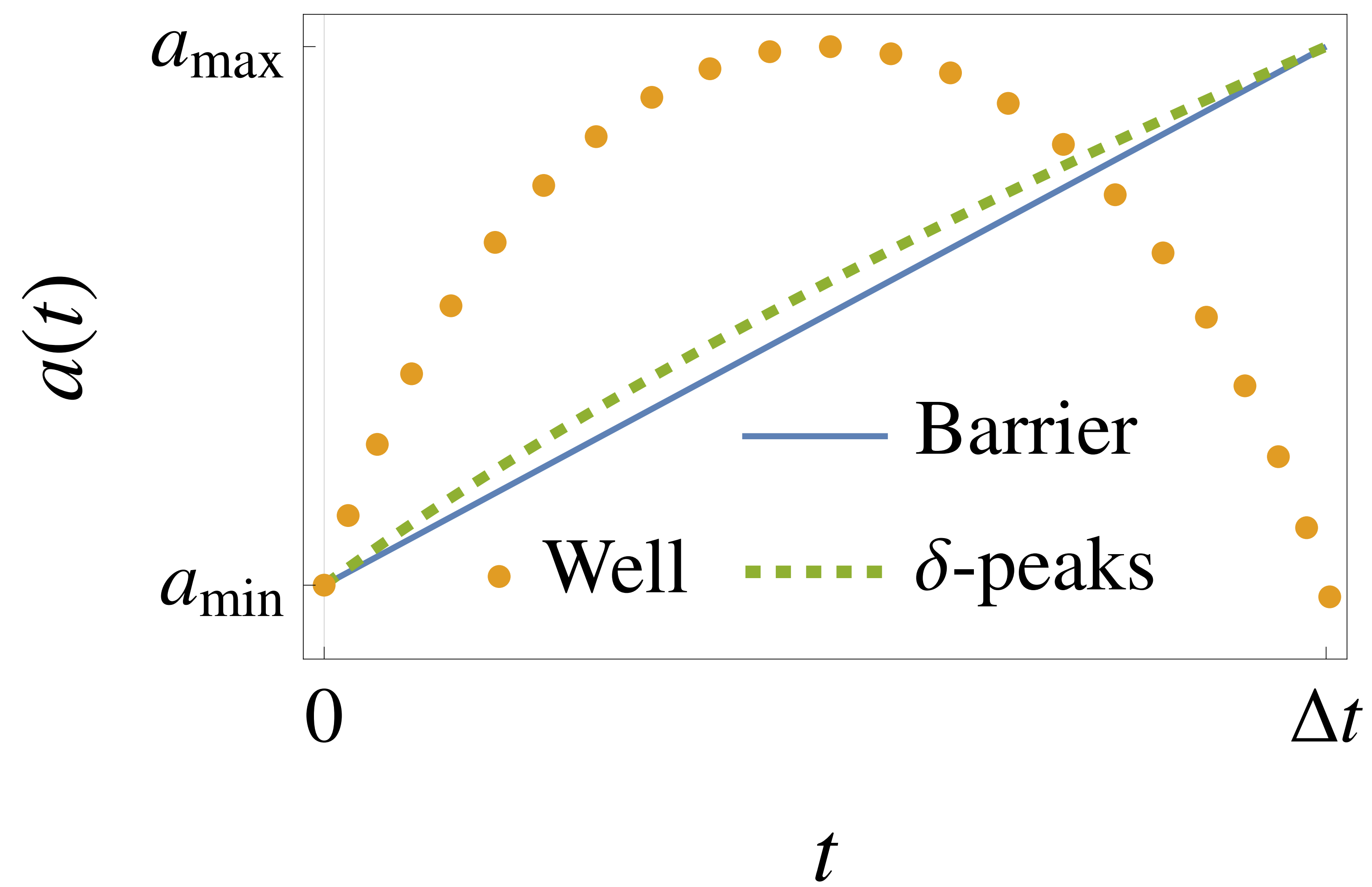
$$\frac{H_0}{2}(\eta_{\rm f} - \eta_{\rm i}) = \arccos\left(2\frac{a_{\min}}{a_{\max}} - 1\right)$$

$$V_{\rm s}(\eta) = \frac{H_0}{2}\sqrt{\frac{a_{\max}}{a_{\min}} - 1}\left[\delta(\eta - \eta_{\rm i}) + \delta(\eta - \eta_{\rm f})\right]$$

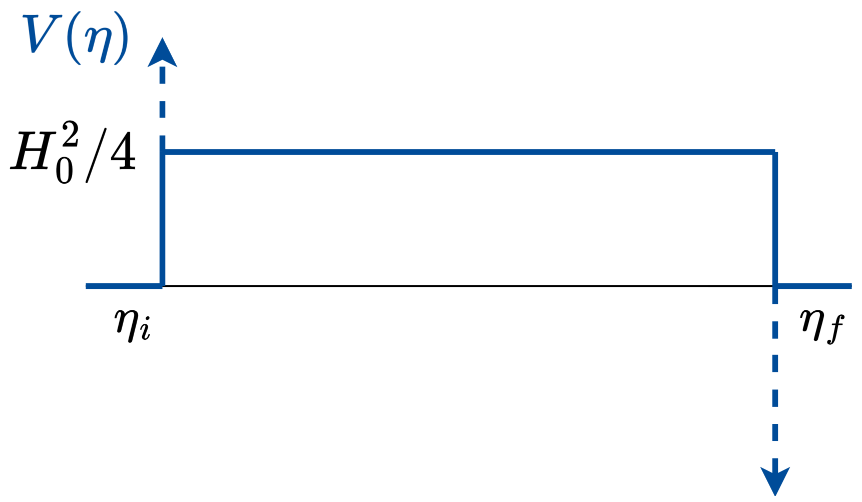
Hubble Parameters

$$H_0 = \frac{1}{c_{\rm s}(a_{\rm s}^{\max})^{q+1}\Delta t}\frac{(a_{\rm s}^{\max}/a_{\rm s}^{\min})^{(q+1)/2} - 1}{q+1}$$

$$H_0 = \frac{1}{c_{\rm s}(a_{\rm s}^{\min})\Delta t}\left[\arccos\left(2\sqrt{\frac{a_{\rm s}^{\min}}{a_{\rm s}^{\max}}} - 1\right) + 2\left(\sqrt{\frac{a_{\rm s}^{\min}}{a_{\rm s}^{\max}}} - \frac{a_{\rm s}^{\min}}{a_{\rm s}^{\max}}\right)^{1/2}\right]$$

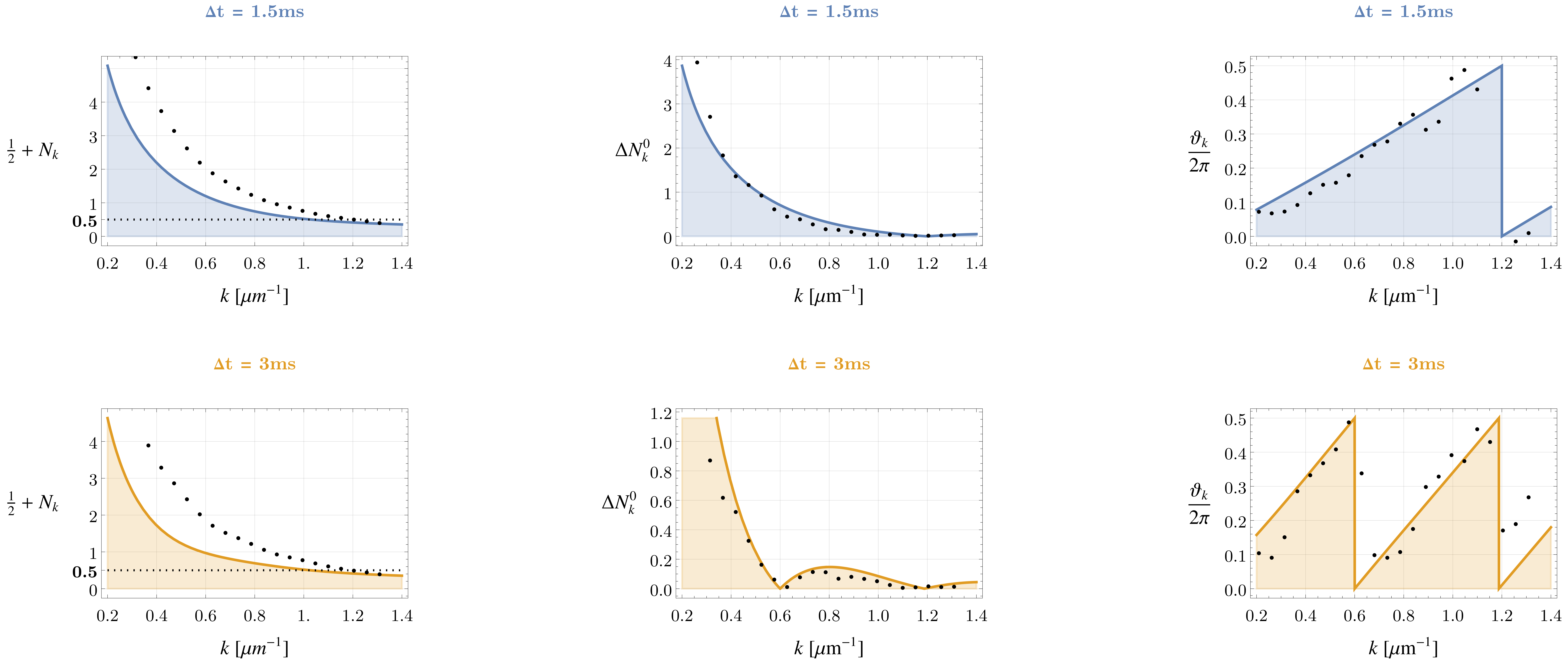


BackUp: PowerLaw (linear)



Spectrum for $\gamma = 1$, α_i= 400, α_f = 50;

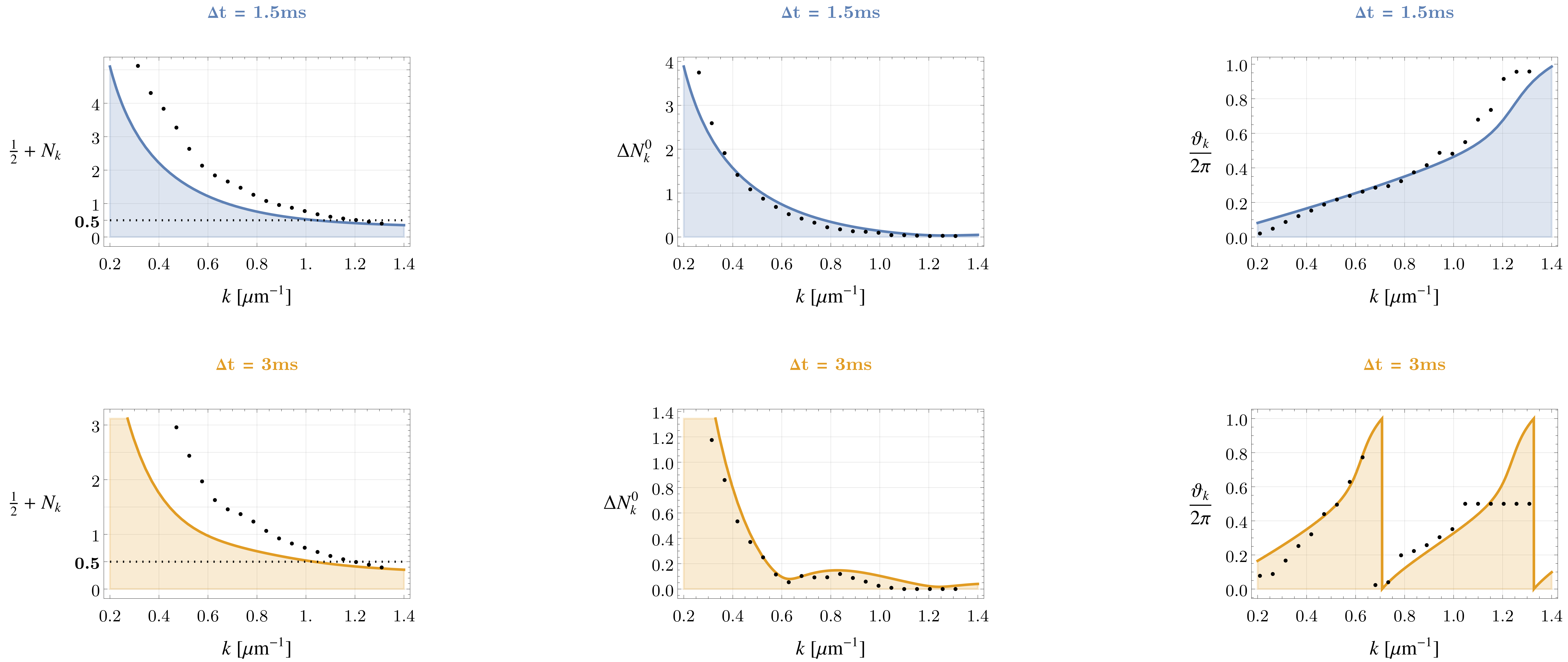
c_s^{fin}= 1.1 μm / ms



BackUp: Power Law (2/3)

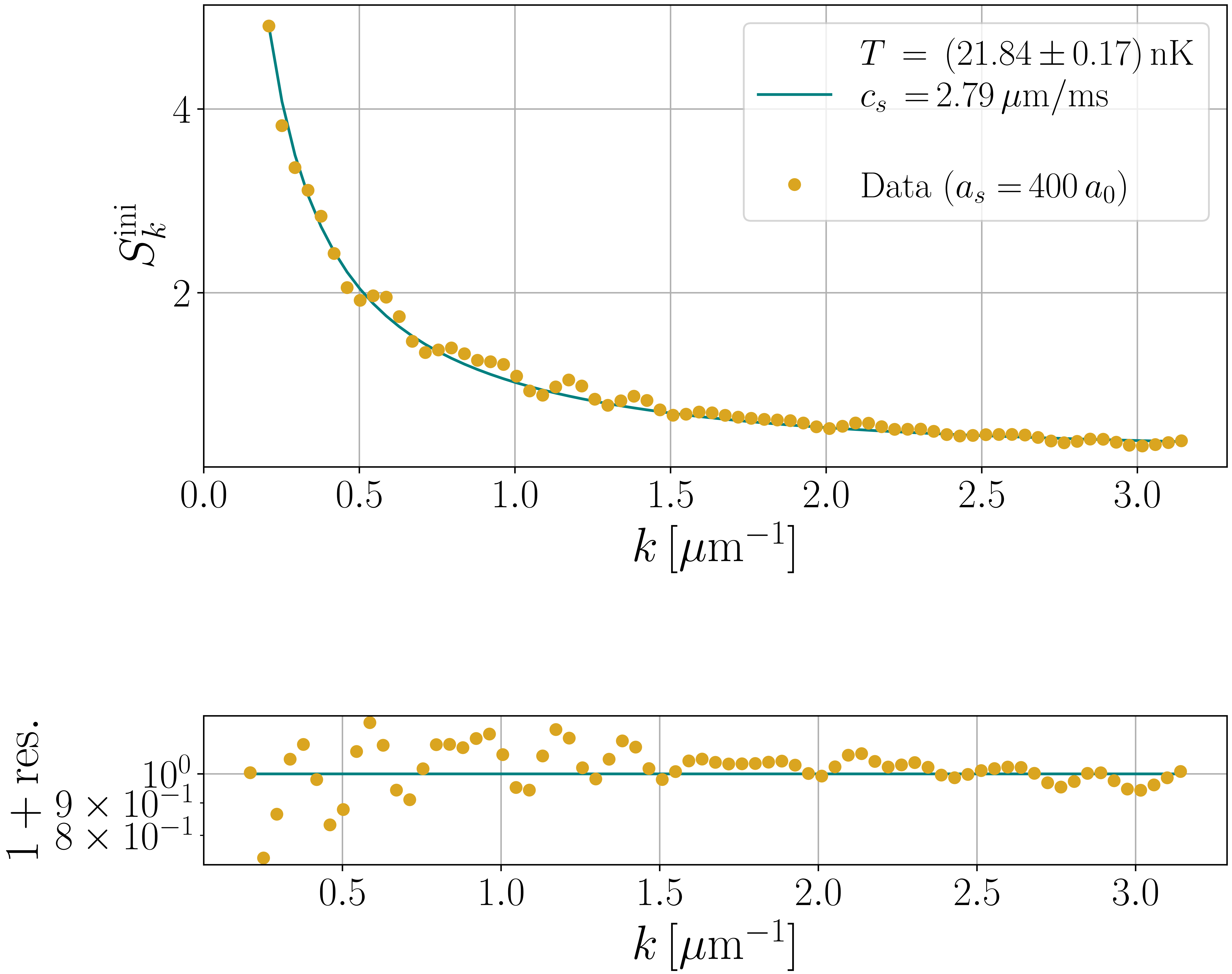
Spectrum for $\gamma = 2/3$, $\alpha_i = 400$, $\alpha_f = 50$;

$c_s^{\text{fin}} = 1.1\ \mu m / ms$



Thermal Initial State

Initial spectrum (linear ramp)



BackUp: Isospectral Cosmologies

$$V(\eta) = \frac{D-1}{2}\left[\frac{a''(\eta)}{a(\eta)} + \frac{D-3}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2\right]$$

minimally coupled, massless

$a_b(t) = \sqrt{1+(H_0 t)^2}$ ⟷ $a_{\text{lin}}(t) = 1 + H_0 t$

$D = 3$ $\quad V(\eta) = H_0^2 \quad$ $D = 3$

$a_{\exp}(t) = \exp(H_0 t)$

Wands (1999) ↕

$D = 3$

$a_{\text{cont}}(t) \propto (-t)^{2/3}$

⟷ $a_{\text{quad}}(t) = \left(1 + \frac{H_0}{2} t\right)^2$

$$V(\eta) = \frac{2}{(\eta-\eta_0)^2}$$

$D = 2$

Find further partners with $y(\eta) \propto a^{(D-1)/2}(\eta)$ $\quad y''(\eta) - V(\eta) y(\eta) = 0$

Or: Find different potentials with equal scattering coefficients

Iso-spectral cosmological backgrounds

Factorize Scattering Hamiltonian (after shift to zero energy solution)

$$H_1 = -\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + V_1(\eta) = A^\dagger A \qquad H_2 = -\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + V_2(\eta) = AA^\dagger$$

with $$A = \frac{\mathrm{d}}{\mathrm{d}\eta} + W(\eta), \quad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}\eta} + W(\eta)$$

→ $$\boxed{V_1(\eta) = W^2(\eta) - W'(\eta)}$$ **is iso-spectral to** $$\boxed{V_2(\eta) = W^2(\eta) + W'(\eta)}$$

[Dunne, Feinberg (1998)]

In scattering analogy

$$W(\eta) = -\frac{y'(\eta)}{y(\eta)} = -\frac{D-1}{2}\frac{a'(\eta)}{a(\eta)}$$ with $$y(\eta) \propto a(\eta)^{(D-1)/2}$$

zero-energy-state

Iso-spectral cosmological backgrounds

Iso-spectrality

$$V_1(\eta) = \frac{D-1}{2}\left[\frac{D-3}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2 + \frac{a''(\eta)}{a(\eta)}\right] \longleftrightarrow V_2(\eta) = \frac{D-1}{2}\left[\frac{D+1}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2 - \frac{a''(\eta)}{a(\eta)}\right]$$

Realize V_2 with different QFT on cosmological background

$$V_2(\eta) \equiv -\tilde{a}^2(\eta)[m^2 + \tilde{\xi}\tilde{R}(\eta)] + \frac{\tilde{D}-1}{2}\left[\frac{\tilde{a}''(\eta)}{a(\eta)} - \frac{3-\tilde{D}}{2}\left(\frac{\tilde{a}'(\eta)}{\tilde{a}(\eta)}\right)^2\right]$$

$$\stackrel{!}{=} \frac{D-1}{2}\left[\frac{D+1}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2 - \frac{a''(\eta)}{a(\eta)}\right]$$