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Kodama-like Vector Fields in Axisymmetric Spacetimes

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based on [PD & R. Verch, arXiv:2402.18993]

Motivation from Spherical Symmetry

The Kodama Vector Field

For a spherically symmetric spacetime

$$ds^2 = g_{ij}(x^1, x^2)\, dx^i dx^j + R^2(x^1, x^2)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

there exists a timelike vector field

$$k^a = \epsilon^{ab}\nabla_b R(x^1, x^2),$$

which gives rise to a covariantly conserved current, i.e.,

$$\nabla_a j^a := \nabla_a \left(G^{ab} k_b\right) = 0.$$

[H. Kodama, *Prog. Theor. Phys.* **63**, 1217–1228 (1980)]

Properties of the Kodama Vector Field

- k^a is divergence-free, i.e.,

$$\nabla_a k^a = 0.$$

- k^a is tangent to constant-R-surfaces, i.e.,

$$k^a \nabla_a R = 0.$$

- The Noether charge of j^a is the gravitational energy of the spacetime, i.e.,

$$Q_k = \frac{1}{8\pi} \int_{\Sigma_R} j^a n_a \,\mathrm{dvol}_{\Sigma_R} = M_{\mathrm{MSH}}.$$

[S. A. Hayward, *Phys. Rev. D* **53**, 1938–1949 (1996)]

Applications of the Kodama Vector Field

- k^a provides a geometrically preferred direction of time in non-stationary spherically symmetric spacetimes.
  [G. Abreu, M. Visser, *Phys. Rev. D* **82**, 044027 (2010)]

- k^a generates the preferred time evolution for the wave equation. [I. Rácz, *Class. Quantum Grav.* **23**, 115–123 (2006)]

- In QFT on evaporating black hole backgrounds, thermal quantum states are defined w.r.t. the Kodama flow.
  [F. Kurpicz *et al.*, *Lett. Math. Phys.* **111** (2021)]

k^a takes the role of the timelike Killing vector field in non-stationary spacetimes.

Example: Vaidya-Bonnor Spacetime

Charged & radiating dynamical black hole metric

$$ds^2 = -\left(1 - \frac{2M(u)}{R} + \frac{Q(u)^2}{R^2}\right) du^2 - 2\,du\,dR + R^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

[W. B. Bonnor, P. C. Vaidya, *Gen. Rel. Grav.* **1**, 127–130 (1970)]

- ”null dust” solution to the Einstein-Maxwell equations.
  [V. Husain, *Phys. Rev. D* **53** (1996)]
- $M(u)$ is the Brown-York quasi-local mass and the ADM mass.
- $M_{\mathrm{MSH}} := \frac{R}{2}\left(1 - \nabla^a R\, \nabla_a R\right) = M(u) - \frac{Q(u)^2}{2R}$

Example: Vaidya-Bonnor Spacetime

– In Vaidya-Bonnor, the Kodama vector field takes the form

$$k^a = \left(\frac{\partial}{\partial u}\right)^a.$$

– The corresponding current is covariantly conserved, i.e.,

$$\nabla_a j^a = \nabla_a \left(G^{ab} k_b\right) = 0.$$

– j^a carries the Noether charge

$$Q_k = \frac{1}{8\pi}\int_{\Sigma_R} j^a n_a \, \mathrm{dvol}_{\Sigma_R} = M(u) - \frac{Q(u)^2}{2R} = M_{\mathrm{MSH}}.$$

Is there a Kodama-like Symmetry in Axisymmetry?

Kerr-Vaidya Spacetime

$$ds^2 = -\left(1 - \frac{2M(v)r}{\rho^2}\right) dv^2 + 2dvdr - \frac{4M(v)ar\sin^2\vartheta}{\rho^2} dvd\psi$$
$$- 2a\sin^2\vartheta\, drd\psi + \rho^2 d\vartheta^2 + \frac{\Gamma^2 - \Delta a^2 \sin^2\vartheta}{\rho^2} \sin^2\vartheta\, d\psi^2$$

$$\rho^2 = r^2 + a^2\cos^2\vartheta, \qquad \Delta = r^2 - 2M(v)r + a^2, \qquad \Gamma = r^2 + a^2$$

- sourced by a type-III energy momentum tensor.
  [P. K. Dahal, D. R. Terno, *Phys. Rev. D* **102** (2020)]
- $M(v)$ is the Brown-York quasi-local mass and the ADM mass.

Horizons in Kerr-Vaidya

- Characterize horizons of dynamical spacetimes by trapping of light [V. Faraoni, Springer, 2015]

- Trapping horizons of Kerr-Vaidya spacetime do not coincide with $\Delta = 0$, i.e.,

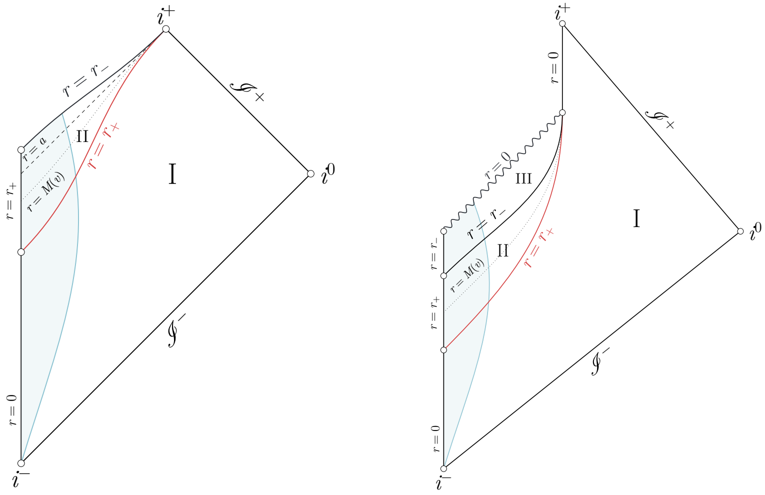
$$r_{\pm} = M(v) \pm \sqrt{M(v)^2 - a^2}.$$

- Future outer trapping horizon can be approximated by

$$r_{\text{foth}} = M(v) + \sqrt{M(v)^2 - a^2} + M'(v)f(v, \vartheta) + \mathcal{O}(M'^2, M'')$$

[P. K. Dahal, *J. Astrophys. Astron.* **42** (2021)]

Kerr-Vaidya-type Black Holes



Kodama-like Symmetry in Kerr-Vaidya

- $K^a = \left(\frac{\partial}{\partial v}\right)^a$ fulfills the Kodama properties in Kerr-Vaidya.

- The current $J^a = G^{ab}K_b$ is covariantly conserved, i.e.,

$$\nabla_a J^a = G^{ab}\nabla_a K_b = 0.$$

- Given a timelike Killing vector field ξ^a in Kerr, it is Kodama-like in the entire exterior of Kerr-Vaidya, if and only if $\xi^a \propto K^a$.

Noether Charge: Gravitational Energy

The current J^a carries the Noether charge

$$Q_K = \frac{1}{8\pi} \int_{\Sigma_r} J^a n_a \, \mathrm{dvol}_{\Sigma_r}$$
$$= M(v) - M'(v) \left(\frac{r^2 + a^2}{2a} \arctan\left(\frac{a}{r}\right) - \frac{r}{2} \right).$$

– For evaporating black holes ($M' < 0$), the additional term is positive.

– In the asymptotically flat region, we have

$$\lim_{r \to \infty} Q_K = M(v)$$

Noether Charge: Angular Momentum

Given the Killing vector field $\phi^a = \left(\frac{\partial}{\partial\psi}\right)^a$, we define the current $I^a = G^{ab}\phi_b$. It carries the Noether charge

$$Q_\phi = \frac{1}{8\pi}\int_{\Sigma_r} I^a n_a \, \mathrm{dvol}_{\Sigma_r}$$
$$= M(v)a - M'(v)\left(\frac{(r^2+a^2)^2}{2a^2}\arctan\left(\frac{a}{r}\right) - \frac{r^3}{2a} - \frac{5ar}{6}\right).$$

- Again, $Q_\phi > M(v)a$ for evaporating black holes.
- At spatial infinity, we find

$$\lim_{r\to\infty} Q_\phi = M(v)a \equiv L_{\mathrm{Kerr.}}$$

Limitations for the Kodama-like Symmetry

- K^a becomes spacelike in the ergoregion.
- The Killing vector field $\zeta^a = \left(\frac{\partial}{\partial v}\right)^a + \Omega_H \left(\frac{\partial}{\partial \psi}\right)^a$ from stationary Kerr spacetime is *not* Kodama-like in Kerr-Vaidya.
- For dynamical rotation parameters $a(v)$, the current J^a is no longer conserved.

$\Rightarrow$ Existence of Kodama-like vector fields in more general spacetimes is rather unlikely.

Kerr-Vaidya-de Sitter Spacetime

$$ds^2 = -\frac{\Delta_\Lambda - \Theta a^2 \sin^2\vartheta}{\rho^2 Z^2} d\tilde{v}^2 + \frac{2}{Z} d\tilde{v} dr - \frac{2a\sin^2\vartheta}{\rho^2 Z^2}\left(\Gamma\Theta - \Delta_\Lambda\right) d\tilde{v} d\tilde{\psi}$$
$$- \frac{2a\sin^2\vartheta}{Z} dr d\tilde{\psi} + \frac{\rho^2}{\Theta} d\vartheta^2 + \frac{\sin^2\vartheta}{\rho^2 Z^2}\left(\Gamma^2\Theta - \Delta_\Lambda a^2 \sin^2\vartheta\right) d\tilde{\psi}^2$$

$$\Delta_\Lambda(\tilde{v}) = \Delta(\tilde{v}) - \frac{\Gamma\Lambda r^2}{3}, \quad \Theta = 1 + \frac{\Lambda a^2 \cos^2\vartheta}{3}, \quad Z = 1 + \frac{\Lambda a^2}{3}$$

- The cosmological horizon $r_C(\tilde{v})$ is dynamical.
- There exists an ergoregion, as before.

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Kodama-like Symmetry in KVdS

- $\tilde{K}^a = \left(\frac{\partial}{\partial \tilde{v}}\right)^a$ is Kodama-like in Kerr-Vaidya-de Sitter.
- Properties are analogous to the asymptotically flat case with some open problems:
- Behaviour of the Kodama-like vector field beyond the cosmological horizon?
- Analytical computation of the Noether charge(s) is not necessarily possible.

Summary of Results

Kodama-like vector fields exist for a class of rotating dynamical black holes.

- In asymptotically flat Kerr-Vaidya spacetime, the charge of the Kodama-like current corresponds to the gravitational energy.
- The charge of the axial Killing current corresponds to the angular momentum.
- Extension to de Sitter asymptotics is possible but poses additional challenges.

Outlook

Obtain new results about dynamical processes:

– Black hole (thermo)dynamics

– quasi-local conservation laws

– Semi-classical evaporation of rotating black holes

Thank you very much!

Backup: Brown-York & ADM Masses

$$M_{\mathrm{BY}} := \frac{1}{8\pi} \int_{S_r} (\mathcal{K}_0 - \mathcal{K}_g)\, d\mathrm{vol}_{S_r}$$

- S_r are closed 2-surfaces.
- $\mathcal{K}_i$ denotes the trace of the extrinsic curvature of S_r isometrically embedded in (M, g) and in $(\mathbb{R}^4, \eta)$.

The Arnowitt-Deser-Misner (ADM) mass is related to M_{BY} via

$$M_{\mathrm{ADM}} = \lim_{S_r \to i^0} M_{\mathrm{BY}}.$$

[E. Poisson, Cambridge University Press, 2004]