Conformal Blocks From the Oscillator Formalism in $d = 4$

Physics Combo Jena

Katharina Wölfl, Jakob Hollweck, Tobias Hössel, Martin Ammon

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Institute for Theoretical Physics
Friedrich-Schiller-University Jena

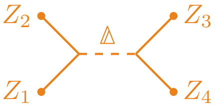
1. Motivation

What?

- decompose CFT correlation functions into conformal blocks

$$\langle \mathcal{O}_1(Z_1)\mathcal{O}_2(Z_2)\mathcal{O}_3(Z_3)\mathcal{O}_4(Z_4)\rangle = \sum_{\Delta} C_{12\Delta}C_{34\Delta}\, G^{(4)}_{\Delta}(Z_1,\dots,Z_4)$$

$$= \sum_{\Delta} C_{1234\Delta}$$



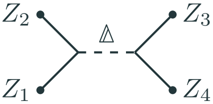
model-dependent conformal data $\{\Delta, C_{ijk}\} \leftrightarrow$ universal building blocks $G^{(4)}_{\Delta}$

→ "universal" within a fixed setting: blocks depend on e. g.

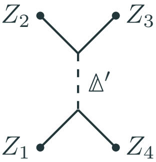
- dimension d of base space
- topology of base space (sphere, torus, ...)

- CFT answer: conformal bootstrap program
  use symmetry to constrain conformal data

$$\sum_{\mathbb{\Delta}} C^{\mathbb{\Delta}}_{\Delta_1\Delta_2\Delta_3\Delta_4} \cdot$$



$$= \sum_{\mathbb{\Delta}'} C^{\mathbb{\Delta}'}_{\Delta_1\Delta_3\Delta_2\Delta_4} \cdot$$



- Holographic answer: AdS/CFT-correspondence Maldacena, 1998
  Conformal blocks are dual to bulk *geodesic Witten diagrams* Witten, 1998; Hijano et al., 2016

How?

- Direct approach: solve Ward identities and Casimir equations (Dolan and Osborn, 2004)
  - difficulty quickly increases with degree of blocks
- Recursive approach: recurrence relation method (Zamolodchikov, 1984)
  - challenging to find closed form expressions
  - only for $d = 2$
- shadow operator method (Ferrara and Parisi, 1972, Ferrara et al., 1972)
  - insert (shadow)projector into correlator

$$\Psi_{\vartriangle}(z_1, \ldots, z_4) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\tilde{P}_{\vartriangle}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle$$

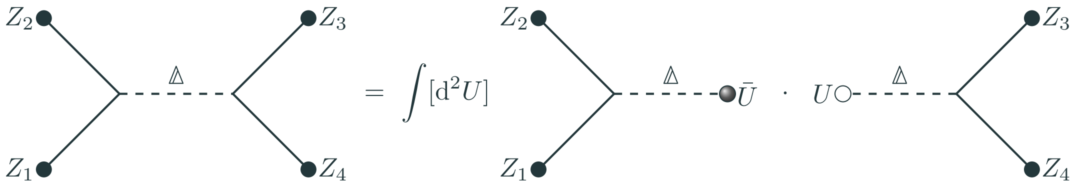
  - gives linear combination of conformal block and shadow block
  - conformal block extracted in an extra step
- oscillator representation method (in $d = 2$: Beşken, Datta, and Kraus, 2020)
  - express projector in terms of generalized coherent states

$$P_{\vartriangle} = \int [\mathrm{d}^2 u] |\bar{u}\rangle\langle u|$$

  - no shadow part!

Computation of the Four-Point Block

$$
\begin{aligned}
&G_{\triangle}^{(4)}(Z_1, Z_2, Z_3, Z_4) \\
&= \langle \mathcal{O}_1(Z_1)\mathcal{O}_2(Z_2) P_{\triangle} \mathcal{O}_3(Z_3)\mathcal{O}_4(Z_4)\rangle = \int [\mathrm{d}^2 U] \langle 0| \mathcal{O}_1(Z_1)\mathcal{O}_2(Z_2) |\bar{U}\rangle \langle U| \mathcal{O}_3(Z_3)\mathcal{O}_4(Z_4) |0\rangle
\end{aligned}
$$



by inserting the projector: $P_{\triangle} = \int [\mathrm{d}^2 U] |\bar{U}\rangle\langle U|$

Oscillator Diagrams

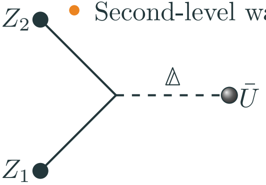
Wavefunctions are determined by differential equations ("oscillator equations").

- First-level wavefunctions:

$$\chi_{\Delta,\mathbb{\Delta}}(Z,\bar{U}) = \big\langle 0\big|\mathcal{O}_{\Delta}(Z)\big|\bar{U}\big\rangle \sim$$

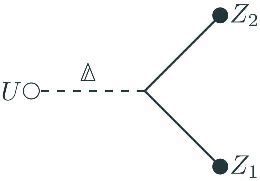
$$\psi_{\Delta,\mathbb{\Delta}}(Z,U) = \langle U|\mathcal{O}_{\Delta}(Z)|0\rangle \sim$$

- Second-level wavefunctions:



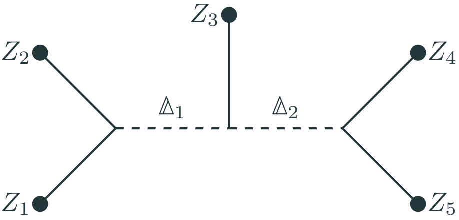
$$\sim \chi_{\Delta_1,\Delta_2,\mathbb{\Delta}}(Z_1,Z_2;\bar{U}) = \big\langle 0\big|\mathcal{O}(Z_1)\mathcal{O}(Z_2)\big|\bar{U}\big\rangle$$

$$\psi_{\Delta_1,\Delta_2,\mathbb{\Delta}}(Z_1,Z_2;U) = \langle U|\mathcal{O}(Z_1)\mathcal{O}(Z_2)|0\rangle \sim$$

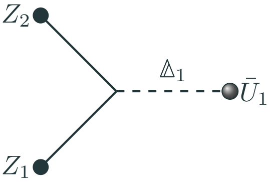


Computate higher-point blocks using $\Omega_{\Delta,\Delta_1,\Delta_2}(Z;U_1,\bar{U}_2) = \langle U_1 | \mathcal{O}_\Delta(Z) | \bar{U}_2 \rangle$

$G^{(5)}_{\Delta_1,\Delta_2} =$



$= \int [\mathrm{d}^2 U_1] \int [\mathrm{d}^2 U_2]$



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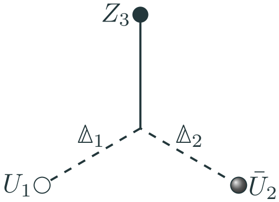

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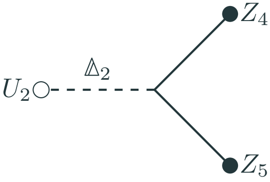


$\rightarrow$ matches (Rosenhaus, 2019)

2. The Oscillator Construction in $d = 4$

Oscillator Construction in $d = 4$

(based on Calixto and Perez-Romero, 2010, 2011, 2014)

- Lorentzian conformal group: SO(4,2)

→ its Lie group isomorphic to $\mathfrak{su}(2,2)$

- Lie algebra generators: $\mathfrak{D}$, $\mathfrak{P}^\mu$, $\mathfrak{K}^\mu$, $\mathfrak{M}^{\mu\nu}$
- Cartan domain $\mathbb{D}_4 = \{U \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \mathbb{1} - UU^\dagger > 0\}$

→ matrix valued oscillator variable: $U \in \mathbb{D}_4$

- unitary representation space: weighted Bergman space $\mathcal{H}L^2(\mathbb{D}_4)$
- Wick rotate to Euclidean setting for adjoint properties

$$(\mathfrak{P}^\mu)^\dagger = -\mathfrak{K}^\mu, \qquad (\mathfrak{M}^{\mu\nu})^\dagger = -\mathfrak{M}^{\mu\nu}, \qquad \mathfrak{D}^\dagger = \mathfrak{D}$$

- Hilbert space with inner product:

$$(f,g) = c_\Delta \int_{\mathbb{D}_4} \frac{\mathrm{d}^4 U}{\det(\mathbb{1} - U^\dagger U)^{4-\Delta}} \overline{f(U)} g(U) \equiv \int [\mathrm{d}^2 U] \overline{f(U)} g(U)$$

Oscillator Construction: Eigenbasis

- orthonormal eigenbasis:

$$\langle U | \varphi^{j,m}_{q_a,q_b} \rangle = \varphi^{j,m}_{q_a,q_b}(U) \sim \det(U)^m \mathcal{D}^j_{q_a,q_b}(U)$$

with generalized Wigner $\mathcal{D}$-matrices

$$\mathcal{D}^j_{q_a,q_b}(U) \quad \sim \sum_{k=\max(0,q_a+q_b)}^{\min(j+q_a,j+q_b)} \binom{j+q_b}{k} \binom{j-q_b}{k-q_a-q_b} u_{11}^k u_{12}^{j+q_a-k} u_{21}^{j+q_b-k} u_{22}^{k-q_a-q_b}$$

- orthogonality relation:

$$\langle \varphi^{j,m}_{q_a,q_b} | \varphi^{j',m'}_{q'_a,q'_b} \rangle = \int_{\mathbb{D}_4} [\mathrm{d}^2 U] \, \overline{\varphi^{j,m}_{q_a,q_b}(U)} \varphi^{j',m'}_{q'_a,q'_b}(U) = \delta_{j,j'} \delta_{m,m'} \delta_{q_a,q'_a} \delta_{q_b,q'_b}$$

- highest-weight state $\varphi^{0,0}_{0,0} = 1$ with $\mathfrak{P}_\mu 1 = 0$ leads to Verma module

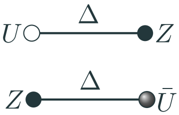
$$\mathcal{V}_{\mathbb{A}} = \mathrm{span}\{ \textstyle\prod_{\mu=1}^{d} \mathfrak{K}_\mu^{n_\mu} 1 : n_\mu = 0, 1, \dots \} \tag{1}$$

Oscillator Construction: Wavefunctions

- wavefunctions as diagrammatic objects:

$$\psi_\Delta(Z,U) = \langle U|\psi\rangle = \langle U|\mathcal{O}_\Delta(Z)|0\rangle$$

$$\chi_\Delta(Z,\bar{U}) = \left\langle \chi \middle| \bar{U} \right\rangle = \left\langle 0 \middle| \mathcal{O}_\Delta(Z) \middle| \bar{U} \right\rangle$$



- derive oscillator equations:

$$0 = \langle U|\,\mathcal{O}_\Delta(Z) L\,|0\rangle = \langle U|\,[\mathcal{O}_\Delta(Z), L] + L\mathcal{O}_\Delta(Z)\,|0\rangle = (-\mathcal{L}^{(\Delta,Z)} + \mathfrak{L}^{*(\Delta,U)})\psi_\Delta(Z;U)$$

with $*$ indicating how the abstract generator changes

$$\langle U|L|\psi\rangle = \mathfrak{L}^* \langle U|\psi\rangle = \mathfrak{L}^*\psi(U) \ :$$

$$\langle U|P^\mu|\psi\rangle = -\mathfrak{K}^\mu \langle U|\psi\rangle\ , \quad \langle U|K^\mu|\psi\rangle = -\mathfrak{P}^\mu \langle U|\psi\rangle\ , \quad \langle U|M^{\mu\nu}|\psi\rangle = -\mathfrak{M}^{\mu\nu} \langle U|\psi\rangle$$

Oscillator Construction: Ward Identities and Reproducing Kernel

$$\left(\mathcal{D}^{(\Delta,z)}+\mathfrak{D}^{(\mathbb{\Delta},\bar{u})}\right)\chi_{\Delta,\mathbb{\Delta}}(z;\bar{u})=0$$
$$\left(\mathcal{P}_\mu^{(\Delta,z)}+\mathfrak{P}_\mu^{(\mathbb{\Delta},\bar{u})}\right)\chi_{\Delta,\mathbb{\Delta}}(z;\bar{u})=0$$
$$\left(\mathcal{K}_\mu^{(\Delta,z)}+\mathfrak{K}_\mu^{(\mathbb{\Delta},\bar{u})}\right)\chi_{\Delta,\mathbb{\Delta}}(z;\bar{u})=0$$
$$\left(\mathcal{M}_{\mu\nu}^{(\Delta,z)}+\mathfrak{M}_{\mu\nu}^{(\mathbb{\Delta},\bar{u})}\right)\chi_{\Delta,\mathbb{\Delta}}(z;\bar{u})=0$$

$$\left(-\mathcal{D}^{(\Delta,z)}+\mathfrak{D}^{(\mathbb{\Delta},u)}\right)\psi_{\Delta,\mathbb{\Delta}}(z;u)=0\,,$$
$$\left(\mathcal{P}_\mu^{(\Delta,z)}+\mathfrak{K}_\mu^{(\mathbb{\Delta},u)}\right)\psi_{\Delta,\mathbb{\Delta}}(z;u)=0\,,$$
$$\left(\mathcal{K}_\mu^{(\Delta,z)}+\mathfrak{P}_\mu^{(\mathbb{\Delta},u)}\right)\psi_{\Delta,\mathbb{\Delta}}(z;u)=0\,,$$
$$\left(\mathcal{M}_{\mu\nu}^{(\Delta,z)}+\mathfrak{M}_{\mu\nu}^{(\mathbb{\Delta},u)}\right)\psi_{\Delta,\mathbb{\Delta}}(z;u)=0\,.$$

$$\Rightarrow \chi_{\Delta,\mathbb{\Delta}}(z;\bar{u})=\left((z_\mu-\bar{u}_\mu)^2\right)^{-\mathbb{\Delta}}\delta_{\Delta,\mathbb{\Delta}}$$

$$\Rightarrow \psi_{\Delta,\mathbb{\Delta}}(z;u)=(1-u\cdot z)^{-2\mathbb{\Delta}}\delta_{\Delta,\mathbb{\Delta}}$$

$$(x_0,x_1,x_2,x_3)\leftrightarrow X=x_0\sigma_0+i(x_1\sigma_1+x_2\sigma_2+x_3\sigma_3)=\begin{pmatrix}x_0+ix_3 & ix_1+x_2\\ ix_1-x_2 & x_0-ix_3\end{pmatrix}$$

with important identification:

$$\det(X)=x^2=\sum_{k=0}^{3}x_k^2$$

Oscillator Construction: Higher-level wavefunctions

- "outer" wavefunctions:

$$\psi_{\Delta_1,\Delta_2,\mathbb{\Delta}}(Z^{(1)},Z^{(2)};U) = \det{}^{\alpha}\left(Z^{(1)}-Z^{(2)}\right)\det{}^{\beta}\left(\mathbb{1}-U\cdot {Z^{(1)}}^{\dagger}\right)\det{}^{\gamma}\left(\mathbb{1}-U\cdot {Z^{(2)}}^{\dagger}\right)$$

$$\chi_{\Delta_1,\Delta_2,\mathbb{\Delta}}(Z^{(1)},Z^{(2)};\bar{U}) = \det{}^{\alpha}\left(Z^{(1)}-Z^{(2)}\right)\det{}^{\beta}\left(Z^{(1)}-\bar{U}\right)\det{}^{\gamma}\left(Z^{(2)}-\bar{U}\right)$$

with exponents given by linear combinations of weights:

$$\alpha=\frac{1}{2}(\mathbb{\Delta}-\Delta_1-\Delta_2),\quad \beta=\frac{1}{2}(\Delta_2-\Delta_1-\mathbb{\Delta}),\quad \gamma=\frac{1}{2}(\Delta_1-\Delta_2-\mathbb{\Delta})$$

- "inner" wavefunction $\Omega_{\Delta,\mathbb{\Delta}_1,\mathbb{\Delta}_2}(Z;U^{(1)},\bar{U}^{(2)}) = \left\langle U^{(1)}\middle|\mathcal{O}_{\Delta}(Z)\middle|\bar{U}^{(2)}\right\rangle$:

$$\Omega_{\Delta,\mathbb{\Delta}_1,\mathbb{\Delta}_2}(Z;U^{(1)},\bar{U}^{(2)}) = \det{}^{\tilde{\alpha}}\left(Z-\overline{U}^{(2)}\right)\det{}^{\tilde{\beta}}\left(\mathbb{1}-U^{(1)}\cdot Z^{\dagger}\right)\det{}^{\tilde{\gamma}}\left(\mathbb{1}-U^{(1)}\cdot{\overline{U}^{(2)}}^{\dagger}\right)$$

with modified exponents $\tilde{\alpha},\ \tilde{\beta},\ \tilde{\gamma}$ depending on $\Delta,\mathbb{\Delta}_1,\mathbb{\Delta}_2$

The extended Schwinger-Master-Theorem

Theorem (Δ-Extended Schwinger-Master-Theorem)

For every $\Delta \in \mathbb{N}$, $\Delta \geq 2$ and every 2×2 matrix $\tilde{X}$, the following identity holds

$$\det{}^{-\Delta}\left(\mathbb{1} - t\tilde{X}\right)$$

$$= \sum_{j \in \mathbb{N}/2} \frac{2j+1}{\Delta - 1} \sum_{n=0}^{\infty} t^{2j+2n} \binom{n + \Delta - 2}{\Delta - 2} \binom{n + 2j + \Delta - 1}{\Delta - 2} \det{}^{n}\left(\tilde{X}\right) \sum_{q=-j}^{j} \mathcal{D}^{j}_{qq}\left(\tilde{X}\right) .$$

$\Rightarrow$ use this to expand wavefunctions in basis functions:

$$\det{}^{-\Delta}\left(\mathbb{1} - ZU^{\dagger}\right) = \sum_{j \in \mathbb{N}/2} \sum_{n=0}^{\infty} \sum_{q,q'=-j}^{j} \varphi^{j,n}_{q,q'}(Z) \varphi^{j,n}_{q,q'}(U^{*})$$

Cautionary Note: $\Delta \in \mathbb{N}$ - but we want $\Delta \in \mathbb{R}$. Careful "analytic continuation" necessary!

The Two-Point Function

Use wavefunctions to calculate two-point function:

$$\langle \mathcal{O}_\Delta(Z)\mathbb{P}_{\mathbb{\Delta}}\mathcal{O}_\Delta(Y)\rangle = \int_{\mathbb{D}_4} [\mathrm{d}^2U]\, \chi_{\Delta,\mathbb{\Delta}}(Z;\bar{U})\psi_{\Delta,\mathbb{\Delta}}(Y;U)$$

$$= z^{-2\Delta} \sum_{j_1,m_1,q_1,q_1'} \sum_{j_2,m_2,q_2,q_2'} z^{-4m_1-4j_1}\varphi^{j_2,m_2}_{q_2,q_2'}(Y)\varphi^{j_1,m_1}_{q_1,q_1'}(Z^*) \int_{\mathbb{D}_4}[\mathrm{d}^2U]\varphi^{j_1,m_1}_{q_1,q_1'}(\bar{U})\varphi^{j_2,m_2}_{q_2,q_2'}(U^*)$$

$$= z^{-2\Delta} \sum_{j_1,m_1,q_1,q_1'} z^{-4m_1-4j_1}\varphi^{j_1,m_1}_{q_1,q_1'}(Z^*) \sum_{j_2,m_2,q_2,q_2'} (\delta^{j_2,m_2,q_2,q_2'}_{j_1,m_1,q_1,q_1'}\ \varphi^{j_2,m_2}_{q_2,q_2'}(Y))\ \varphi^{j_1,m_1}_{q_1,q_1'}(Y)$$

$$= \det{}^{-\Delta}(Z)\ \det{}^{-\Delta}\left(\mathbb{1} - YZ^{-1}\right) = \det{}^{-\Delta}(Z-Y)$$

$$= \left((z_0-y_0)^2 + (z_1-y_1)^2 + (z_2-y_2)^2 + (z_3-y_3)^2\right)^{-\Delta}$$

$\Rightarrow$ same result as by directly solving the Ward identities

The Four-Point Block: Preparation

- $n = 4$-point block depends on two independent cross-ratios with $z_{ij} = \left|z^{(i)} - z^{(j)}\right|$:

$$u = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\,, \qquad v = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2}\,,$$

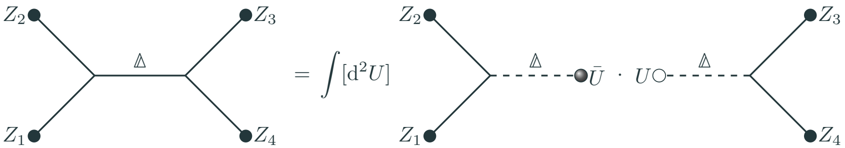
- choose point configuration using conformal transformations:

$$\begin{aligned} z^{(1)} &\to \infty\,, \\ z^{(2)} &\to (1,0,0,0)\,, \\ z^{(3)} &\to (0,0,0,0)\,, \\ z^{(4)} &\to (x,0,0,y) \end{aligned}$$

- cross-ratios simplify to

$$\begin{aligned} u &\to \left|z^{(4)}\right|^2 = x^2 + y^2 = z\bar{z} \\ v &\to \left|z^{(2)} - z^{(4)}\right|^2 = (1-x)^2 + y^2 = (1-z)(1-\bar{z}) \end{aligned}$$

The Four-Point Block: Expansion



Expand second-level wavefunctions using the extended SMT:

$$\chi_{\Delta_1,\Delta_2,\mathbb{\Delta}}(\infty, \mathbb{1}; \bar{U}) = \Lambda^{-2\Delta_1} \sum_{q_a,q_b}^{j,m} {}_{(\gamma)}\varphi^{j,m}_{q_a,q_b}(\bar{U})\, {}_{(\gamma)}\varphi^{j,m}_{q_a,q_b}(\mathbb{1})$$

with cutoff Λ for the limit $Z^{(1)} \to \infty$

$$\psi_{\Delta_3,\Delta_4,\mathbb{\Delta}}(Z^{(4)}, \mathbf{0}; U) = \det{}^{-\varepsilon}(Z^{(4)}) \sum_{q_a,q_b}^{j,m} {}_{(\varepsilon)}\varphi^{j,m}_{q_a,q_b}(U^*)\, {}_{(\varepsilon)}\varphi^{j,m}_{q_a,q_b}(Z^{(4)}) \qquad (2)$$

with $\gamma = \frac{1}{2}(\mathbb{\Delta} + \Delta_2 - \Delta_1)$ and $\varepsilon = \frac{1}{2}(\mathbb{\Delta} + \Delta_3 - \Delta_4)$

The Four-Point Block: Integration

$$G_{\mathbb{A}} = \int [d^2U]\chi_{\Delta_1,\Delta_2,\mathbb{A}}(\infty,\mathbb{1};\bar{U})\psi_{\Delta_3,\Delta_4,\mathbb{A}}(Z^{(4)},\mathbf{0};U)$$

$$= \det{}^{-\varepsilon}(Z^{(4)}) \sum_{q_a,q_b}^{j,m} \sum_{q'_a,q'_b}^{j',m'} {}_{(\gamma)}\varphi^{j,m}_{q_a,q_b}(\mathbb{1})_{(\varepsilon)}\varphi^{j',m'}_{q'_a,q'_b}(Z^{(4)}) \cdot \int [\mathrm{d}^2U_{\mathbb{A}}]_{(\gamma)}\varphi^{j,m}_{q_a,q_b}(\bar{U})_{(\varepsilon)}\varphi^{j',m'}_{q'_a,q'_b}(U^*)$$

$$= \det{}^{-\varepsilon}(Z^{(4)}) \sum_{q_a,q_b}^{j,m} \sum_{q'_a,q'_b}^{j',m'} {}_{(\gamma)}\varphi^{j,m}_{q_a,q_b}(\mathbb{1})_{(\varepsilon)}\varphi^{j',m'}_{q'_a,q'_b}(Z^{(4)}) \cdot \frac{\mathcal{N}^{j,m}_{\gamma}\mathcal{N}^{j,m}_{\varepsilon}}{\left(\mathcal{N}^{j,m}_{\mathbb{A}}\right)^2}\delta^{j',m',q'_a,q'_b}_{j,m,q_a,q_b}$$

Use that $\mathbb{1}$ and $Z^{(4)}$ are diagonal matrices:

$$= \det{}^{-\varepsilon}(Z^{(4)}) \sum_{q_a,q_b}^{j,m} \frac{\left(\mathcal{N}^{j,m}_{\gamma}\right)^2\left(\mathcal{N}^{j,m}_{\varepsilon}\right)^2}{\left(\mathcal{N}^{j,m}_{\mathbb{A}}\right)^2}\delta_{q_a,q_b}\det{}^m(Z^{(4)})\mathcal{D}^j_{q_a,q_b}(Z^{(4)})$$

$$= \frac{(z\bar{z})^{\frac{1}{2}(\mathbb{A}+\Delta_3-\Delta_4)}}{z-\bar{z}} \sum_{m=0}^{\infty}\sum_{j\in\mathbb{N}/2} \frac{\left(\mathcal{N}^{j,m}_{\gamma}\right)^2\left(\mathcal{N}^{j,m}_{\varepsilon}\right)^2}{\left(\mathcal{N}^{j,m}_{\mathbb{A}}\right)^2} z^m\bar{z}^m\left(z^{2j+1}-\bar{z}^{2j+1}\right)$$

The Four-Point Block: Recovering the Hypergeometric Functions

- bare block (strip away leg factor):

$$g_{\mathbb{A}} = \sum_{m=0}^{\infty} \sum_{j \in \mathbb{N}/2} \frac{\left(\mathcal{N}_{\gamma}^{j,m}\right)^2 \left(\mathcal{N}_{\varepsilon}^{j,m}\right)^2}{\left(\mathcal{N}_{\mathbb{A}}^{j,m}\right)^2} z^m \bar{z}^m \left(z^{2j+1} - \bar{z}^{2j+1}\right)$$

with

$$\left(\mathcal{N}_{\alpha}^{j,m}\right)^2 = \frac{2j+1}{\alpha - 1} \binom{m + \alpha - 2}{\alpha - 2} \binom{m + 2j + \alpha - 1}{\alpha - 2}$$

- careful analysis of sums allows to recover:

$$z \, {}_2F_1 \left(\begin{matrix} -\gamma, -\varepsilon \\ \mathbb{A} \end{matrix} ; z \right) {}_2F_1 \left(\begin{matrix} -\gamma - 1, -\varepsilon - 1 \\ \mathbb{A} - 2 \end{matrix} ; \bar{z} \right) - \bar{z} \, {}_2F_1 \left(\begin{matrix} -\gamma, -\varepsilon \\ \mathbb{A} \end{matrix} ; \bar{z} \right) {}_2F_1 \left(\begin{matrix} -\gamma - 1, -\varepsilon - 1 \\ \mathbb{A} - 2 \end{matrix} ; z \right)$$

with $\gamma = \frac{1}{2}(\mathbb{A} + \Delta_2 - \Delta_1)$ and $\varepsilon = \frac{1}{2}(\mathbb{A} + \Delta_3 - \Delta_4)$

$\Rightarrow$ matches result of (Dolan and Osborn, 2004, 2001)

3. Conclusion and Outlook

Oscillator Formalism: From $d = 2$ to $d = 4$

	CFT in $d = 2$	CFT in $d = 4$
conf. algebra	$\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{su}(2,2)$
Generators	$L_{-1},\ L_0,\ L_1 \quad (3)$	$D,\ P^\mu,\ K^\mu,\ M^{\mu\nu} \quad (15)$
Cartan domain	$\mathbb{D} = \mathrm{SU}(1,1)/\mathrm{U}(1)$ $\cong \{u \in \mathbb{C} : 1 - u^* u > 0\}$	$\mathbb{D}_4 = \mathrm{SU}(2,2)/(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$ $\cong \{U \in \mathbb{C}^{2\times 2} : 1 - U^\dagger U > 0\}$
ONB	Monomials $\varphi_n(u) \sim u^n$	Generalized Wigner-$\mathcal{D}$-matrices $\varphi^{j,m}_{q_a,q_b}(U) \sim \det(U)^m \mathcal{D}^j_{q_a,q_b}(U)$
Wavefunctions	$\psi_h(z;u) = (1 - zu)^{-2h}$ $\chi_h(z,\bar{u}) = (z - \bar{u})^{-2h}$ $\psi(z_1,z_2;u),\ \chi(z_1,z_2;\bar{u})$	$\psi_\Delta(z;u) = (1 - z_\mu \cdot u^\mu)^{-2\Delta}$ $\chi_\Delta(z;\bar{u}) = (z_\mu - \bar{u}_\mu)^{-2\Delta}$ $\psi(z^{(1)},z^{(2)};u),\ \chi(z^{(1)},z^{(2)};\bar{u})$
Conformal blocks	up to n-point blocks (Comb channel)	4-point block: ✓ higher point blocks: ?

Take-Home Message and Outlook

The oscillator construction provides an adaptable formalism for the computation of conformal blocks:

- intuitive diagrammatic formulation
- applicable for various topologies (sphere, torus, ...)
- applicable in $d = 2$ and $d = 4$

Future directions:

- derive closed form solutions for higher-point blocks in $d = 4$
- implement spin in $d = 4$
- further explore different settings (channels, dimensions, $\ldots$)

Thank you for your attention!

4. References

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