Geometry from Quantum Field Theories
— "AdS/CFT" by a conformal flow —

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Toward quantum simulation of gauge/gravity duality and lattice gauge theory
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I. Introduction

- Our scope -

Quantum Gravity

One of the most challenging/unsolved problems in theoretical physics.

Holographic principle   't Hooft, Susskind

Gravity is encoded on a lower dimensional boundary.

ex. black hole thermodynamics

AdS/CFT correspondences   Maldacena 1997

D3 branes in superstring theories in d=10

open string interpretation   ← duality →   closed string interpretation

weak coupling N=4 SYM
(+ SUGRA on flat d=10)

↔ equivalent ?

SUGRA on AdS_5 x S^5
(+ SUGRA in flat d=10)

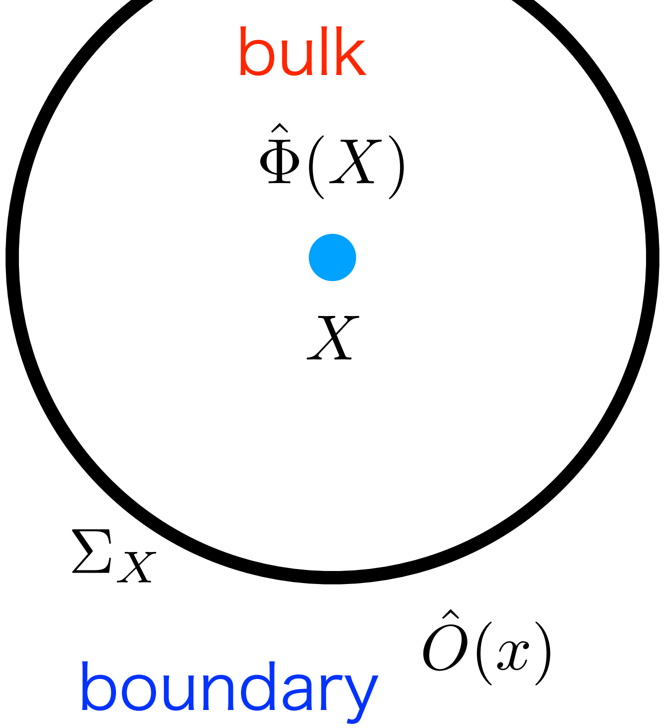
Gravity is constructed from Gauge theory (?) (Gauge/Gravity duality) .

Theme of this workshop

understand AdS/CFT form field theories without using string theory.

ex. HKLL bulk reconstruction Hamilton-Kabat-Lifschytz-Lowe 2006

free massive scalar field operator on AdS $\hat{\Phi}(X)$ with $m^2 = (\Delta - d)\Delta$



$$\lim_{z \to 0} z^{-\Delta} \hat{\Phi}(z, x) = \hat{O}(x)$$

BDHM relation

Banks-Douglas-Horowitz-Martinec 1998

CFT field operator at the boundary $\hat{O}(x)$

express $\Phi(X)$ in terms of $O(x)$

$$\hat{\Phi}(X) = \int_{\Sigma_X} d^d y \, K(X, y) \hat{O}(y)$$

c.f. S. Terashima, 2021: CFT<->AdS without BDHM relation in the large N.

An different point of view

How can we map properties of QFT to a "geometry" ?

In this talk, to answer this question, we exclusively consider a scalar CFT with Euclidean signature, which is quantized by the path-integral.

We will provide a prototype of AdS/CFT correspondence without using string duality (gauge/gravity correspondence). Therefore results may differ from usual understanding. We however hope that our attempt may provide deeper understanding of the standard AdS/CFT correspondence.

We can easily extend our method to a general scalar (non-conformal) QFT, which may provide an answer to the above question.

Unfortunately, it is not so easy to handle gauge theories in this approach analytically. A numerical methods may be needed for gauge/gravity duality in this approach.

Content of this talk

I. ~~Introduction - Our scope -~~

II. Holography, conformal flow and BDHM relation

III. Bulk (quantum) space and GKP-Witten relation

IV. Bulk geometry

V. Summary and Discussion

II. Holography, conformal flow
and
BDHM relation

Holography

Starting point We consider an O(N) scalar CFT in d dimensions.

A non-singlet primary field satisfies

$$\langle 0|\hat{\varphi}^a(x)\hat{\varphi}^b(y)|0\rangle = \delta^{ab}\frac{C_0}{|x-y|^{2\Delta}}$$

Smeared field We smear the field as

$$\hat{\sigma}^a(X) = \int d^d y\, h(z, x-y)\hat{\varphi}^a(y)$$

field in $d+1$ dimensions $X := (x, z)$

$h(z, x-y)$: smearing function

z is an extra direction, which corresponds to an energy scale of CFT.

$$z = 0 \text{ (UV) and } z = \infty \text{ (IR)}$$

QFT(CFT) in d-dimension + energy scale → d+1 dimensional bulk space

"Holography"

Smearing by flow equation

$$\hat{\sigma}^a(X) = \int d^d y\, h(z, x-y)\hat{\varphi}^a(y)$$

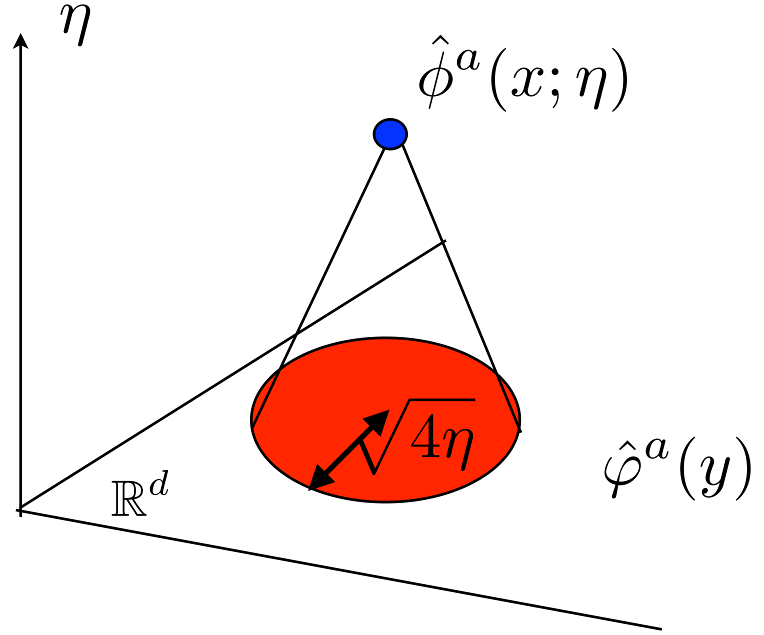
(1) We smear the field in CFT using the "flow" equation as

$$(-\alpha\eta\partial_\eta^2 + \beta\partial_\eta)\hat{\phi}^a(x;\eta) = \Box_x \hat{\phi}^a(x;\eta), \quad \hat{\phi}^a(x;0) = \hat{\varphi}^a(x)$$

This is a generalization of the "gradient flow".

Ex. $\alpha = 0$, $\beta = 1$ (Gaussian flow)

$$\hat{\phi}^a(x;\eta) = e^{\Box\eta}\hat{\varphi}^a(x) = \int d^d y\, g(\eta, x-y)\hat{\varphi}^a(y) \qquad g(\eta, x-y) := \frac{e^{-(x-y)^2/4\eta}}{(4\pi\eta)^{d/2}}$$



Heat kernel

"Gradient flow" was introduced to smooth UV fluctuations in lattice QCD.

Narayanan-Neuberger 2006, Luescher 2010

(2) We then normalized smeared field as

$$\hat{\sigma}^a(X) := \frac{\hat{\phi}^a(x;\eta)}{\sqrt{\langle 0|\hat{\phi}^2(x;\eta)|0\rangle}}$$

$|0\rangle$: CFT vacuum

$X := (x, z) \qquad \eta = \frac{\alpha}{4} z^2$

→ $\langle 0|\hat{\sigma}^2(X)|0\rangle = 1$

$\langle 0|\hat{\phi}^2(x;\eta)|0\rangle$ is well-defined without UV divergence.

Smeared field	smearing UV fluctuations	→ **define a kind of ``RG" transformation**
Normalization	field renormalization	

Sonoda-Suzuki (PTEP2019(2019)033B05) Gaussian flow = Exact RG

Conformal flow

$$(-\alpha\eta\partial_\eta^2 + \beta\partial_\eta)\hat{\phi}^a(x;\eta) = \Box_x \hat{\phi}^a(x;\eta), \quad \hat{\phi}^a(x;0) = \hat{\varphi}^a(x)$$

An “optimal” choice is given by $\nu := 1 + \frac{\beta}{\alpha} = \frac{d}{2} - \Delta$

With this choice, a conformal transformation on $\hat{\varphi}^a(x)$ generates a coordinate transformation on $\hat{\sigma}^a(X)$ as

$$\delta^{\rm conf}\hat{\varphi}^a(x) = -\delta x^\mu \partial_\mu \hat{\varphi}^a(x) - \frac{\Delta}{d}(\partial_\mu \delta x^\mu)\hat{\varphi}^a(x)$$

$$\downarrow$$

$$\delta x^\mu := a^\mu + w^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b\cdot x)$$

Poincare scale special conformal

$$\delta^{\rm conf}\hat{\sigma}^a(X) = -\delta X^A \partial_A \hat{\sigma}^a(x) \qquad \delta X^\mu := \delta x^\mu + z^2 b^\mu \qquad \delta X^{d+1} := (\lambda - 2b\cdot x)z$$

This coordinate transformation is nothing but the AdS isometry, and we call it conformal(-AdS) flow.

Aoki-Balog-Onogi-Yokoyama, PTEP **2023**(2023) 013B03.

(a seed of) “AdS/CFT correspondence”

BDHM relation

The kernel of the conformal flow is given by

$$h(z,x) = \Sigma_0 \left(\frac{z}{x^2 + z^2} \right)^{d-\Delta}, \qquad \Delta < \frac{d}{2}$$

The condition $\Delta < \frac{d}{2}$ is needed to satisfy $\hat{\phi}^a(x,0) = \hat{\varphi}^a(x)$.

The kernel $h(z, x-y)$ agree with $K(X,y)$ of HKLL but for $\Delta > d-1$.

HKLL, PRD74(2006)066009

$\hat{\sigma}^a(X) = \int d^d y\, h(z, x-y) \hat{\varphi}^a(y)$ satisfies EOM of a free scalar field on AdS as

$$(\Box_{\text{AdS}} - m^2)\hat{\sigma}^a(X) = 0 \qquad \text{where } m^2 = (\Delta - d)\Delta < 0.$$

Furthermore $\lim_{z\to 0} h(z,x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ ⇒ $\lim_{z\to 0} h(z,x) = \frac{\Sigma_0}{\Lambda} z^{\Delta} \delta^{(d)}(x).$

∴ $$\lim_{z\to 0} z^{-\Delta} \hat{\sigma}(X) = \lim_{z\to 0} \int d^d y\, z^{-\Delta} h(z, x-y) \hat{\varphi}^a(y) = \frac{\Sigma_0}{\Lambda} \hat{\varphi}^a(x)$$ **BDHM relation**

BDHM relation is correctly reproduced.

III. Bulk (quantum) space
and
GKP-Witten relation

Symmetry

boundary → bulk

$$U\varphi^a(y)U^\dagger := \tilde{\varphi}^a(y) = J(y)^\Delta \varphi^a(\tilde{y}) \longrightarrow U\sigma^a(X)U^\dagger := \tilde{\sigma}^a(X) = \sigma^a(\tilde{X})$$

conformal transformation → **coordinate transformation**

1. translation

$$\tilde{y}^\mu = y^\mu + a^\mu,\ \ J(y) = 1 \qquad \tilde{x}^\mu = x^\mu + a^\mu,\ \ \tilde{z} = z$$

2. rotation

$$\tilde{y} = \Omega^\mu{}_\nu y^\nu,\ \ J(y) = 1 \qquad \tilde{x}^\mu = \Omega^\mu{}_\nu x^\mu,\ \ \tilde{z} = z$$

3. dilatation

$$\tilde{y}^\mu = \lambda y^\mu,\ \ J(y) = \lambda \qquad \tilde{X}^A = \lambda X^A$$

4. inversion

$$\tilde{y}^\mu = \frac{y^\mu}{y^2},\ \ J(y) = \frac{1}{y^2} \qquad \tilde{X}^A = \frac{X^A}{X^2}$$

$$\mathrm{SO}(d+1,1) \qquad \mathrm{SO}(d+1,1)$$

AdS isometry

(Quantum) bulk space

Operators in the boundary and the bulk enjoy these symmetries as

$$\langle 0| \prod_{i=1}^{m} G^i_{A^i_1\cdots A^i_{n_i}}(\tilde{X}_i) \prod_{j=1}^{s} O^j_{\mu^j_1\cdots\mu^j_{l_j}}(\tilde{y}_j)|0\rangle = \prod_{i=1}^{m} \frac{\partial X_i^{B^i_1}}{\partial \tilde{X}_i^{A^i_1}} \cdots \frac{\partial X_i^{B^i_{n_i}}}{\partial \tilde{X}_i^{A^i_{n_i}}} \prod_{j=1}^{s} J(y_j)^{-\Delta_j} \frac{\partial y_j^{\nu^j_1}}{\partial \tilde{y}_j^{\nu^j_1}} \cdots \frac{\partial y_j^{\nu^j_{l_j}}}{\partial \tilde{y}_j^{\nu^j_{l_j}}}$$

$$\times \langle 0| \prod_{i=1}^{m} G^i_{B^i_1\cdots B^i_{n_i}}(X_i) \prod_{j=1}^{s} O^j_{\nu^j_1\cdots\nu^j_{l_j}}(y_j)|0\rangle$$

bulk operator (with an arbitrary spin) $G^i_{A^i_1\cdots A^i_{n_i}}(X_i)$ $X \to \tilde{X}$

coordinate transformation

boundary operator (with an arbitrary spin) $O^j_{\mu^j_1\cdots\mu^j_{l_j}}(y_j)$ $y \to \tilde{y}$

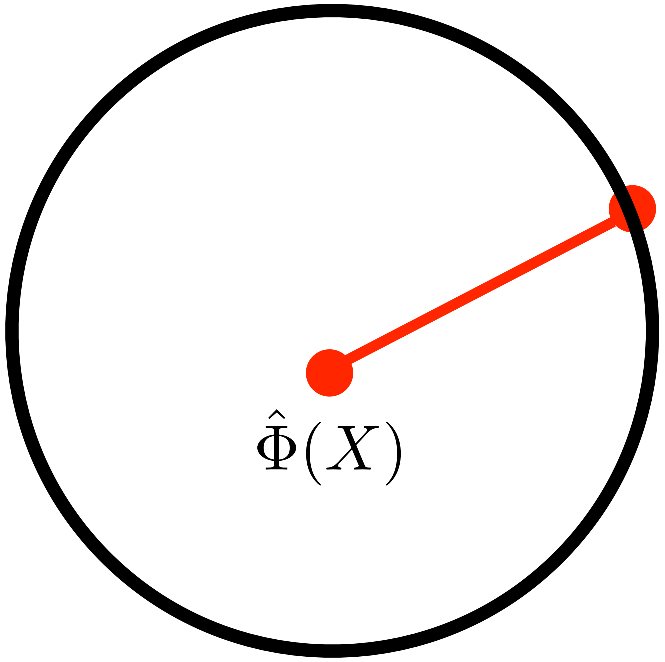
conformal transformation

Correlation functions including all quantum corrections are defined in the bulk.

(Quantum) bulk space

This is different from the standard AdS/CFT correspondence, where the bulk is classical in the large N limit.

Bulk to boundary propagator



$$F(X,y) := \langle 0|\hat\Phi(X)\hat S(y)|0\rangle_c$$

a singlet scalar primary field of weight Δ_S: $\hat S(y)$

a singlet bulk scalar field: $S(X) := \hat\sigma^a(X)\hat\sigma^a(X)$ composite

exact result

Symmetries(bulk & boundary) ⟶ $F(X,y) = C_S\left(\frac{z}{(x-y)^2+z^2}\right)^{\Delta_S}$

unknown (non-perturbative) constant

dilatation $F(\lambda X,\lambda y) = \lambda^{-\Delta_S} F(X,y)$

inversion $(x-y)^2+z^2 \to \frac{(x-y)^2+z^2}{X^2y^2}$ ⟶ $F(X,y) \to \left(\frac{1}{y^2}\right)^{-\Delta_S} F(X,y)$

$$J(y) = \frac{1}{y^2}$$

In addition, $F(X,y)$ satisfies a "free" KG equation on AdS as

$$\left(\Box^X_{\rm AdS} - m_S^2\right) F(X,y) = 0, \qquad m_S^2 := (\Delta_S - d)\Delta_S$$

GKP-Witten relation

Gubser-Klebanov-Polyakov 1998, Witten 1998

A free scalar on AdS with $m^2 = (\Delta_S - d)\Delta_S$ in the presence of a source

$$\Phi_J(X) \xrightarrow{z\to 0} z^{d-\Delta_S} J(x) + z^{\Delta_S} \langle 0|S(x)|0\rangle_J + \cdots$$

GKP-Witten relation

$J(x)$: source of the boundary scalar $S(x)$ with a conformal weight Δ_S

In our setup, let us consider

$$\Phi_J(X) := \langle 0| : \hat{\Phi}(X) : \exp\left[\int d^d y\, J(y)\hat{S}(y)\right] |0\rangle$$

At the leading order of small J, we obtain

$$F(X,y) := \langle 0|\hat{\Phi}(X)\hat{S}(y)|0\rangle_c$$

$$\Phi_J(X) = \int d^d y\, F(X,y) J(y) + O(J^2)$$

Since

$$C_S \left(\frac{z}{(x-y)^2 + z^2}\right)^{\Delta_S} \overset{z\to 0}{\simeq} \hat{C}_S z^{d-\Delta_S} \delta^{(d)}(x-y) + C_S \frac{z^{\Delta_S}}{|x-y|^{2\Delta_S}}$$

we obtain

$$\Phi_J(X) \overset{z\to 0}{\simeq} \hat{C}_S z^{d-\Delta_S} J(X) + C_S z^{\Delta_S} \int d^d y\, \frac{J(y)}{|x-y|^{2\Delta_S}}$$

$$= \int d^d y\, J(y) \langle 0|\hat{S}(x)\hat{S}(y)|0\rangle \simeq \langle 0|\hat{S}(x) \exp\left[\int d^d y\, J(y)\hat{S}(y)\right] |0\rangle$$

$$\langle 0|\hat{S}(x)|0\rangle = 0 \text{ for CFT}$$

$$\Phi_J(X) \overset{z\to 0}{\simeq} \hat{C}_S z^{d-\Delta_S} J(X) + C_S z^{\Delta_S} \langle 0|\hat{S}(x)|0\rangle_J + O(J^2)$$

GKP-Witten relation is reproduced at small J .

IV. Bulk geometry

Geometry ?

What we obtain so far is

CFT + energy scale via conformal flow ⟶ bulk (quantum) space

but we do not define "geometry" up to now.

How can we determine a geometric structure of the bulk space ?

Possibilities

- a geometry which makes the (scalar) propagator solution to a free Klein-Gordon equation.
- a geometry which makes the (boundary) entanglement entropy equal to the minimal surface in the bulk. (Ryu-Takayanmagi)
- others

They are rather complicated. We instead consider a more direct method.

We determine a bulk geometry using Bures information metric.

Bures Information metric

define a distance between a density matrix ρ and $\rho + d\rho$ (ρ can be mixed state) as

$$d^2(\rho, \rho + d\rho) := \frac{1}{2} \mathrm{Tr} \left(d\rho \hat{G} \right)$$ **Bures metric** where $\hat{G}$ satisfies $$\rho \hat{G} + \hat{G} \rho = d\rho$$

Our case

State dependent density matrix $|S\rangle$: some state in CFT

$$\rho_S(X) := \sum_{a=1}^{N} \hat{\sigma}_S^a(X) |S\rangle\langle S| \hat{\sigma}_S^a(X)$$

N entangled pairs (mixed state)

$$\mathrm{tr}\, \rho_S(X) = 1 \longrightarrow \langle S | \hat{\sigma}_S^2(X) | S\rangle = 1$$

$$\longrightarrow \hat{\sigma}_S^a(X) := \frac{\hat{\sigma}^a(X)}{\sqrt{\langle S|\hat{\sigma}^2(X)|S\rangle}} = \frac{\hat{\phi}^a(x;\eta)}{\sqrt{\langle S|\hat{\phi}^2(x;\eta)|S\rangle}}$$

normalized for the state $|S\rangle$

$O(N)$ symmetry $\longrightarrow$ $$\langle S|\hat{\sigma}_S^a(X)\hat{\sigma}_S^b(X)|S\rangle = \frac{1}{N}\delta^{ab}$$

Bures information metric for $\rho_S(X)$

$$\rho_S^2(X) = \hat{\sigma}_S^a(X)|S\rangle\langle S|\hat{\sigma}_S^a(X)\hat{\sigma}_S^b(X)|S\rangle\langle S|\hat{\sigma}_S^b(X) = \frac{1}{N}\rho_S(X)$$

$\longrightarrow$ $\rho_S N d\rho_S + N d\rho_S \rho_S = d\rho_S$ $\longrightarrow$ $\hat{G} = N d\rho_S(X) = N dX^A \partial_A \rho_S(X)$

$$\rho\hat{G} + \hat{G}\rho = d\rho$$

$\longrightarrow$ $$\frac{1}{2}\mathrm{Tr}\left[d\rho_S(X)\hat{G}\right] = \frac{N}{2}\mathrm{Tr}\left[\partial_A\rho_S(X)\partial_B\rho_S(X)\right]dX^A dX^B$$

We then define the line element as $ds^2 := \frac{2\ell^2}{N}d^2(\rho_S, \rho_S + d\rho_S)$ $\quad \ell$: some length scale

➡ **Our metric** $$g_{AB}^S(X) := \ell^2 \sum_{a=1}^{N}\langle S|\partial_A\hat{\sigma}_S^a(X)\partial_B\hat{\sigma}_S^a(X)|S\rangle$$

The metric (geometry) is (CFT) state dependent.

If we introduce the metric operator as $\hat{g}_{AB}^S(X) := \ell^2\partial_A\hat{\sigma}_S^a(X)\partial_B\hat{\sigma}_S^a(X)$

$\longrightarrow$ $$g_{AB}^S(X) = \langle S|\hat{g}_{AB}^S(X)|S\rangle$$

Some remarks

1. The metric g^S_{AB} is finite at $z > 0$, thanks to the smearing.

On the other hand, we can not define the metric directly in CFT as $\langle S | \partial_\mu \hat{\varphi}^a \partial_\nu \hat{\varphi}^b | S \rangle$ due to UV divergence.

2. The metric operator $\hat{g}^S_{AB}$ becomes classical in the large N limit as

$$\langle S|\hat{g}^S_{AB}(X)\hat{g}^S_{CD}(Y)|S\rangle = g^S_{AB}(X)g^S_{CD}(Y) + O(1/N)$$

large N factorization

$\Rightarrow$

$$\langle S|R_{AB}(\hat{g}^S_{CD})|S\rangle = R_{AB}(g^S_{CD}) + O(1/N)$$

Ricci tensor

The classical geometry emerges after quantum averaging.

Vacuum metric

Bulk geometry is state dependent. There is no unique geometry for bulk (quantum) space.

Vacuum case $|0\rangle$ $$\hat{\sigma}_0^a(X) = \hat{\sigma}^a(X) = \frac{\hat{\phi}^a(x;\eta)}{\sqrt{\langle 0|\hat{\phi}^2(x;\eta)|0\rangle}}$$

$\longrightarrow$ $$g_{AB}^{\rm vac}(X) = \ell^2 \frac{\Delta(d-\Delta)}{d+1}\frac{\delta_{AB}}{z^2}$$ AdS metric in the Poincare coordinate

A vacuum state in an arbitrary CFT for a primary with $\Delta < \frac{d}{2}$ $\longrightarrow$ AdS metric

AdS/CFT correspondence

Even if we use Gaussian flow, this correspondence emerges for an arbitrary Δ .

Aoki-Yokoyama, PTEP **2018**(2018) 031B01

What is a metric for excited states ?

Metric for excited states

Finite temperature (Thermo field double state)

$$|\mathrm{TFD}\rangle := \frac{1}{\sqrt{Z_T}} \sum_n e^{-E_n/2T} |E_n\rangle \otimes |\widetilde{E_n}\rangle \qquad Z_T := \sum_n e^{-E_n/T}$$

If the $O(N)$ model is free, the metric becomes the asymptotic AdS.

In the UV region ($z \to 0$), the metric is a classical solution to $f(R)$ gravity.

Aoki-Shimada-Balog-Kawana, PRD109(2024) 4, 04606. e-Print:2308.01076

Scalar state

$|S\rangle = |\Phi\rangle$: scalar state with a conformal dimension Δ_Φ

In general, the metric becomes the asymptotic AdS.

In the case of free $O(N)$ model with $\Delta_\phi = 2\Delta$, we have

$$g^{\Phi}_{AB}(X) = g^{\mathrm{vac}}_{AB}(X) + \frac{1}{N}\delta g_{AB}(X) + O(1/N^2)$$

Asymptotic AdS by $1/N$ correction

Aoki-Balog-Shimada, work in progress

IV. Summary and discussion

Summary

Bulk (quantum) space

CFT $\hat{\varphi}^a(y), \quad \Delta < \frac{d}{2}$ non-singlet, elementary operator conformal symmetry

↓ conformal flow $\hat{\sigma}^a(X) = \int d^d y\, h(z, x-y)\hat{\varphi}^a(y)$ ↓

$\hat{\sigma}^a(X)$ bulk non-singlet, elementary (“quarks”) bulk “quantum” symmetry

BDHM relation $\hat{\sigma}^a(X) \stackrel{z\to 0}{\longrightarrow} z^{\Delta}\hat{\varphi}^a(x)$

bulk singlet composite operators (“hadrons”)

Ex. composite scalar $\hat{\Phi}(X) := \hat{\sigma}^a(X)\hat{\sigma}^a(X)$

correlation functions among composite operators are controlled by symmetries.

$\langle 0|\hat{\Phi}(X_1)\hat{\Phi}(X_2)\cdots S(y_1)S(y_2)\cdots|0\rangle$ bulk (quantum) space

⟶ GKP-Witten relation

(state-dependent) geometry

state dependent Bures metric ← state dependent normalization

$$g^S_{AB}(X) := \ell^2 \sum_{a=1}^{N} \langle S|\partial_A \hat{\sigma}^a_S(X) \partial_B \hat{\sigma}^a_S(X)|S\rangle$$

$$\langle S|\hat{\sigma}^2_S(X)|S\rangle = 1$$

metric becomes classical in the large N limit → "geometry"

vacuum state → AdS   bulk symmetry = isometry

"AdS/CFT correspondence"

excited states → asymptotic AdS

bulk "quantum" symmetry controls properties in the bulk.

Discussions

1. State dependent geometry

Is a concept of the state-dependent geometry compatible with the conventional AdS/CFT correspondence ?

2. Conformal flow vs. Gaussian flow

$\Delta < \frac{d}{2}$ for the conformal flow, while $^\forall\Delta$ for the Gaussian flow.

A choice of the metric operator is limited for the conformal flow. Good (?)

3. unknown bulk ?

The flow method may be useful to guess the unknown bulk geometry from a given field theory.

4. Geometry -> Gravity

How can we derive gravitational interactions in this approach ?

5. Extension to gauge theory

Can we apply the conformal flow to gauge theories ?

$$(-\alpha\eta\partial_\eta^2 + \beta\partial_\eta)\hat{\phi}(x;\eta) = D^\mu D_\mu \hat{\phi}(x;\eta)$$

Gauge/Gravity duality ?

Due to interactions in D_μ, we can not solve the flow equation analytically.

Numerical evaluation is required. Lattice gauge theories ?

Your ideas/attempts are very welcome.

6. de Sitter ?

de Sitter space is constructed from non-unitary CFT by Gaussian flow.

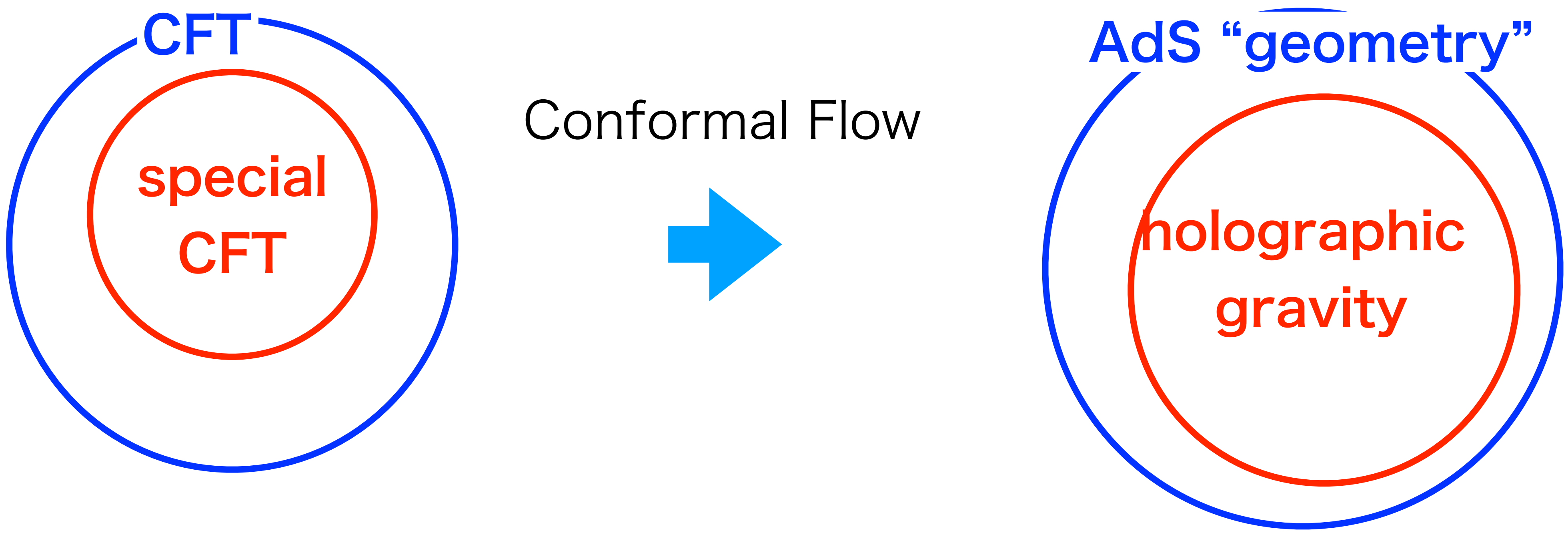
S. Yokoyama, PTEP2020(2020)10,103B05

Thank you for your attention.

Back up

Possibilities

1. Conformal flow method contains the standard AdS/CFT.



2. Conformal flow method has nothing to do with the standard AdS/CFT.