Quantum energy inequalities in the thermal sector

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Outline

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However, when classical fields are quantised (QFT), the necessity to avoid divergences in the definition of the energy density (**Wick ordering**), turns out to be incompatible with the positiveness request (Fewster 2012),(Epstein, Glaser, and Jaffe 1965).

$\implies$ Necessity to find lower bounds on the expectation value of the (time averaged) quantised version of the energy density.

Algebraic Quantum Field Theory

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In the real scalar case ($P\phi = (\Box + m^2)\phi = 0$), the fundamental object is the abstract $*$-**algebra $\mathcal{A}$**, polynomially generated by the smeared fields $\phi(f), f \in \mathcal{C}_0^\infty(\mathbb{M})$, that satisfy:

- linearity, $\phi(\lambda f + g) = \lambda\phi(f) + \phi(g), \lambda \in \mathbb{C}$.
- hermiticity, $\phi(f)^* = \phi(\overline{f})$.
- weak solution of field equation, $\phi(Pf) = 0$.
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States ω are positive, normalised linear functionals over $\mathcal{A}$.

In **quantum mechanics** (finitely many degrees of freedom), Gibbs states ω^G describe thermal equilibrium states inside the vacuum representation:

$$\omega^G(A) := \frac{\mathrm{Tr}(Ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})} := \frac{\mathrm{Tr}(Ae^{-\beta H})}{Z}.$$

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In **quantum field theory** on infinite volume (infinitely many degrees of freedom), Gibbs states are not well defined $\implies$ the notion of **KMS states** (ω^β) is introduced, as a generalization of Gibbs states to infinite volume systems:

Definition

A state ω^β satisfies the KMS condition with respect to the time evolution τ_t if:

$$\omega^\beta((\tau_t A)B) = \omega^\beta(B(\tau_{t+i\beta}A)),$$

and the function $z \in \mathbb{C} \to \omega^\beta(B\tau_z(A))$ is analytic inside the strip $\Im z \in [0, \beta]$ and continuous on the border.

General program

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- We construct the representation induced by the KMS state (**purification procedure**).
- We identify therein the energy density operator.
- We study the expectation value of this operator in this representation. $\Longrightarrow$ necessity to introduce mathematical tools as **modular theory** and **non-commutative** L^p **spaces**.

Outline

One particle structure

Given a $*$-algebra $\mathcal{A}$ (e.g. associated to the symplectic space $(\mathcal{C}_0^\infty(\mathbb{M}), \Delta)$) and a state ω over $\mathcal{A} \implies$ representation of $\mathcal{A}$ via **GNS construction**.

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The GNS representation induced by ω^β is the Fock representation over the symmetrised Fock space $\mathcal{F}^s$ (**purification procedure**):

$$\mathcal{F}^s(L^2 \oplus L^2) \simeq \mathcal{F}^s(L^2) \otimes \mathcal{F}^s(L^2),$$

where $\mathcal{F}^s(L^2)$ is the usual bosonic Fock space over the Hilbert space of L^2 functions on the mass hyperboloid.

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Test functions are mapped into the Hilbert space via the map K:

$$K(f)(\mathbf{k}) = \frac{\overline{\hat{f}|_{H_m^+}}(\mathbf{k})}{\sqrt{e^{\beta\omega_{\mathbf{k}}} - 1}} \oplus \frac{\hat{f}|_{H_m^+}(\mathbf{k})}{\sqrt{1 - e^{-\beta\omega_{\mathbf{k}}}}} =: \mathscr{B}_{\mathbf{k}}^- \overline{\hat{f}|_{H_m^+}}(\mathbf{k}) \oplus \mathscr{B}_{\mathbf{k}}^+ \hat{f}|_{H_m^+}(\mathbf{k}).$$

Thermal representation

We get explicit expression for the smeared **field** $\phi(f)$ (and for its commuting $\tilde{\phi}(g)$) in terms of the usual particles (and holes) creation and annihilation "operators" $b^{\#}$ $(a^{\#})$:

$$[a_{\mathbf{k}}, a_{\mathbf{p}}^{\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k}); \quad [b_{\mathbf{k}}, b_{\mathbf{p}}^{\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k});$$

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[\mathscr{B}_{\mathbf{k}}^{-} a_{\mathbf{k}} e^{ikx} + \mathscr{B}_{\mathbf{k}}^{-} a_{\mathbf{k}}^{\dagger} e^{-ikx} + \mathscr{B}_{\mathbf{k}}^{+} b_{\mathbf{k}} e^{-ikx} + \mathscr{B}_{\mathbf{k}}^{+} b_{\mathbf{k}}^{\dagger} e^{ikx} \right];$$

$$\tilde{\phi}(y) = \phi(y)|_{a \leftrightarrow b, a^{\dagger} \leftrightarrow b^{\dagger}}; \quad [\phi(x), \tilde{\phi}(y)] \equiv 0.$$

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In addition, the generator of time evolution $:\hat{H}:$ (thermal Hamiltonian or **Liouvillian**) and its **space density** $:\widehat{T_{00}}:(x)$ are given by:

$$:\hat{H}: = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}} - a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} \right); \quad :\widehat{T_{00}}:(x) = :T_{00}:(x) - :\tilde{T}_{00}:(x),$$

where (and accordingly for $:\tilde{T}_{00}:(x)$):

$$:T_{00}: = \frac{1}{2}:(\partial_0\phi)^2: + \frac{1}{2}\sum_{i=1}^{3} :(\partial_i\phi)^2: + \frac{1}{2}m^2:(\phi)^2:.$$

Energy density quantum inequalities

Study the expectation value of $:\widehat{T_{00}}:(f) = :T_{00}:(f) - :\tilde{T}_{00}:(f)$, i.e. the Liouvillian density smeared in space *and* time with a positive test function $f \in \mathcal{C}_0^\infty(\mathbb{M})$:

$$\left(\Psi, :\widehat{T_{00}}:(f)\Psi\right), \text{ with } \Psi \in \mathcal{F}^s(L^2) \otimes \mathcal{F}^s(L^2), (\Psi, \Psi) = N_\Psi^2.$$

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- State independent QEI for the term $:T_{00}:(f)$ (analogous to (Fewster 2012),(Fewster and Eveson 1998)). Smearing in time with a test function $|g(t)|^2$ (at $\mathbf{x} = \mathbf{0}$) we get:

$$\left(\Psi, :T_{00}:(|g|^2)\Psi\right) \geq$$
$$- \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\omega_\mathbf{k}}{2} \left[|\hat{g}(\omega + \omega_\mathbf{k})|^2 (\mathscr{B}_\mathbf{k}^+)^2 + |\hat{g}(\omega - \omega_\mathbf{k})|^2 (\mathscr{B}_\mathbf{k}^-)^2\right] N_\psi^2$$

where the integrals are convergent for every $\beta \in \mathbb{R}$ and $g \in \mathcal{C}_0^\infty(\mathbb{R})$ (**Hadamard property** of the KMS state ω^β).

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We try to obtain a **state dependent QEI** for the operator $-{:}\tilde{T}_{00}{:}(f)$. We restrict the attention to the set of vectors F obtained perturbing the vacuum state Ω (representing the KMS state ω^{β}) with operators that belong to the field $*$-algebra $\mathcal{A}$:

$$F := \left\{\Psi \in \mathcal{F}^s(L^2) \otimes \mathcal{F}^s(L^2) : \Psi = A\Omega, \text{for some } \; A \in \mathcal{A}\right\}.$$

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Remark: If a state dependent inequality exists for $-{:}\tilde{T}_{00}{:}(f)$ in terms of $\| \cdot \|_s$, than it extends to the operator ${:}\widehat{T_{00}}{:}(f)$.

Outline

Modular theory in a nutshell

Let $\mathcal{H}$ be a **Hilbert space**, $\mathcal{M}$ a **von Neumann algebra** (with commutant $\mathcal{M}'$) and Ω be a **cyclic and separating vector** ($\mathcal{M}\Omega, \mathcal{M}'\Omega$ are dense in $\mathcal{H}$). $\implies$ **modular theory** can be constructed (Borchers 2000):

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- Let $S = J\Delta^{1/2}$ be the polar decomposition of S. Δ (**modular operator**) defines an **automorphism** for $\mathcal{M}$ via the adjoint action of the unitary group $\Delta^{it}, t \in \mathbb{R}$:

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- The state defined by the vector Ω (via $\omega(\cdot) = (\Omega, \cdot\Omega)$) is KMS respect to the **modular evolution** implemented by $\mathrm{Ad}\Delta^{it}(\cdot)$. The opposite is also true.

Non commutative L^p spaces in a nutshell

Noncommutative $\mathbf{L^p}$ spaces generalize usual L_p spaces from integration theory (commutative v.N. algebras) to general v.N. algebras. Given $\mathcal{M}$, we can construct a family $L^p(\mathcal{M}), 1 \leq p \leq \infty$ of Banach spaces following different approaches ((Araki and Masuda 1982),(Haagerup 1979),(Kosaki 1984)).

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- $L^\infty(\mathcal{M}, \Omega) \equiv \mathcal{M}$ and $L^2(\mathcal{M}, \Omega) \simeq \mathcal{H}$ with same norm.
- For all $1 \leq p' \leq p \leq \infty$

$$\mathcal{M} \subseteq L^p(\mathcal{M}, \Omega) \subseteq L^{p'}(\mathcal{M}, \Omega) \subseteq \mathcal{M}_* \text{ with } \mathcal{M} \text{ dense in each } L^p;$$
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- For $a \in \mathcal{M}$ (corresponding to $a\Omega \in L^2$), seen as an element of L^4, it holds:

$$\|a\|_4 := \|\Delta_\Omega^{1/4} a^* a\Omega\|^{1/2}.$$

Affiliated operator

We have stated modular theory and non commutative spaces for (v.N.) algebras of bounded operators. We would like to extend this results to more general situations. We want to extend them to **unbounded affiliated** operators (Bratteli and Robinson 1987):

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We can prove the following two technical lemmas:

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If $A\eta\mathcal{M}$ and $\Omega \in \mathcal{D}(A), \mathcal{D}(A^*)$(with Ω cyclic and separating vector), we have $A\Omega \in \mathcal{D}(S)$, with S the Tomita operator relative to Ω and the following equality holds:

$$S A\Omega = A^*\Omega.$$

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Lemma

Let be $A\eta\mathcal{M}$ and $\Omega \in \mathcal{D}(A)$. If $A\Omega \in \mathcal{D}(A^*)$, $A\Omega$ belongs to the non-commutative L^4 space and we have:

$$\|A\|_4^2 = \|\Delta_\Omega^{1/4} A^* A\Omega\|,$$

Outline

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We can now prove the main result of the work, in a general and abstract form:

Theorem

Let be $\tilde{T}$ symmetric and affiliated with the commutant $\mathcal{M}'$ of the v.N. algebra $\mathcal{M}$, and suppose $\Omega \in \mathcal{D}(\tilde{T})$. Then, for every operator $A \eta \mathcal{M}$ s.t. $\Omega \in \mathcal{D}(A)$, $\Omega \in \mathcal{D}(A^* A)$ and $A\Omega \in \mathcal{D}(\tilde{T})$, the following inequality is satisfied:

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(Naive) Proof: We have:

$$\left|\left(A\Omega, \tilde{T}A\Omega\right)\right| = \left|\left(A\Omega, A\tilde{T}\Omega\right)\right| = \left|\left(A^*A\Omega, \tilde{T}\Omega\right)\right| = \left|\left(\Delta^{1/4}A^*A\Omega, \Delta^{-1/4}\tilde{T}\Omega\right)\right|.$$

Using Cauchy–Schwarz inequality:

$$\left|\left(A\Omega, \tilde{T}A\Omega\right)\right| \leq \left\|\Delta^{1/4}A^*A\Omega\right\| \left\|\Delta^{-1/4}\tilde{T}\Omega\right\| = C\|A\|_4^2.$$

This concludes the proof.

Application to TFT

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We directly prove that $-{:}\tilde{T}_{00}{:}(f)$ (smeared also in time) is affiliated to $\mathcal{M}'$. $\Longrightarrow$ We get a **trivial state dependent** inequality for $-{:}\tilde{T}_{00}{:}(f)$:

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The non triviality can be checked via direct examples and descends from the unboundesness of the operator ${:}T_{00}{:}(f)$.

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Possible future outlook: application of the abstract theorem to other situations in which a similar structure in terms operator affiliated to an algebra and to their commutant is manifest (e.g. Rindler wedge, entanglement aspects in information theory).

Araki, Huzihiro and Tetsu Masuda (1982). "Positive Cones and L p -Spaces for von Neumann Algebras". In: *Publications of The Research Institute for Mathematical Sciences.*

Borchers, H. J. (2000). "On revolutionizing quantum field theory with Tomita's modular theory". In: *J. Math. Phys.*

Bratteli, O. and D.W. Robinson (1987). *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras. Symmetry Groups. Decomposition of States.* Operator Algebras and Quantum Statistical Mechanics.

Epstein, H., V. Glaser, and A. Jaffe (1965). "Nonpositivity of energy density in Quantized field theories". In: *Nuovo Cim.*

Fewster, C. J. (2012). *Lectures on quantum energy inequalities.* eprint: 1208.5399.

Fewster, C. J. and S. P. Eveson (1998). "Bounds on negative energy densities in flat spacetime". In: *Physical Review D* 58.

Haagerup, Uffe (1979). "L^p-spaces associated with an arbitrary von Neumann algebra". In: *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977).*

Kay, Bernard S. and Robert M. Wald (1991). "Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon". In: *Physics Reports.* ISSN: 0370-1573.

Kosaki, Hideki (1984). "Applications of the complex interpolation method to a von Neumann algebra: Non-commutative Lp-spaces". In: *Journal of Functional Analysis.*