Towards a deeper understanding of relativistic dissipative hydrodynamics in numerical relativity: Introducing the BDNK theory.

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Introduction

- Motivation
- Non-relativistic Hydrodynamics
- Relativistic Hydrodynamics
- BDNK theory
- Discussion
- Final remarks

Motivation

- Dissipative effects
- Mergers/accretion/ejecta
- Magnetohydrodynamics (MHD)

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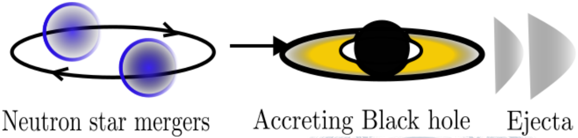

Figure 1: Neutron star mergers produce accreting black holes and ejecta.

Motivation

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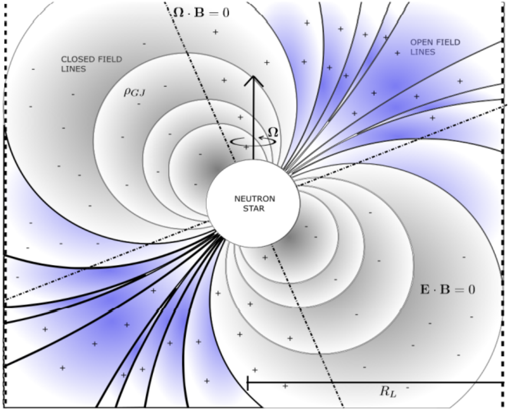

Figure 2: Force-free model of a pulsar's magnetosphere

Non-relativistic Hydrodynamics

- Defining fluids
- Thermodynamics and equilibrium
- Equation of motion

Fluids

- $\ell \ll L$
- $f = f(t, x^i, v^i)$
- $f = f_{(0)} + \tau f_{(1)} + \mathcal{O}(\tau^2)$

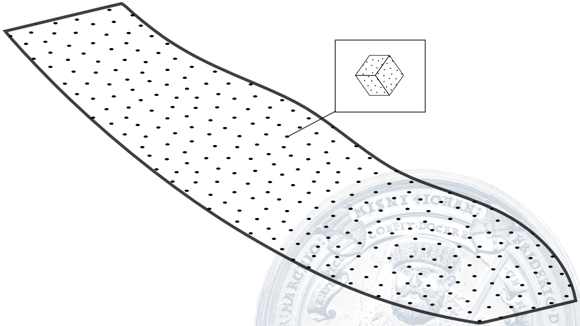

Figure 3: A system with N particles, where the width represents the characteristic length.

Non-relativistic Hydrodynamics

- Defining fluids
- Thermodynamics and equilibrium
- Equation of motion

Thermodynamics

- Fluids are not closed systems.
- Volume elements are closed systems.
- ε, ρ, and v^i represent fields dependent on position and time.

Non-relativistic Hydrodynamics

- Defining fluids
- Thermodynamics and equilibrium
- Equation of motion

$$j^i_\varepsilon = \left(\varepsilon + p + \frac{\rho v^2}{2}\right) v^i - \eta \sigma^{ij} v_j - \zeta v^i \partial_k v^k$$

$$\Pi^{ij} = p\delta^{ij} + \rho v^i v^j - \eta\sigma^{ij} - \zeta \delta^{ij} \partial_k v^k.$$

Note: Thermal conductivity has been set equal to zero.

Equation of motion

$$\partial_t \rho + \partial_i(\rho v^i) = 0$$
$$\partial_t(\rho v^i) + \partial_j \Pi^{ij} = 0$$
$$\partial_t\left(\varepsilon + \frac{\rho v^2}{2}\right) + \partial_i j^i_\varepsilon = 0.$$

Relativistic Hydrodynamics

- Conserved quantities
- Eckart and Landau-Lifshitz frames
- Müller-Isreal-Stewart theory
- Relativistic Navier Stoke Equations

$$T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu}$$
$$J^\mu_{(0)} = n u^\mu$$

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

The e.o.m are found by contracting with u_ν and $\Delta_{\mu\nu}$

Energy-momentum tensor and four current

T^{00} : energy density

T^{ii} : pressure

T^{ij} : *stress*

T^{0i} : energy current

J^0 : charge density

J^i : charge current

Relativistic Hydrodynamics

- Conserved quantities
- Eckart and Landau-Lifshitz frames
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- Relativistic Navier Stoke Equations

$$\text{Im } \omega(\kappa) \leq 0$$

$$0 \leq \frac{\text{Re } \omega(\kappa)}{\kappa} < 1$$

- Classical theories.
- Added dissipative effects
- **Eckart**

$$u_\mu u_\nu T^{\mu\nu}_{(1)} = 0 \qquad J^\mu_{(1)} = 0$$

- **Landau-Lifshitz**

$$u_\mu T^{\mu\nu}_{(1)} = 0 \qquad u_\mu J^\mu_{(1)} = 0$$

- Unstable and acausal; NOT GOOD!

Relativistic Hydrodynamics

- Conserved quantities
- Eckart and Landau-Lifshitz frames
- Müller-Isreal-Stewart theory
- Relativistic Navier Stoke Equations

- Most popular fix
- Introduces extended fields
- Stable and causal
- Not classical theory
- Further complication for numerical relativity.

Relativistic Hydrodynamics

- Conserved quantities
- Eckart and Landau-Lifshitz frames
- Müller-Isreal-Stewart theory
- Relativistic Navier Stoke Equations

Question:

- Classical theory
- Stable and causal
- Locally well posed

(Not a strict requirement)

- Strong hyperbolic

BDNK theory

- Effective field theory
- Setting up the general frame
- Constitution relations
- Frame transformation

BDNK: Bemfica, Disconzi, Noronha, and Kovtun

BDNK: Is locally well posed and strongly hyperbolic [1]

Essentials

- μ, T and u^μ
- $\nabla_\mu T^{\mu\nu} = 0$, $\nabla_\mu J^\mu = 0$
- Gradient expansion

Translation and global $U(1)$

BDNK theory

- Effective field theory
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- Frame transformation

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + 2\mathcal{Q}^{(\mu} u^{\nu)} + t^{\mu\nu}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu$$

Decomposition

$$\mathcal{E} = u_\mu u_\nu T^{\mu\nu} \qquad \mathcal{P} = \frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$$
$$\mathcal{Q}^\mu = \Delta^\mu{}_\nu u_\sigma T^{\nu\sigma} \qquad j^\mu = \Delta^\mu{}_\nu J^\nu$$
$$\mathcal{N} = u_\mu J^\mu$$

$$t^{\mu\nu} = \frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right) T^{\alpha\beta}$$

BDNK theory

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$$\mathcal{E} : \varepsilon + \epsilon_i \qquad \mathcal{P} : p + \pi_i \qquad \mathcal{N} : n + \tau_i$$
$$\mathcal{Q} : q_i \qquad j^\mu : l_i \qquad t^{\mu\nu} : \eta$$

Example

$$\mathcal{E} = \varepsilon + \epsilon_1 \frac{1}{T}\dot{T} + \epsilon_2 \nabla_\mu u^\mu + \epsilon_3 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right)$$

$$\dot{T} = u^\mu \nabla_\mu T$$

There are 3 coefficients at zero order, and 16 at first order.

BDNK theory

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$$\delta T = a_1 \frac{1}{T} u^\mu \nabla_\mu T + a_2 \nabla_\mu u^\mu + a_3 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right)$$

$$\epsilon_i' \to \epsilon_i - \left(\frac{\partial \varepsilon}{\partial T}\right)_\mu a_i - \left(\frac{\partial \varepsilon}{\partial \mu}\right)_T b_i$$

Out of equilibrium

- μ, T and u^μ: No microscopic definiton
- $T \to T + \delta T$
- 7 Genuine transport coefficients

$$f_i \qquad \ell_i \qquad \eta$$

- $\ell_1 = \ell_2$ [2]

BDNK: Constraints

- Entropy Current
- Interpretation of transport coefficients
- Stability and causality

$$\nabla_\mu S^\mu = \nabla_\mu \left(\frac{1}{T} u_\nu T^{\mu\nu} \right) + \nabla_\mu \left(\frac{\mu}{T} J^\mu \right) \geq 0$$

$$\zeta = p_{,\varepsilon} f_1 - f_2 + \frac{p_{,n}}{T} f_3$$

$$\sigma = \frac{n}{w} \ell_1 - \frac{1}{T} \ell_3$$

Procedure

- Derivative at all order
- On-shell
- Three physical coefficients

$\zeta \geq 0 \qquad \sigma \geq 0 \qquad \eta \geq 0$

BDNK: Constraints

- Entropy Current
- Interpretation of transport coefficients
- Stability and causality

Philosophy of BDNK

- Only three physical coefficients
- $14 - 3 = 11$ non physical transport coefficients
- "UV regulators"
- "strict constraints"

Only a few transport coefficients have constraints; most can be set to zero.

BDNK: Constraints

- Entropy Current
- Interpretation of transport coefficients
- Stability and causality

Dispersion relations

- Fluctuations
- Follow plane waves
- Physical and unphysical modes
- Routh-Hurwitz critera

$$\omega_{\text{shear}} = -\frac{i\eta}{w}\kappa^2 + \mathcal{O}(\kappa^3)$$

$$\omega_{\text{sound}} = \pm v_s\kappa - \frac{i}{2}\omega_2\kappa^2 + \mathcal{O}(\kappa^3)$$

$$\omega_{\text{heat}} = -iD\kappa^2 + \mathcal{O}(\kappa^3)$$

$$D = \frac{\sigma w p_{;\epsilon}^2}{v_s^2 w n_{;\mu} - n^2 v_s^2}$$

$$\omega_2 = \frac{4\eta + \zeta}{w} + \frac{\sigma}{w v_s^2} p_{;n}^2$$

$$Im\ \omega(\kappa) \leq 0$$

$$0 \leq \frac{\text{Re}\omega(\kappa)}{\kappa} < 1$$

BDNK: Magnetohydrodynamics

Dispersion relations BDNK: MHD

- 'Complete theory'
- Two extra degrees of freedom
- Stable and causal [3]
- More realistic

Discussion

- BDNK or MIS
- Dissipation in numerical relativity
- BDNK Limitations

BDNK or MIS

- MIS is well established.
- Numerically suitable.
- Handling of shock.[1] [4]
- Low viscosity: BDNK $\sim$ MIS [4]

Rankine-Hugoniot Condition: Not satisfied by MIS.

Discussion

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Dissipation in NR

- Gravitation waves: unlikely.
- Accretion disk: likely
- Ejecta: Likely.
- MHD application: Important.

Rankine-Hugoniot Condition: Not satisfied by MIS.

Discussion

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BDNK Limitations

- Non-linear fluctuation: non-Markovian behavior. [5]
- Limited study of the theory
- Perhaps more to come

Final remarks

- Finish analytical work
- Testing validity of frames
- Applying to neutron star merges (with coupled magnetic field)
- BDNK: magnetohydrodynamics

Reference

[1] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *First-order general-relativistic viscous fluid dynamics*, 2022 (cited on pages 13, 21).

[2] P. Kovtun, “First-order relativistic hydrodynamics is stable”, Journal of High Energy Physics **2019**, 10.1007/jhep10(2019)034 (2019) (cited on page 16).

[3] J. Armas and F. Camilloni, “A stable and causal model of magnetohydrodynamics”, Journal of Cosmology and Astroparticle Physics **2022**, 039 (2022) (cited on page 20).

[4] A. Pandya and F. Pretorius, “Numerical exploration of first-order relativistic hydrodynamics”, Phys. Rev. D **104**, 023015 (2021) (cited on page 21).

[5] L. Gavassino, N. Mullins, and M. Hippert, *Consistent inclusion of fluctuations in first-order causal and stable relativistic hydrodynamics*, 2024 (cited on page 23).