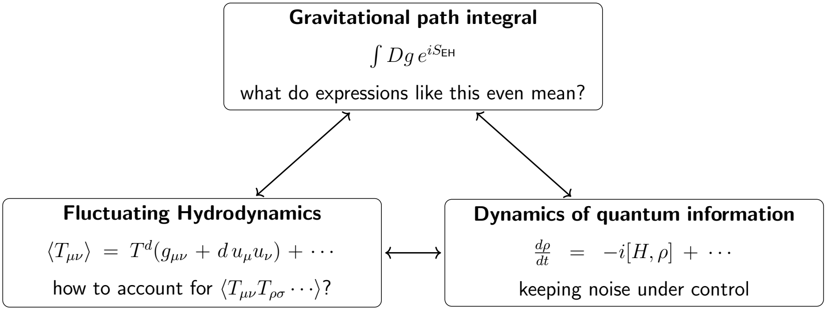
# Open quantum systems:

# From Brownian motion to quantum gravity

Julio Virrueta

TPI, FSU Jena

# The big ticket questions



**Gravitational path integral**

$$\int Dg\, e^{iS_{\mathsf{EH}}}$$

what do expressions like this even mean?

**Fluctuating Hydrodynamics**

$$\langle T_{\mu\nu}\rangle \;=\; T^d(g_{\mu\nu} + d\,u_\mu u_\nu) + \cdots$$

how to account for $\langle T_{\mu\nu}T_{\rho\sigma}\cdots\rangle$?

**Dynamics of quantum information**

$$\frac{d\rho}{dt} \;=\; -i[H,\rho] \,+\, \cdots$$

keeping noise under control

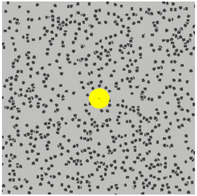
# Return to the basics (I am not very smart)

Modeling dissipative dynamics: the **Langevin equation**

$$\underbrace{m\,\ddot{x} + \frac{dV(x)}{dx}}_{\text{Good old dynamics}} + \overbrace{\eta\,\dot{x}}^{\text{Dissipation}} = \overbrace{F(t)}^{\text{External force}}$$

$$\langle F(t) \rangle = 0$$

$$\langle F(t)F(t') \rangle \propto T\delta(t-t')$$



**Many applications:** Brownian motion, Johnson noise, etc.

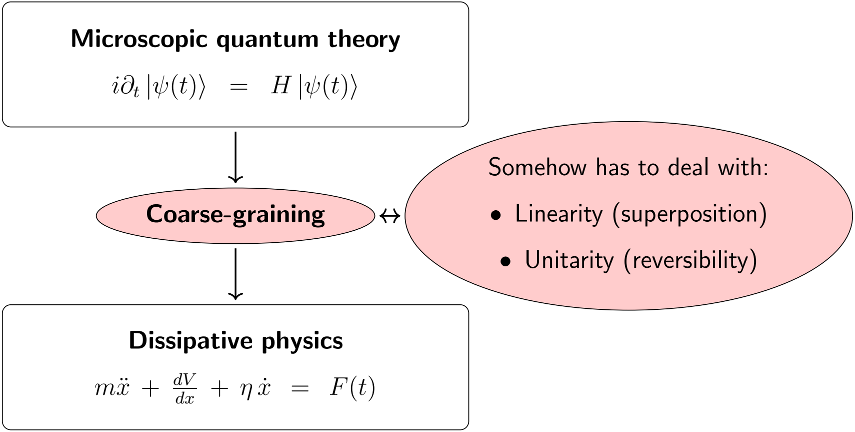
Dissipation also plays a role in theories at local equilibrium (hydrodynamics)

$$T_{\mu\nu} = \underbrace{(\epsilon + p)\, u_\mu u_\nu + p\, g_{\mu\nu}}_{\text{Ideal fluid}} \underbrace{- \eta\, \partial_{(\mu} u_{\nu)} - \zeta\, (\partial \cdot u)\, (g_{\mu\nu} + u_\mu u_\nu)}_{\text{Dissipative terms}} + \cdots$$

but what about stochastic fluctuation?

# Microscopic origin of dissipation

The two situations before are **effective** descriptions.



**Microscopic quantum theory**

$$i\partial_t \left|\psi(t)\right\rangle \;=\; H \left|\psi(t)\right\rangle$$

$\downarrow$

**Coarse-graining** $\leftrightarrow$ Somehow has to deal with:

- Linearity (superposition)
- Unitarity (reversibility)

$\downarrow$

**Dissipative physics**

$$m\ddot{x} + \tfrac{dV}{dx} + \eta\,\dot{x} \;=\; F(t)$$

# A solution: open quantum systems

Consider a QM system composed of two distinct parts:

$$H = \overbrace{H_A}^{\text{System of interest}} + \underbrace{H_B}_{\text{Environment}} + \overbrace{H_I}^{\text{Interaction}}$$

with unitary time evolution $\rho(t) = e^{-iHt}\,\underbrace{\rho(0)}_{\text{Could be taken to be pure... but don't.}}\,e^{iHt}$.

The Hilbert space is taken to be of the form

$$\mathcal{H} = \overset{\text{Spanned by } \{|x\rangle\}}{\mathcal{H}_A} \bigotimes \underset{\text{Spanned by } \{|R\rangle\}}{\mathcal{H}_B}$$

The state at $t = 0$ can be taken to be a tensor product state

$$\rho(0) = \underbrace{\rho_A(0) \otimes \rho_B(0)}_{\text{Very simple (SRE) state: } I(A,B) = S(AB) - S(A) - S(B) = 0}$$

The matrix elements of $\rho(t)$ in the given basis are

$$\langle x,R|\,\rho(t)\,|y,Q\rangle = \sum\!\!\!\!\!\!\!\int \; \overset{K(x,r;t|x',R';0)}{\langle x,R|\,e^{-iHt}\,|x',R'\rangle}\; \langle x',R'|\,\rho(0)\,|y',Q'\rangle$$

$$\times\;\; \underset{K^*(y,Q;t|y',Q';0)}{\langle y',Q'|\,e^{iHt}\,|y,Q\rangle}$$

and the propagators admit a path integral representation

$$K(x,r;t|x',R';0) = \int_{\tilde{x}(0)=x',\tilde{R}(0)=R'}^{\tilde{x}(t)=x,\tilde{R}(t)=R} D\tilde{x}D\tilde{R}\,e^{iS[\tilde{x},\tilde{R}]}$$

Mind the limits

## All together

Difference between $K$ and $K^*$

$$\langle x, R|\,\rho(t)\,|y, Q\rangle = \int D\tilde{x} D\tilde{R}\, D\tilde{y} D\tilde{R}\, e^{iS[\tilde{x},\tilde{R}]\;-\;iS[\tilde{y},\tilde{Q}]}$$

$$\times\,\langle x', R'|\,\rho(0)\,|y', Q'\rangle$$

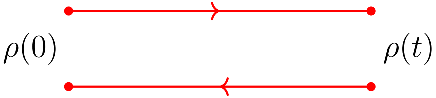
or abusing short-hand

$$\rho(t) = \int_{\rho(0)} DX_R\;\; DX_L\;\; e^{iS[X_R]-iS[X_L]}$$

$X_R = \{\tilde{x}, \tilde{R}\}$

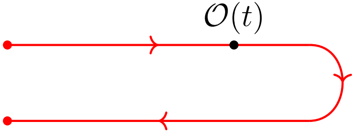
$X_L = \{\tilde{y}, \tilde{Q}\}$

We can also use diagrams:



and it is clear also how to compute correlators

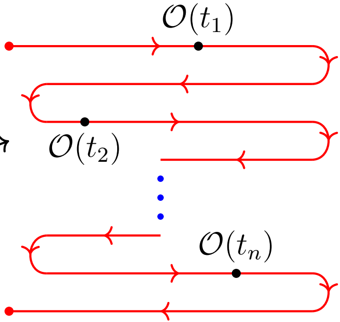
$$\mathsf{Tr}\,(\rho(0)\mathcal{O}(t)) =$$



This is known as the Schwinger-Keldysh contour.

This story can be extended for higher order correlators

$$\mathrm{Tr}\left(\rho(0)\mathcal{O}(t_1)\mathcal{O}(t_2)...\mathcal{O}(t_n)\right) =$$



More general situation.

Often can be simplified

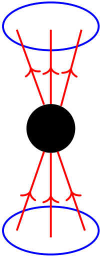
using unitarity.

- All this can be “easily” applied to QFT.

# Some comments about SK contours

- At least double the DoF $X \to \{X_R, X_L\}$ (purification).
- In-in formalism: only input is $\rho(0)$ (out-of-equilibrium).
- Any operator ordering (retarded/advance correlators).

Compare against $S-$matrix calculations (in-out formalism).

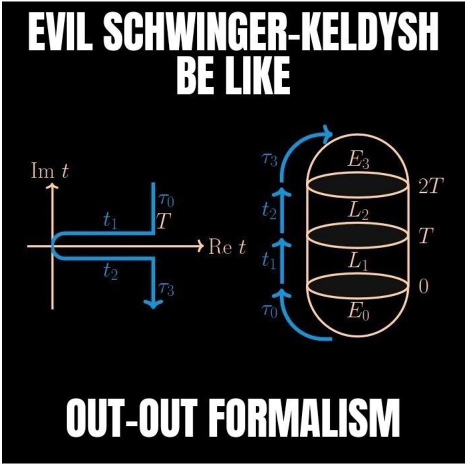


$t \to \infty$ $|out\rangle \sim a_i^\dagger |0\rangle$

$\langle in|out\rangle \propto (\text{LSZ}) \langle \mathcal{T}\mathcal{O}(t_1)\mathcal{O}(t_2)\cdots\rangle$

$t \to -\infty$ $|in\rangle \sim a_j^\dagger |0\rangle$

Of course, we could also have



**credit:** Anonymous undergraduate.

# Partial traces and influence functionals

We do not care about the full density matrix, $\rho(t)$, but just $\rho_A(t) = \mathrm{Tr}_B \rho(t)$ where the environment is ignored$^{\text{American style}}$.

In terms of matrix elements:

Only A's DoF

$$\begin{aligned}
\langle x|\,\rho_A(t)\,|y\rangle &= \sum\!\!\!\!\!\!\!\int dR\,\langle x,R|\,\rho(t)\,|y,R\rangle \\
&= \sum\!\!\!\!\!\!\!\int dx'dy' \int_{x',y'}^{x,y} D\tilde{x}D\tilde{y}\,e^{iS_A[\tilde{x}]-iS_A[\tilde{y}]}\,\langle x'|\,\rho_A(0)\,|y'\rangle \\
&\times \sum\!\!\!\!\!\!\!\int dRdR'dQ' \int_{R',Q'}^{R} D\tilde{R}D\tilde{Q}\,e^{iS_B[\tilde{R}]-iS_B[\tilde{Q}]} \\
&\times e^{iS_I[\tilde{x},\tilde{R}]-iS_I[\tilde{y},\tilde{Q}]}\ \langle R'|\,\rho_B(0)\,|Q'\rangle
\end{aligned}$$

Interaction

Only B's DoF

where we used $\rho(0) = \rho_A(0)\otimes\rho_B(0)$.

Or in much more compact notation

$$\rho_A(t) = \int_{\rho_A(0)} Dx_R\, Dx_L \,\mathsf{exp}\left[\, \overbrace{iS_A[x_R] - iS_A[x_L]}^{\dot{\rho}_A = -i[H_A, \rho_A]} \;+\; \underbrace{iF[x_R, x_L]}_{\text{Dissipation \& noise}} \,\right]$$

where we introduce the *influence functional*

$$e^{iF[x_R,x_L]} = \int_{\rho_B(0)} DQ_R DQ_L\, e^{iS_B[Q_R]-iS_B[Q_L]+iS_I[x_R,Q_R]-iS_I[x_L,Q_L]}$$

# An old friend, rediscovered

Consider a very simple interaction:

$$S_I = \lambda \int_0^t dt' x(t') Q(t')$$

then

$$e^{iF[x_R, x_L]} = \int_{\rho_B(0)} DQ_R DQ_L \, e^{iS_B[Q_R] - iS_B[Q_L]}$$

$$\times \; e^{i\lambda \int dt' (x_R \, Q_R - x_L \, Q_L)}$$

$$= \left\langle e^{i\lambda \int dt' (x_R \, Q_R - x_L \, Q_L)} \right\rangle_{\rho_B}$$

The influence functional = generating functional of correlators.

# Effective action for Fluctuating Hydro

A special case: $\rho_B = \frac{1}{Z(\beta)} e^{-\beta H}$

- The effect of system $A$ on $B$ is to take it out of equilibrium!
- The response is captured by $e^{iF[x_R, x_L]}$, in the limit $\omega, |\vec{k}| \ll T$.
- From the influence functional we get $\langle T_{\mu\nu} T_{\rho\sigma} \cdots \rangle$

# The only example ever: harmonic oscillators

Let us evaluate the influence functional for:

$$H = \frac{p^2}{2M} + V(x) + \frac{1}{2}\sum_k \left( \frac{P_k^2}{m} + m\omega_k^2 R_k \right) + x \sum_k \lambda_k R_k$$

we then evaluate

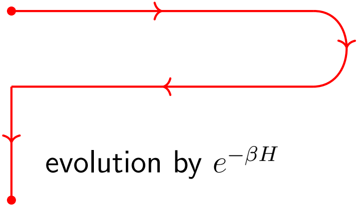
$$e^{iF[x_R, x_L]} = \int_{R',Q'}^{R} DQ_R DQ_L \, e^{\frac{i}{2} m \sum_k \int dt' \left( \dot{Q}_R^2 - \omega_k^2 Q_R^2 + \lambda_k x_R Q_R \right) - (R \to L)}$$

$$\times \;\; \langle R' | \, \rho_B(0) \, | Q' \rangle$$

$$\langle R' | \, e^{-\beta H} \, | Q' \rangle = K(R'; -i\beta | Q'; 0)$$

Diagramatically

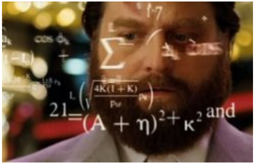
$$e^{iF[x_R,x_L]} =$$



evolution by $e^{-\beta H}$

with a lot of insertions along the Lorentzian segments.

# Doing the path integral



Final answer:

$$e^{iF[x_R,x_L]} = \exp\left[-\int d\tau ds\,(x_R(\tau) - x_L(\tau))\right.$$
$$\left.\times\;(\alpha(\tau - s)x_R(s) - \alpha^*(\tau - s)x_L(s))\right]$$

where

$$\alpha(t) = \sum_k \frac{\lambda_k}{2m\omega_k}\left[e^{-i\omega_k t} + \frac{e^{-i\omega_k t} + e^{i\omega_k t}}{e^{\beta\omega_k} - 1}\right]$$

It is better to write all this as:

$$e^{iF[x_a,x_d]} = \exp\Bigg[- \int d\tau ds\,(\alpha_R(\tau - s)x_d(\tau)x_d(s)$$

$$+2i\alpha_I(\tau - s)x_d(\tau)x_a(s))]$$

where

$$x_a \equiv \frac{1}{2}\left(x_R + x_L\right)\;, \quad x_d \equiv x_R - x_L$$

This term is important: KMS

$$\alpha_R(\tau) = \mathsf{Re}\left(\alpha(\tau)\right) = \sum_k \frac{\lambda_k^2}{2m\omega_k}\;\coth\left(\frac{\omega_k\beta}{2}\right)\;\cos\left(\omega_k t\right)$$

$$\alpha_I(\tau) = \mathsf{Im}\left(\alpha(\tau)\right) = \sum_k \frac{\lambda_k^2}{2m\omega_k}\sin\left(\omega_k t\right)$$

# What it all means? Recovering Brownian motion

Ok... we got a big expression for the propagator of $A$:

$$J(x,y;t|x'y';0) = \int Dx_R\, Dx_L\, e^{iS[x_R]-iS[x_L]-\ i\int x_d \alpha_I x_a\ -\ \int x_d \alpha_R x_d}$$

How can we interpretate it?

Compare with expectation: system under random external force

$$J(x,y;t|x'y';0) = \int Dx_R\, Dx_L\, \underbrace{DF\, p(F)}_{\text{Distribution of random force (take it gaussian)}}\ e^{iS[x_R]-iS[x_L]+i\ \int x_d F(t)}$$

Just as before, we change perspective

$$Z_F[x_d] \;=\; \int DF\, p(F)\, e^{i\int x_d F}$$

Generating functional of correlators, $\langle F(t_1)F(t_2)\cdots\rangle$

For gaussian distributions $p(F) \propto e^{-F^2/A}$:

$$Z_F[x_d] = e^{-\int d\tau ds\, x_d(\tau)A(\tau - s)x_d(s)}$$

where $\langle F(\tau)F(s)\rangle = A(\tau - s)$.

We can compare the two expressions:

Pheno.

Real exponentials

$$J(x,y;t|x'y';0) = \int Dx_R\, Dx_L\, e^{iS[x_R]-iS[x_L]- \int x_d A\, x_d}$$

$$J(x,y;t|x'y';0) = \int Dx_R\, Dx_L\, e^{iS[x_R]-iS[x_L]- i\int x_d \alpha_I x_a - \int x_d \alpha_R x_d}$$

Micro.

Additional part: micro. dynamics

and identify:

$$\langle F(t)F(0)\rangle = \alpha_R(t)$$

In the thermodynamic limit

$$\alpha_R(t) \overset{\omega \ll T}{\sim} \frac{T}{m}\sum_k \frac{\lambda_k^2}{\omega_k^2}\cos\omega_k t \underset{\text{Large volume}}{\sim} \frac{T}{m}\int_0^\infty \rho(\omega)\frac{\lambda(\omega)^2}{\omega^2}\cos\omega t$$

With the right choice of $\rho(\omega)$:

$$\alpha_R(t) \sim \frac{2\eta T}{\pi}\frac{\sin\Omega t}{t} \sim 2\eta T\delta(t)$$

We recover Langevin from micro. dynamics!

# A closer look: what about $\alpha_I$

Once again, take the view of the enviroment:

$$Z[x_a, x_d] = \left\langle e^{i\int(Q_R x_R - Q_L x_L)} \right\rangle_\beta = \left\langle e^{i\int(Q_d x_a + Q_a x_d)} \right\rangle_\beta$$

$$= e^{-i\int x_d\,\alpha_I\,x_a - \int x_d\,\alpha_R\,x_d}$$

and taking functional derivatives:

Fluctuating force

$$\langle Q_a(t)\,Q_a(0)\rangle = \frac{1}{2}\,\langle\{Q(t), Q(0)\}\rangle = \frac{1}{i}\frac{\delta}{\delta x_d}\frac{1}{i}\frac{\delta}{\delta x_d}Z[x_a, x_d] = \alpha_R(t)$$

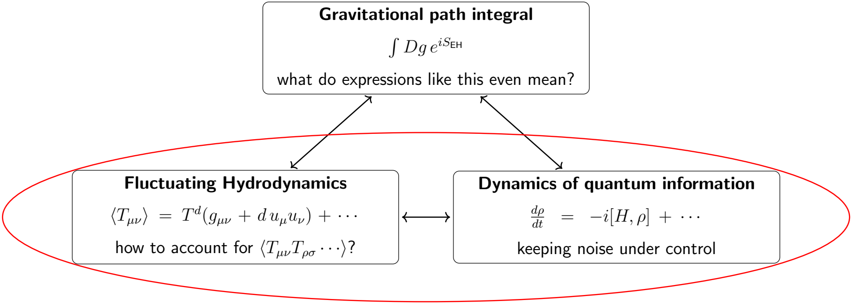
$$\langle Q_a(t)\,Q_d(0)\rangle = \langle[Q(t), Q(0)]\rangle = \frac{1}{i}\frac{\delta}{\delta x_a}\frac{1}{i}\frac{\delta}{\delta x_d}Z[x_a, x_d] = i\alpha_I(t)$$

Retarded response

# Lessons learned

- Influence functional: key object in open quantum systems.
- The $x_d\,x_d$ coefficient captures the fluctuations (they average to the semi-classical force in Langevin).
- The $x_a\,x_d$ coefficient gives the response of the environment (dissipation, when conserved).
- Notice, there is no $x_a\,x_a$ term (unitarity).

# Taking stock



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keeping noise under control

**Open quantum systems**

# The gravitational path integral

Old story: gravitational systems can be seen as thermal:

$$\int_{\mathcal{M}\times S^1_\beta} Dg\, e^{-S_E} \sim \sum_{g^*|\propto \mathcal{M}\times S^1_\beta} e^{-S(g^*)} \sim \mathsf{Tr}\left(e^{-\beta H_{\mathsf{bdy.}}}\right)$$

Mind the limits

Sum over solutions

“Newer” story: the holographic GKPW dictionary

$$\left\langle e^{-\int J(x)\mathcal{O}(x)}\right\rangle_{\mathsf{CFT}} = \int_{\mathsf{AdS}} DgD\phi\, e^{-S_E} \sim e^{-S^{\mathsf{on\text{-}shell}}[\phi\to J]}$$

# Unpacking GKPW

The generating functional has two pieces of data

$$Z_{\mathsf{CFT}}[J] = \left\langle e^{-\int \overset{\text{Sources}}{\downarrow\atop J(x)}\;\mathcal{O}(x)} \right\rangle_{\underset{\text{State}}{\rho}}$$

This has to be encoded in the dynamics on asymptotically AdS spacetimes

$$ds^2 = \frac{dr^2}{r^2} + r^2 g_{\mu\nu}(r,x)dx^\mu\, dx^\nu$$

Generic behavior in AdS$_{d+1}$:

First identification: leading behavior = source

$$\Phi(r,x) = r^{d-\Delta}\ J(x)\ (1+\cdots) + r^{-\Delta}\ \langle\mathcal{O}(x)\rangle_{\rho}\ (1+\cdots)$$

Second identification: sub-leading behavior = Exp. Val.

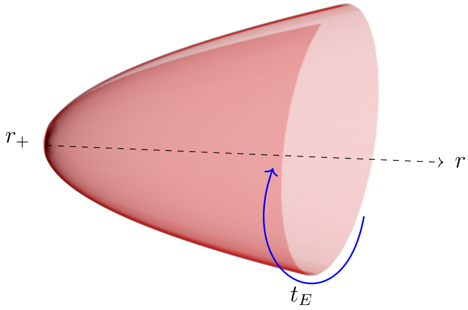
- The leading behavior is universal → sources are fixed external inputs.
- The sub-leading behavior depends on the interior geometry → correlations are dynamical responses (holographic reconstruction).

# A special case: the thermal state

- **Observation:** In QFT, $Z(\beta) = \mathrm{Tr} e^{-\beta H}$ can be obtained as a path integral over $\mathbb{R}^{d-1} \times S^1_\beta$.
- **GKPW:** Sum over geometries with boundary condition $\mathbb{R}^{d-1} \times S^1_\beta$.

**Dominant saddle:** Euclidean cigar ("black hole"):

$$ds^2 = r^2 f(r) dt_E^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}_{d-1}^2\,, \quad f(r) = 1 - \frac{r_+^d}{r^d}$$



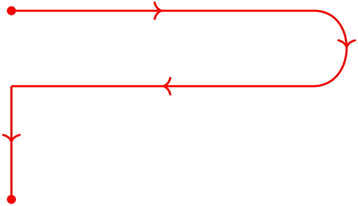
# The Lorentzian GKPW dictionary

- GKPW was only formulated for Euclidean geometries.
- Long history of Lorentzian generalizations [Herzog, Son, Starinets, Skenderis, van Rees...].
- **Main obstacle:** Additional data due to time-ordering (causality).
- **Solution:** Lesson from before, the Schwinger-Keldysh contour.

# The LHS: what do we want to compute?

From our previous study, the real-time generating functional is:

$$Z[J_R, J_L] = \left\langle P e^{i\int_{\mathbb{R}^{d-1}\times\mathcal{C}_{SK}}(J_R\mathcal{O}_R - J_L\mathcal{O}_L)} \right\rangle_\beta =$$



# The gravitational SK (grSK) geometry

Generalized GKPW prescription:

$$\left\langle P e^{i\int_{\mathbb{R}^{d-1}\times\mathcal{C}_{SK}}(J_R\mathcal{O}_R - J_L\mathcal{O}_L)}\right\rangle_\beta = \int_{g|_{\text{bdy.}}\sim\mathbb{R}^{d-1}\times\mathcal{C}_{SK}} Dg\, e^{iS}$$

$$= e^{iS[g_*]}$$

where $g_*$ is a solution of Einstein's equations with bdy. condition $\mathbb{R}^{d-1}\times\mathcal{C}_{SK}$.

# Constructing the solution

- Euclidean insight: dominant saddle in the imaginary-time section is $ds_E^2 = r^2 f(r) dt_E^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}_{d-1}^2 \, .$
- Dominant solution in Lorentzian segments: $ds_\pm^2 = -r^2 f(r) dt_\pm^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}_{d-1}^2 \, .$
- Building new solutions by copy-paste (poor's man solution):

$$\left[ g_{\mu\nu}^{(i)} - g_{\mu\nu}^{(j)} \right]_\Sigma = 0 \, , \quad \left[ K_{\mu\nu}^{(i)} - K_{\mu\nu}^{(j)} \right]_\Sigma = 0 \, .$$

- **Easy part**: pasting to the Euclidean segment.

  Simply take $\Sigma$ to be the surface $t_E = 0 \sim 0 + \beta$, and $t_+ \sim 0$, $t_- = 0 + i\beta$.

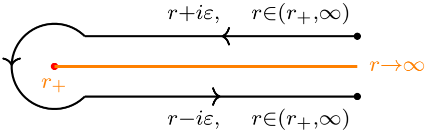
- **Tricky part**: pasting of Lorentzian segments across their horizon.

# The CGL prescription

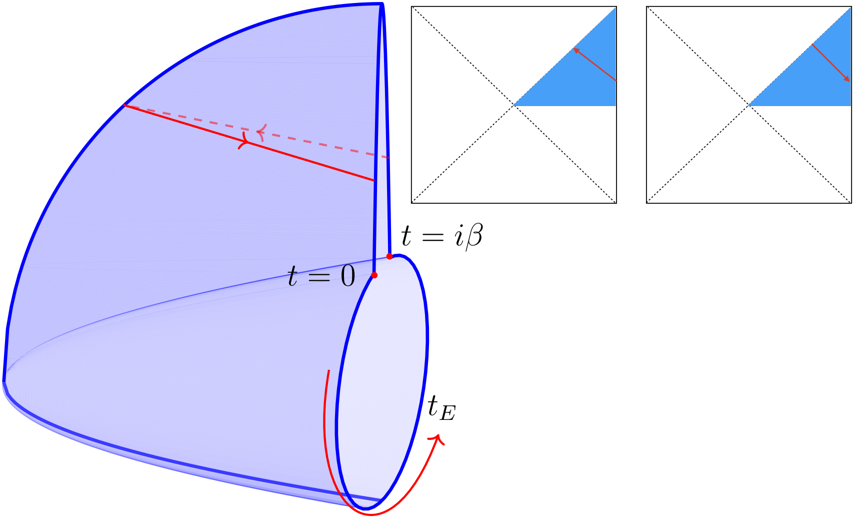
To paste the Lorentzian segments, make $r$ complex[Glorioso, Crossley, Liu]

$$ds^2 = -r^2 f dv^2 + 2dvdr + r^2 d\vec{x}^2$$

$$r - r_+ = \epsilon e^{i\theta}\,, \quad \theta \in (0, 2\pi)$$



We then restrict $r \in \mathbb{C}$ to live on the Hankel contour above.



**Credit:** Heroic tikz work by Tom Angrick

# All in one patch: Mock tortoise coordinate

It is convenient to introduce the *mock tortoise coordinate* [Jana, Loganayagam, Rangamani]:

$$ds^2 = -r^2 f dv^2 + i\beta r^2 f dv d\zeta + r^2 d\vec{x}^2 \,,$$

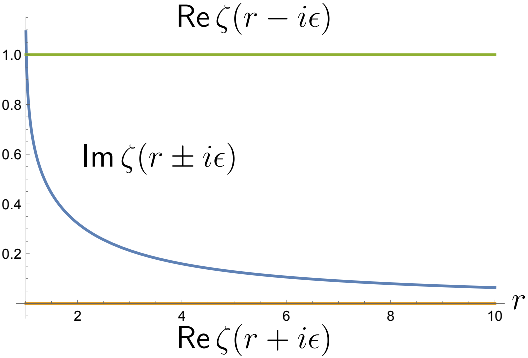
$$\frac{d\zeta}{dr} = \frac{2}{i\beta} \; \frac{1}{r^2 f(r)} \,,$$

Convinience prefactor

Good ol' tortoise coordinate

**Main property:** $\zeta(r) \propto \log(r - r_+)$, logarithmic branch cut!

The prefactors have been chosen such that



In particular

$$\lim_{r\to\infty+i\epsilon} \zeta(r) = 0\,, \qquad \lim_{r\to\infty-i\epsilon} \zeta(r) = 1\,.$$

# Scalar dynamics in grSK

To test the prescription, consider

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}g^{AB}\partial_A\Phi\partial_B\Phi$$

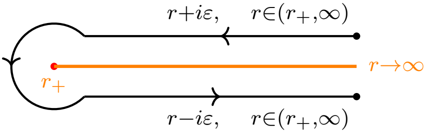
For $r \to \infty \pm i\epsilon$, nothing really changes

F.T.: $e^{-i\omega v + i\vec{k}\cdot\vec{x}}$

$$\Phi(r,k) \sim J_{R/L}(k)\left(1+\cdots\right) + \frac{\phi^{(1)}_{R/L}(k)}{r^d}\left(1+\cdots\right)\ , \qquad k = (\omega, \vec{k})$$

But for $r \to r_+$

$$\Phi(r,k) \sim \overset{\text{Regular at horizon}}{A_{R/L}(k)}\ (1+\cdots) + B_{R/L}(k)\underset{\text{Discontinuous around horizon}}{(r-r_+)}^{\frac{i\beta\omega}{2\pi}}\ (1+\cdots)$$



Across the horizon $r - r_+ \to (r-r_+)e^{2\pi i}$.

Continuity of the full solution then requires:

$$A_R(k) = A_L(k)\,, \qquad B_R(k) = B_L(k)e^{-\beta\omega}.$$

It is convenient to write linear combinations, a fully regular solution:

$$\begin{aligned}\Phi_{\mathsf{in}}(r,k) &\equiv \Phi_R(r,k) - e^{-\beta\omega}\Phi_L(r,k)\\ &\sim \begin{cases} A(k)(1+\cdots) & r \to r_+ \\ -(1+n_\omega)J_R(k) + n_\omega J_L\,(1+\cdots) & r \to \infty + i\epsilon \end{cases}\end{aligned}$$

and a fully irregular one:

$$\Phi_{\mathsf{out}}(r,k) \equiv \Phi_R(r,k) - \Phi_L(r,k)$$
$$\sim \begin{cases} B(k)\,(r-r_+)^{\frac{i\beta\omega}{2\pi}}\,(1+\cdots) & r\to r_+ \\ -n_\omega\,(J_R(k)-J_L(k))\,(1+\cdots) & r\to\infty+i\epsilon \end{cases}$$

where $n_\omega = \frac{1}{e^{\beta\omega}-1}$ is the familiar Boltzmann factor. Notice a different linear combination produces:

$$J_a = \frac{1}{2}\left(J_R + J_L\right)\,, \qquad J_d = J_R - J_L\,.$$

# Solving the EoM

With the boundary conditions sorted out, we can solve

$$-\frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}g^{AB}\partial_B\Phi\right)=0\,,$$

or in Fourier domain

$$\frac{1}{r^{d-1}}\mathbb{D}_+\left(r^{d-1}\mathbb{D}_+\Phi(r,k)\right)+\left(\omega^2-k^2f\right)\Phi(r,k)=0\,.$$

where $\mathbb{D}_+=r^2f\frac{d}{dr}-i\omega$ is motivated by time-reversal.

We define the *ingoing propagator* $G^{\text{in}}(r,k)$ as a solution satisfying:

$$\lim_{r\to\infty\pm i\epsilon} G^{\text{in}}(r,k) = 1\,, \qquad \lim_{r\to r_+} G^{\text{in}}(r,k) = \text{ regular}$$

Due to regularity, it's the same in both branches

Using time-reversal symmetry, $v \to i\beta\zeta - v$, the *outgoing propagator* is

$$G^{\text{out}}(r,\omega,\vec{k}) = \; e^{-\beta\omega\zeta} \;\; G^{\text{in}}(r,-\omega,\vec{k})$$

Fully irregular, $\zeta \to 0, 1$

Fully regular

The general solution to the EoM in grSK geometry is

$$\Phi(\zeta,\omega,k) = \left(J_a + \left(n_\omega + \frac{1}{2}\right) J_d\right) G^{\mathsf{in}}(r,\omega,\vec{k})$$
$$- n_\omega J_d e^{\beta\omega(1-\zeta)} G^{\mathsf{in}}(r,-\omega,k)$$

Notice, we *only* need to compute $G^{\mathsf{in}}(r,\omega,k)$!

A quick check:

$$\lim_{\zeta\to 0} \Phi(\zeta,\omega,k) = \left(J_a + \left(n_\omega + \frac{1}{2}\right) J_d\right) - n_\omega e^{\beta\omega}\, J_d = J_L$$
$$\lim_{\zeta\to 1} \Phi(\zeta,\omega,k) = \left(J_a + \left(n_\omega + \frac{1}{2}\right) J_d\right) - n_\omega\, J_d = J_R$$

# Evaluating the on-shell action

With the solution on hand, we evaluate:

$$
\begin{aligned}
S &= \mathsf{EoM} - \int d^{d+1}x\partial_A\left(\sqrt{-g}g^{AB}\Phi\partial_B\Phi\right) \\
&= \frac{1}{2}\int_k \Phi(r,k)\Pi(r,-k)\,\Big|_{\zeta=0}^{\zeta=1}
\end{aligned}
$$

There are two boundaries

where

$$\Pi(r,k) \equiv -\sqrt{-g}g^{rB}\partial_B\Phi = -r^{d-1}\mathbb{D}_+\Phi$$

Defining

$$K^{\mathsf{in}}(\omega,k) = -\lim_{r\to\infty}\left(r^{d-1}\mathbb{D}_+ G^{\mathsf{in}}(r,\omega,k) + \mathsf{c.t.}\right)$$

We find

$$S_{\mathsf{on\text{-}shell}} = -\frac{1}{2}\int_k J_d^\dagger(k)\, K^{\mathsf{in}}(\omega,k)\left(J_a(k) + \left(n_\omega + \frac{1}{2}\right) J_d(k)\right) + \mathsf{c.c.}$$

which, by GKPW, gives us the generating functional of connected correlators:

$$iW[J_a,J_d] = \log Z[J_a,J_d] \sim iS_{\mathsf{on\text{-}shell}}[J_a,J_d]$$

# A reassuring answer

We can compare:

$$S_{\textsf{on-shell}} = -\frac{1}{2}\int_k J_d^\dagger(k)\, K^{\textsf{in}}(\omega,k)\left(J_a(k) + \left(n_\omega + \frac{1}{2}\right) J_d(k)\right) + \text{c.c.}$$

with the harmonic oscillator answer

$$\log Z[x_a, x_d] = -i\int x_d\,\alpha_I\,x_a - \int x_d\,\alpha_R\,x_d$$

It is the same structure!

- From the H.O. analysis, the $J_d^\dagger J_d$ coefficient gives the average force:

$$\langle \mathcal{O}(-k)\mathcal{O}(k)\rangle^{\text{Kel.}} = -\frac{1}{2}\ \coth\left(\frac{\beta\omega}{2}\right) \text{Im} K^{\text{in}}(\omega,\vec{k})$$

Same factor we found in the H.O. analysis!

- While the $J_d^\dagger J_a$ coefficient gives the retarded response:

$$\langle \mathcal{O}(-k)\mathcal{O}(k)\rangle^{\text{Ret.}} = iK^{\text{in}}(\omega,\vec{k})$$

- The two results are related:

$$\langle \mathcal{O}(-k)\mathcal{O}(k)\rangle^{\textsf{Kel.}} = \frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right)\textsf{Re}\,\langle \mathcal{O}(-k)\mathcal{O}(k)\rangle^{\textsf{Ret.}}$$

  which is the fluctuation-dissipation theorem.

- Notice there is no $J_a^\dagger J_a$ term, as expected from unitarity.

- Our gravitational calculation agrees completely with the QM expectation.

# All together: Holographic open quantum systems

Observer/measuring system

$$S = S_A + S_B + \int d^d x \, J(x) \;\; \mathcal{O}(x)$$

Holographic system, for example $\mathcal{N} = 4$ SYM

Influence functional:

Fancy-pants harmonic oscillator

$$\left\langle e^{i\int(J_R\mathcal{O}_R - J_L\mathcal{O}_L)}\right\rangle_{\beta}^{\mathcal{N}=4\,\mathsf{SYM}} = \int Dg D\Phi \, e^{iS[g,\Phi]} \sim e^{iS_{\text{on-shell}}[J_R,J_L]}\Big|_{\mathsf{grSK}}$$

[Jana,Loganayagam,Rangamani][Loganayagam,Rangamani,JV]

# The real deal: metric fluctuations

Going beyond toy scalars:

$$\begin{aligned}\left\langle e^{i\int (J_R\mathcal{O}_R - J_L\mathcal{O}_L)}\right\rangle_{\beta}^{\mathcal{N}=4\,\mathsf{SYM}} &\sim e^{iS_{\text{on-shell}}}\,\text{(metric fluctuations)}\\ &= e^{iS_{\text{on-shell}}}\int Dh\, e^{iS^{(2)}[h]+iS^{(3)}[h]+\cdots}\end{aligned}$$

where $g = g_{\mathsf{grSK}} + h$, with $|h| \ll 1$, and $S^{(n)}[h]$ come from expanding the Einstein-Hilbert action.

# Humble beginnings: gaussian fluctuations

Start with only

$$Z_2[h] = \int Dh\, e^{iS^{(2)}[h]} \overset{\text{Gaussian integration}}{=} e^{iS^{(2)}_{\text{on-shell}}[h|_{\text{bdy.}}]}$$

Generic behavior:

$$h_{AB}(r,x)dx^A dx^B \sim \frac{dr^2}{r^2} + r^2 \left( \underset{\text{Source}}{\gamma_{\mu\nu}(x)} + \frac{\overset{\text{One-point function}}{\langle T_{\mu\nu}(x)\rangle}}{r^d} \right) dx^\mu dx^\nu + \cdots$$

# Metric fluctuations as open quantum systems

Comparing with the scalar results:

Generating functional of stress tensor correlators

$$\left\langle e^{i\int\left(\gamma^R_{\mu\nu}\,T^{\mu\nu}_R-\gamma^L_{\mu\nu}\,T^{\mu\nu}_L\right)}\right\rangle^{\mathcal{N}=4\,\mathsf{SYM}}_{\beta} = e^{iS^{(2)}_{\text{on-shell}}[h|_{\text{bdy.}}\sim\gamma]}+\cdots$$

Perturbation around grSK geometry

**Connection with Hydrodynamics:** $\langle T_{\mu\nu}T_{\rho\sigma}\cdots\rangle$ in the limit $\omega, |\vec{k}| \ll T$.

# Handling metric fluctuations: designer scalars

$g = g_{\mathsf{grSK}} + h$

$SO(d-2)$ harmonic descomposition

Diff. invariance

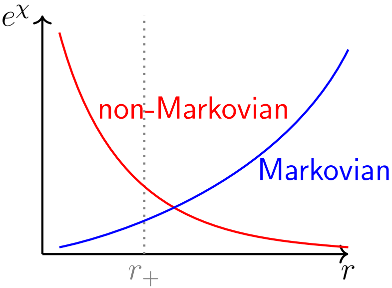
Constraint equations

$$S_{\mathcal{M}} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}e^{\chi(r)}\nabla_A\Phi_{\mathcal{M}}\nabla^A\Phi_{\mathcal{M}}, \quad \sqrt{-g}e^{\chi(r)} \sim r^{\mathcal{M}}$$

[Ghosh,Loganayagam,Prabhu,Rangamani,Sivakumar,Vishal][He,Loganayagam,Rangamani,Sivakumar,JV]

## The dilaton makes all the difference

- Tensors: $e^{\chi(r)} = 1$.
- Vectors: $e^{\chi(r)} = \frac{1}{r^{2(d-1)}}$ ,
- Scalars: $e^{\chi(r)} = \frac{1}{r^{2(d-2)}\Lambda_k^2}$ , $\quad \Lambda_k = k^2 + \frac{1}{2}(d-1)r^3 f'$.



# Dynamic of designer scalars in SK

- Let us study the system (scalars are hard):

$$S_{\text{designer}} = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,r^{\mathcal{M}-d+1}\,\nabla_A\Phi_{\mathcal{M}}\nabla^A\Phi_{\mathcal{M}}+S_{\text{bdy.}}$$

- The equation of motion is

$$r^{-\mathcal{M}}\mathbb{D}_+\left(r^{\mathcal{M}}\mathbb{D}_+\Phi_{\mathcal{M}}\right)+(\omega^2-k^2f)\Phi_{\mathcal{M}}=0\,.$$

- For $d > 2$, this equation can be solved in a gradient expansion:

$$\frac{\omega}{r_+},\frac{k}{r_+}\ll 1$$

- Asymptotic behavior:

$$\Phi_{\mathcal{M}} \sim c_1 + \frac{c_2}{r^{\mathcal{M}+1}}$$

- For $\mathcal{M} > -1$, we impose Dirichlet conditions ($S_{\textsf{bdy.}} = 0$).
- For $\mathcal{M} < -1$, compute the conjugate momentum:

$$\pi_{\mathcal{M}} = -r^{\mathcal{M}} \mathbb{D}_+ \Phi_{\mathcal{M}} \sim \tilde{c}_1 + \tilde{c}_2 r^{\mathcal{M}}$$

- Quantization is done using Neumann conditions ($S_{\textsf{bdy.}} = \int d^d x \Phi_{\mathcal{M}} \pi_{\mathcal{M}}$).

# Markovian fields (tensors)

Same old scalar:

$$\Phi_{\mathcal{M}}(\zeta, w, \vec{k}) = J_a G^{\mathsf{in}}_{\mathcal{M}} + \left[\left(n_\beta + \frac{1}{2}\right) G^{\mathsf{in}}_{\mathcal{M}} - n_\beta e^{\beta(\zeta-1)} G^{\mathsf{rev}}_{\mathcal{M}}\right] J_d$$

$$S_{\mathsf{on\text{-}shell}}[J_a, J_d] = -\int_k J_d^\dagger \mathcal{K}^{\mathsf{in}}_{\mathcal{M}} \left[J_a + \left(n_\beta + \frac{1}{2}\right) J_d\right]$$

$$K^{\mathsf{in}}_{\mathcal{M}} = -iw + \frac{k^2}{1-\mathcal{M}} + \Delta^{2,0}_{\mathcal{M}}(r_+) w^2 + ...$$

**Note:** Correlations are analytic in $\omega$ and $|\vec{k}| \Rightarrow$ there are no conserved tensor-like currents!

# Non-Markovian fields (vectors and scalars*)

Here things are trickier

- We could compute $\log Z[J_a, J_d] = iS_{\text{on-shell}}[J_a, J_d]$, using Newmann conditions.
- Alternatively, we could still use Dirichlet

  $$\Phi_{\mathcal{M}}(\zeta, w, \vec{k}) = \check{\Phi}_a G_{\mathcal{M}}^{\text{in}} + \left[\left(n_\beta + \frac{1}{2}\right) G_{\mathcal{M}}^{\text{in}} - n_\beta e^{\beta(\zeta-1)} G_{\mathcal{M}}^{\text{rev}}\right] \check{\Phi}_d$$

  fixing the one-point function (very non-kosher).

# Old friend: Quantum Effective Action

- What does it mean to fix the one-point function?
- Gravity side: changing from Newmann to Dirichlet:

$$\tilde{S} = \overbrace{-\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,r^{\mathcal{M}-d+1}\,\nabla_A\Phi_{\mathcal{M}}\nabla^A\Phi_{\mathcal{M}} + \int d^dx\Phi_{\mathcal{M}}\pi_{\mathcal{M}}}^{\text{Original action: Newmann conditions}}$$

$$\underbrace{-\int d^dx\Phi_{\mathcal{M}}\pi_{\mathcal{M}}}_{\text{Legendre transform to Dirichlet conditions}}$$

- Field theory side:

$$i\Gamma[\check{\Phi}_a, \check{\Phi}_d] = \overset{\overset{iS_{\text{on-shell}}[J_a, J_d]}{\downarrow}}{\log Z[J_a, J_d]} - \overset{\overset{\int d^d x \Phi_{\mathcal{M}} \pi_{\mathcal{M}} \big|_{\zeta-0}^{\zeta=1}}{\downarrow}}{i \int d^d x \, \left( J_R \check{\Phi}_R - J_L \check{\Phi}_L \right)}$$

- The Legendre transform of the generating functional is the Quantum Effective Action.

$$\begin{aligned}\Gamma[\check{\Phi}_a, \check{\Phi}_d] &= S^{\text{Dirichlet}}_{\text{on-shell}}[\check{\Phi}_a, \check{\Phi}_d] \\ &= -\int_k \check{\Phi}_d^\dagger \mathcal{K}^{\text{in}}_{-\mathcal{M}} \left[ \check{\Phi}_a + \left( n_\beta + \frac{1}{2} \right) \check{\Phi}_d \right]\end{aligned}$$

# Interpreation: Wilsonian Influence Functional

- Before, influence functional:

$$e^{iF[x_R,x_L]} = \int_\rho DQ_R DQ_L \, e^{iS_R - iS_L}$$

- Now, Wilsonian Influence Functional

Short-Lived modes

Long-Lived modes

$$\begin{aligned} e^{iF[x_R,x_L]} &= \int_\rho \left( DQ_R^{SL} DQ_L^{SL} \right) \left( DQ_R^{LL} DQ_L^{LL} \right) e^{iS_R - iS_L} \\ &= \int \left( DQ_R^{SL} DQ_L^{SL} \right) \; e^{i\mathcal{S}_{\mathsf{WIF}}} \end{aligned}$$

LT wrt Long-Lived modes ⇒ QEA ⇒ Fluid EFT

- In total:

$$\mathcal{S}_{\mathsf{WIF}} \propto -\sum_{\alpha=1}^{N_V}\int_k \left(\check{\mathcal{P}}_d^{\alpha}\right)^{\dagger} K^{\mathsf{in}}_{\mathsf{Mom.}}\left[\check{\mathcal{P}}_a^{\alpha} + \left(n_\beta+\frac{1}{2}\right)\check{\mathcal{P}}_d^{\alpha}\right] - \int_k \left(\check{\mathcal{Z}}_d\right)^{\dagger} K^{\mathsf{in}}_{\mathsf{Sound}}\left[\check{\mathcal{Z}}_a + \left(n_\beta+\frac{1}{2}\right)\check{\mathcal{Z}}_d\right]$$

Famous result $\mathcal{D}=\frac{\eta}{e+p}\rightarrow\frac{\eta}{s}=\frac{1}{4\pi}$

$$K^{\mathsf{in}}_{\mathsf{Mom}} = -i\omega + \frac{1}{d}\,k^2 + \cdots$$

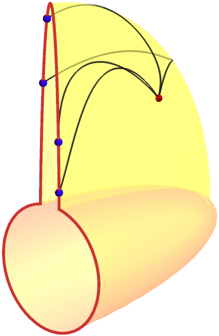
$$K^{\mathsf{in}}_{\mathsf{Sound}} = -\omega^2 + \frac{k^2}{d-1} + \nu_s k^2 \Gamma_s(\omega,k)$$

Hawking sound

# Beyond the Gaussian level:

# Witten diagrams in SK geometry

What about $S^{(n)}[h]$ for $n > 1$



[Loganayagam,Rangamani,JV] and work in progress [Ammon,Rangamani,Specht,JV]

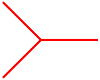
# The ingredients:

- Ingoing Bulk-to-Bdy Prop.: $G_{\mathsf{in}}(\zeta, k)$.
- Outgoing Bulk-to-Bdy Prop.: $G_{\mathsf{out}}(\zeta, k) = e^{-\beta\omega\zeta} G_{\mathsf{in}}(\zeta, \bar{k})$
- Bulk-to-Bulk Prop:

  $G_{\mathsf{bb}}(\zeta, \zeta'; k) = \mathcal{N}(k) e^{\beta\omega\zeta'} G_{\mathsf{L}}(\zeta_>, k) G_{\mathsf{R}}(\zeta'_<, k)$.

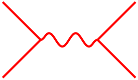
**Important:** $G_{\mathsf{in}}(\zeta + 1, k) = G_{\mathsf{in}}(\zeta, k)$.

- **Contact diagram:**



$$= \oint d\zeta \, \mathfrak{L}(\zeta) = \int_{r_{\mathsf{H}}}^{r_c} dr \, (\mathfrak{L}(\zeta(r)+1) - \mathfrak{L}(\zeta(r)))$$

- **Exchange diagram**



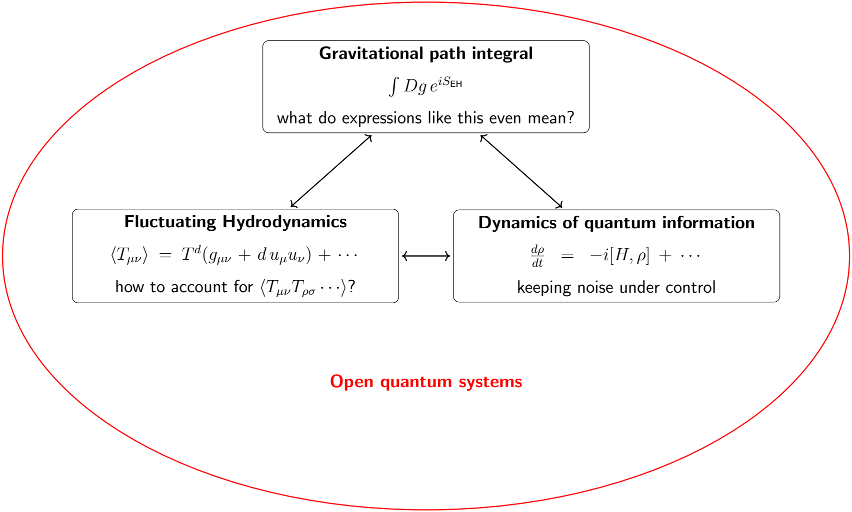
$$\begin{aligned} &= \oint d\zeta \oint d\zeta' \, [F_1(\zeta,\zeta')\Theta(\zeta-\zeta') + F_2(\zeta,\zeta')\Theta(\zeta'-\zeta)] \\ &= \int_{r_{\mathsf{H}}}^{r_c} dr \int_{r_{\mathsf{H}}}^{r_c} dr' \, [\mathfrak{F}_1\theta(r-r') + \mathfrak{F}_2\theta(r'-r)] \end{aligned}$$

where

$$\mathfrak{F}_1 = F_1(\zeta,\zeta') - F_1(\zeta+1,\zeta') + F_2(\zeta+1,\zeta'+1) - F_2(\zeta,\zeta'+1)\,,$$

$$\mathfrak{F}_2 = F_1(\zeta+1,\zeta'+1) - F_1(\zeta+1,\zeta') + F_2(\zeta,\zeta') - F_2(\zeta,\zeta'+1)\,.$$

# Taking stock, part II



**Gravitational path integral**

$$\int Dg\, e^{iS_{\text{EH}}}$$

what do expressions like this even mean?

**Fluctuating Hydrodynamics**

$$\langle T_{\mu\nu}\rangle = T^d(g_{\mu\nu} + d\, u_\mu u_\nu) + \cdots$$

how to account for $\langle T_{\mu\nu} T_{\rho\sigma} \cdots \rangle$?

**Dynamics of quantum information**

$$\frac{d\rho}{dt} = -i[H, \rho] + \cdots$$

keeping noise under control

**Open quantum systems**

# Work in progress

- Higher order SK correlators for $d > 2$ [Ammon, Rangamani, Specht, JV].
- Connections to non-linear fluid actions [Rangamani, JV].
- Beyond the SK contour: OTOC [Ammon, Germerodt, Sieling, JV].

# Final musings

- More complicated thermal systems: Kerr black holes, superfluids,...

  (For charged black holes see [He, Loganayagam, Rangamani, JV])

- Transition amplitudes: beyond saddle point approximation

  (Golden dream: topology changes)

- Non local-probes: entanglement entropy, reflected entropy, negativity,... [Colin-Ellerin, Dong, Marolf, Rangamani, Wang][Pelliconi, Sonner][Mezei,JV]

- Effective actions for chaotic systems [Blake, Lee, Liu][Haehl, Rozali]
- Gravitational SK contours for more general spacetimes (see recent work on dS correlators[Di Pietro, Gorbenko, Komatsu][Loganayagam, Shetye]).
- Is there something to be said in asymptotically flat spacetimes?

Thank You!