Quantum algorithms from algebraic Hilbert spaces

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Title : : Quantum algorithms from algebraic Hilbert spaces.

Abstract: The structure of composite operators in gauge quantum field theories with matrix or tensor degrees of freedom is controlled by hidden symmetries which organise the combinatorics of gauge invariants. These include group algebras of symmetric groups and associated natural generalisations. Dualities in string theory, in particular gauge-string duality, motivate the formulation of new classical and quantum algorithms based on structural properties of these algebras. I will describe an interesting number sequence $k_*(n)$ associated with symmetric groups on n elements, which plays an important role in these structural properties and determines the complexities of the associated quantum algorithms. The talk will be based on https://arxiv.org/abs/1911.11649 and https://arxiv.org/abs/2303.12154

Based on

[1] Garry Kemp, Sanjaye Ramgoolam, " **BPS states, conserved charges and centres of symmetric group algebras** " arXiv[2303.12154], JHEP

[2] Joseph Ben Geloun, Sanjaye Ramgoolam, **"Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients,"** Algebraic Combinatorics, Volume 6 (2023) no. 2, pp. 547-594 ; arXiv:2010.04054v3 [hep-th] for this version)

[3] Joseph Ben Geloun, Sanjaye Ramgoolam, " **The quantum detection of projectors in finite-dimensional algebras and holography**" arXiv[2303.12154]

Introduction : Algebraic Hilbert spaces

Finite group G. A finite set of elements $\{g_1, \cdots, g_{|G|}\}$.
Associative product, identity, inverse. $|G|$ is the order of G.

The group algebra $\mathbb{C}(G)$:

$$a = \sum_{i=1}^{|G|} a_i g_i$$

The $a_i \in \mathbb{C}$.

This is a complex vector space of dimension $|G|$.

It is also equipped with a product. Example - $|G| = 2$:

$$\begin{aligned} a &= a_1 g_1 + a_2 g_2 \\ b &= b_1 g_1 + b_2 g_2 \\ a.b &= (a_1 g_1 + a_2 g_2).(b_1 g_1 + b_2 g_2) \\ &= a_1 b_1 (g_1 . g_1) + (a_1 b_2)(g_1 . g_2) + (a_2 b_1)(g_2 . g_1) + a_2 b_2 (g_2 . g_2) \end{aligned}$$

Introduction : The group algebra $\mathbb{C}(G)$

$\mathbb{C}(G)$ is an associative algebra - i.e. a vector space with an associative product - with a unit, the identity element g_0 with coefficient 1.

There is a map $\mathbb{C}(G) \to \mathbb{C}$

$$\begin{aligned} \delta(g) &= 1 \quad \text{if} \quad g = g_0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

This is the character of the regular representation

$$tr(D^{\text{reg}}(g)) = |G|\delta(g)$$

We can define an inner product on $\mathbb{C}(G)$ by using the delta function and extending by regularity

$$\langle \sum_i a_i g_i, \sum_j b_j g_j \rangle = \sum_{i,j} a_i^* b_j \delta(g_i g_j^{-1}) = \sum_i a_i b_i^*$$

This makes $\mathbb{C}(G)$ a Hilbert space, which is also an unital associative algebra.

Introduction : The centre of the group algebra : $\mathcal{Z}(\mathbb{C}(G))$

The centre $\mathcal{Z}(\mathbb{C}(G))$ of the group algebra is the subspace which commutes with all G, equivalently with all $\mathbb{C}(G)$.

$$\begin{aligned} &\text{Dimension of } \mathcal{Z}(\mathbb{C}(G)) \\ = \; &\text{Number of conjugacy classes of } G \\ = \; &\text{Number of irreducible representations of } G \end{aligned}$$

Bases in the centre of the group algebra : $\mathcal{Z}(\mathbb{C}(G))$

There is a basis labelled by conjugacy classes. Let $(\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_K)$ be the conjugacy classes. Within each conjugacy class, any pair of elements g_1, g_2 are related by $g_1 = g g_2 g^{-1}$, for some $g \in G$.

$$T_{\mathcal{C}_i} = \sum_{g \in \mathcal{C}_i} g$$

$T_{\mathcal{C}_i}$ is evidently in $\mathbb{C}(G)$. One checks it is in the centre $\mathcal{Z}(\mathbb{C}(G))$, i.e.

$$hT_{\mathcal{C}_i} = T_{\mathcal{C}_i} h \quad \text{for all} \quad h \in G$$

Basis labelled by irreps R

Let R label the irreducible reps.

$$D^R(g) : V_R \to V_R$$

$$\chi^R(g) = \mathrm{tr}_{V_R}(D^R(g))$$

Projectors labelled by irreps R

$$P_R = \frac{d_R}{|G|} \sum_g \chi^R(g) \;\; g^{-1}$$

form a basis for $\mathcal{Z}(\mathbb{C}(G)$, where $d_R =$ Dimension V_R.

Basis labelled by irreps R

$$P_R = \frac{d_R}{|G|} \sum_{i=1}^{K} \chi^R_{g_i^{-1}} T_{\mathcal{C}_i}$$

g_i is a group element in class $\mathcal{C}_i$.

This is a change of basis between the $T_{\mathcal{C}_i}$ and the P_R given by the $K \times K$ matrix $\chi^R_{\mathcal{C}_i}$.

Symmetric groups S_n

$$\begin{aligned} \text{Dimension of } \mathbb{C}(S_n) &= n! \\ \text{Dimension of } \mathcal{Z}(\mathbb{C}(S_n)) &= \text{ Number of partitions of } \; n = p(n) \end{aligned}$$

Asymptotics

$$\begin{aligned} n! &\sim n^n e^{-n} \\ p(n) &\sim e^{\sqrt{n}} \end{aligned}$$

First main result: Quantum algorithms of complexity polynomial in n

Algorithms motivated by AdS/CFT.
The CFT is four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory, with $U(N)$ gauge group. Among the matter fields, a complex matrix quantum field Z of size $N \times N$, playing an important role in the maximally super-symmetric (half-BPS) sector of the theory.
Classifying gauge-invariant composite local operators made from Z is usefully done with the help $\mathbb{C}(S_n)$ representation theory.

Background: Connection between $U(N)$ and $\bigoplus_n \mathbb{C}(S_n)$

The key to the connection is that irreps of $U(N)$, can be constructed from the fundamental V_N by taking tensor products and decomposing into irreps. The irreps of $U(N)$ appearing in $V_N^{\otimes n}$ come from different symmetry types of tensors, labelled by Young diagrams R having n boxes.

The precise relation is given by Schur-Weyl duality

$$V_N^{\otimes n} = \bigoplus_{\substack{R \vdash n \\ l(R) \leq N}} V_R^{U(N)} \otimes V_R^{S_n}$$

Quantum algorithms motivated by ADS/CFT

There are composite operators made from Z, closely related to the projectors P_R, which are dual to large semi-classical objects in the ADS dual of the CFT.

AdS physics motivates the question of "identification of the semi-classical object" using small probes.

This leads to the definition of a "quantum detection of projectors" in the algebraic set-up.

Other Algebraic Hilbert spaces :$\mathbb{C}_H(G)$

$\mathbb{C}_H(G)$: Sub-algebra of $\mathbb{C}(G)$ which commutes with all the elements $h \in H$ for a sub-group $H \subset G$.

$$a = \sum_{i=1}^{K} a_i g_i$$

$$ha = ah \text{ for all } h \in H \subset G$$

Multi-matrix composites in CFT/ADS : $\mathbb{C}_{S_m \times S_n}(S_{m+n})$

Take S_{m+n} to be the permutations of $\{1, \cdots, m, m+1, \cdots, m+n\}$.

It has a sub-group S_m which permutes $\{1, \cdots, m\}$ leaving fixed $\{m+1, \cdots, m+n\}$.

Also a sub-group S_n which permutes $\{m+1, \cdots, m+n\}$ while leaving fixed $\{1, \cdots, m\}$.

The subgroup $S_m \times S_n$ mixes the first m and the last n but no mixing between the two subsets.

Multi-matrix composites in CFT/ADS : $\mathbb{C}_{S_m \times S_n}(S_{m+n})$

There are similar - "quantum detection of projectors" - algorithms with complexity polynomial in m, n, despite the exponentially growing orders of the groups involved.

This is of interest in connection with complexity of Little-wood-Richardson coefficients.

Composites of 3-index tensors : $\mathbb{C}_{S_n}(S_n \times S_n)$

Composites of tensor fields Φ_{ijk} transforming in the fundamental $V_N \otimes V_N \otimes V_N$, along with their conjugates $\bar{\Phi}_{ijk}$, lead to the algebra $\mathbb{C}_{S_n}(S_n \times S_n)$.

These algebras are related to Kronecker coefficients.

And there are corresponding polynomial "quantum detection of projectors" algorithms.

OUTLINE OF TALK

PART 1 : A sequence $k_*(n)$ related to character tables of S_n .. and an eigenvalue problem in $\mathcal{Z}(\mathbb{C}(S_n))$.

PART 2 : 1-matrix gauge invariant composites and AdS/CFT.

PART 3: Quantum Projector detection in $\mathcal{Z}(\mathbb{C}(S_n))$ and its complexities.

Part 4 : The AdS dual and a holographically dual algorithm.

Part 5 : Generalisations - Littlewood-Richardson and Kronecker coefficients.

PART 1 : Conjugacy classes of S_n - cycle structures.

S_3 : permutations of $\{1, 2, 3, \}$.

$$(1)(2)(3)...[1, 1, 1]$$
$$(1, 2)(3), (1, 3)(2), (2, 3)(1)....[2, 1]$$
$$(1, 2, 3), (1, 3, 2)...[3]$$

PART 1 : Distinguishing irreps with characters

In [2]:
```
## defining s for Schur polynomials
s = SymmetricFunctions(QQ).schur()
```

In [3]:
```
## defining p for power sum symmetric functions ; multi-traces
p = SymmetricFunctions(QQ).power()
```

In [13]:
```
# CHARACTER TABLE FOR S3
[[ s(r).scalar(p(q))  for q in Partitions (3) ] for r in Partitions (3)]
```

Out[13]:
```
[[1, 1, 1], [-1, 0, 2], [1, -1, 1]]
```

In [39]:
```
# character^R ([2,1])   f0r conjugacy class [2,1] for all irreps R of S3
q2 = Partition ( [ 2,1])
[ s(r).scalar(p(q2))    for r in Partitions (3)]
```

Out[39]:
```
[1, 0, -1]
```

In [41]:
```
#  character^R ([2,1])/dimension(r)  conjugacy class [2,1] for all irreps R of S3
q2 = Partition ( [ 2,1])
[ s(r).scalar(p(q2))/dimension(r)    for r in Partitions (3)]
```

Out[41]:
```
[1, 0, -1]
```

In [42]:
```
## character^R ([2,1])   f0r conjugacy class [2,1] for all irreps R of S4
q2 = Partition ( [ 2,1,1])
[ s(r).scalar(p(q2))   for r in Partitions (4)]
```

Out[42]:
```
[1, 1, 0, -1, -1]
```

In [43]:
```
### character^R ([2,1]) / dim R   for conjugacy class [2,1] for all irreps R of S4
q2 = Partition ( [ 2,1,1])
[ s(r).scalar(p(q2))/dimension(r)    for r in Partitions (4)]
```

Out[43]:
```
[1, 1/3, 0, -1/3, -1]
```

PART 1 : Distinguishing irreps with characters

In [44]:
```
### character^R ([2,1]) / dim R   for conjugacy class [2,1] for all irreps R of S5
q2 = Partition ( [ 2,1,1,1])
[ s(r).scalar(p(q2))/dimension(r)    for r in Partitions (5)]
```

Out[44]:
```
[1, 1/2, 1/5, 0, -1/5, -1/2, -1]
```

In [22]:
```
q2 = Partition ( [ 2,1,1,1,1])
[ s(r).scalar(p(q2))/dimension(r)    for r in Partitions (6)]
```

Out[22]:
```
[1, 3/5, 1/3, 1/5, 1/5, 0, -1/5, -1/5, -1/3, -3/5, -1]
```

In [45]:
```
[ [ s(r).scalar(p(q))/dimension(r)  for q in { Partition([2,1,1,1,1]), Partition([
```

Out[45]:
```
[[1, 1],
 [3/5, 2/5],
 [1/3, 0],
 [1/5, 1/10],
 [1/5, -1/5],
 [0, -1/8],
 [-1/5, 1/10],
 [-1/5, -1/5],
 [-1/3, 0],
 [-3/5, 2/5],
 [-1, 1]]
```

PART 1 : Distingushing irreps with characters

test-tables - Jupyter Notebook

In [46]:
```
q2 = Partition ( [ 2,1,1,1,1,1])
[ s(r).scalar(p(q2))/dimension(r)    for r in Partitions (7)]
```

Out[46]: [1, 2/3, 3/7, 1/3, 2/7, 1/7, 0, 1/21, -1/21, -1/7, -1/3, -2/7, -3/7, -2/3, -1]

In [ ]:

PART 1 : Normalised characters as eigenvalues

Recall the conjugacy-class-labelled elements in $\mathcal{Z}(\mathbb{C}(G))$.

Combinations of this form

$$T_p = \frac{1}{|\mathcal{C}_p|} \sum_i \sigma_i^{(p)}$$

live in the centre of the group algebra, denoted $\mathcal{Z}(\mathbb{C}(S_n))$.

As p ranges over all partitions of n, the T_p form a basis for $\mathcal{Z}(\mathbb{C}(S_n))$.

Projector basis for $\mathcal{Z}(\mathbb{C}(S_n))$

The projector basis of $\mathbb{C}(S_n)$ is labelled by Young diagrams :

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma)\ \sigma = \frac{d_R}{n!} \sum_{p \vdash n} \chi^R(\sigma^{(p)})\ (T_p)$$

$$P_R P_S = \delta_{RS} P_S$$

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

P_R satisfy eigenvalue equations, with known simple eigenvalues.

$$T_p P_R = \frac{\chi^R(T_p)}{d_R} P_R$$

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

A subset of the T_p are interesting: when p corresponds to a partition associated with a single cycle of length k and remaining cycles of length 1, i.e. $p = [k, 1^{n-k}]$:

$$T_{p=[k,1^{n-k}]} \equiv T_k$$

The eigenvalues for T_k have nice expressions in terms of power sums of contents of boxes in Young diagrams.

Math papers by Lasalle and by Corteel, Goupille, Schaeffer

NUmber of $T_k = n$.

The eigenvalues of the set of T_k, for $k = 1 \cdots n$, uniquely specify the P_R.

Kemp, Ramgoolam, "BPS states in $N = 4$ SYM theory and centres of symmetric group algebras" JHEP

Part 1 : Detecting P_R in $\mathcal{Z}(\mathbb{C}(S_n))$

In fact, for any given finite n, we only need a small number of T_k to distinguish the P_R

E.g. for $n \in \{2, 3, 4, 5, 7\}$ it suffices to know the eigenvalue of T_2, i.e. $\frac{\chi^R(T_2)}{d_R}$.

With $\{T_2, T_3\}$ we can distinguish all P_R for n up to 14.

For general n, there is some $k_*(n)$, such that

$$\{T_2, T_3, \cdots, T_{k_*(n)}\}$$

uniquely specify the R. Such subsets non-linearly generate the centre $\mathcal{Z}(\mathbb{C}(S_n))$.
We computed this for n up to 80 .. e.g. $k_*(n = 80) = 6.$!

Kemp, Ramgoolam, "BPS states in $N = 4$ SYM theory and centres of symmetric group algebras" JHEP-2020

PART 2 : AdS/CFT and composite operators made from matrix fields

String theory on $AdS_5 \times S^5$ geometry and its fluctuations are captured by $\mathcal{N} = 4$ SYM with $U(N)$ gauge group.

CFT operators : Half-BPS

General half-BPS gauge invariant operators of dimension n can be parameterised by using permutations $\sigma \in S_n$:

$$\mathcal{O}_\sigma(Z) = \sum_{i_1,\cdots,i_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

Different choices of permutation σ give rise to different trace structures.

For example, for $n = 3$,

$$\sigma = (1,2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} = \mathrm{tr}(Z^3)$$

$$\sigma = (1,2)(3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} = \mathrm{tr}(Z^2)\mathrm{tr}(Z)$$

$$\sigma = (1)(2)(3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$
$$\mathcal{O}_\sigma(Z) = \sum_{i_1,i_2,i_3} Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_3} = \mathrm{tr}(Z)\mathrm{tr}(Z)\mathrm{tr}(Z) = (\mathrm{tr}Z))^3$$

CFT operators :

For fixed $n < N$, the number of trace structures is equal to the number of partitions of n (more on general n shortly).

From the permutation parameterisation,

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z)$$

Operators associated to conjugate permutations are the same. Conjugacy classes in S_n are in 1-1 correspondence with cycle structures. These in turn correspond to trace structures :

$$(\mathrm{tr}(Z))^{p_1}(\mathrm{tr}(Z^2))^{p_2}\cdots(\mathrm{tr}(Z^n))^{p_n} = \mathcal{O}_{\sigma^{(p)}}(Z) = \frac{1}{|\mathcal{C}_p|}\sum_{i=1}^{|\mathcal{C}_p|}\mathcal{O}_{\sigma_i^{(p)}}(Z)$$

Projectors and Schur-basis of CFT operators

The projectors are directly related to the Schur-basis operators

$$\mathcal{O}_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma) \; \mathcal{O}_\sigma(Z)$$

which were defined and shown to have orthogonal 2-point functions in free-field $\mathcal{N} = 4$ SYM :

$$\langle \mathcal{O}_R(Z(x_1)) \mathcal{O}_S(Z^\dagger(x_2)) \rangle = \frac{\delta_{RS}}{|x_1 - x_2|^{2n}} f_R(N)$$

Corley, Jevicki, Ramgoolam (CJR2001), "Exact Correlators of Giant Gravitons from dual N=4 SYM", ATMP-2001,

https://arxiv.org/abs/hep-th/0111222

The orthogonality of projectors is directly related (diagrammatic) to the orthogonality of the two-point functions.

Corley, Ramgoolam, "Finite Factorization equations and Sum Rules for BPS correlators in N=4 SYM theory"

JHEP-2002, https://arxiv.org/abs/2201.12917

Young-diagram bases, finite N effects and AdS/CFT

In Young diagram bases for composite operators, the finite N effects are crisply captured by

$$l(R) \leq N$$

This makes the Young diagram bases useful for identifying semiclassical dual of large composite operators, since AdS has striking finite N physics, e.g. giant gravitons (Mc Greevy, Susskind, Toumbas, 2001).

Young diagrams and giants.

It was argued that the Young diagram basis allows the identification of operators dual to semi-classical giants which are large in the AdS as well as the sphere directions (CJR-2001).

There is an underlying free-fermion picture for this half-BPS sector which sheds further light on the dictionary between Young diagram operators and giants (CJR2001, Berenstein2004).

Berenstein “A toy model for the AdS/CFT correspondence” , https://arxiv.org/abs/hep-th/0403110

Extensive evidence in subsequent papers : David Berenstein, Robert de Mello Koch, SR, many others ... multi-matrix operators related to strings attached to giants ...

Many giants and LLM geometries

Giant gravitons are described in space-time in terms of solutions to 3-brane actions with embedding in $AdS_5 \times S^5$ bulk space-time. They correspond to operators of dimension $n \sim N$.

For $n \sim N^2$, the giant gravitons back-react and produce large deformations of space-time, with $AdS_5 \times S^5$ asymptotics. The general half-BPS super-gravity solutions were characterised by Lin, Lunin, Maldacena (2004) - LLM-2004.

A distinguished role is played by a 2-dimensional plane in space-time, which can be identified with a free-fermion phase space. Colourings of the plane by concentric black rings in a white background can be mapped to Young diagram row lengths and equivalently excitations of N free fermions in a 1D harmonic oscilliator potential, making contact with the free-fermion/Young-diagram connection from the matrix model of a complex scalar.

Part 3: $\mathcal{Z}(\mathbb{C}(S_n))$ as a Hilbert space

There is an inner product on $\mathbb{C}(S_n)$ where the group elements are defined to be orthonormal.

This induces an inner product on the central subspace $\mathcal{Z}(\mathbb{C}(S_n))$.

Multiplication by T_k defines hermitian operators. The eigenvalue problems above can be approached using standard quantum computation techniques.

Part 3: Quantum detection of projectors P_R

The task of identifying the projector P_R using the T_k eigenvalue equations lends itself to standard quantum algorithms - quantum phase estimation, which come with associated complexity estimates (query and gate complexity).

JBG-SR-2023 : J. Ben Geloun and S. Ramgoolam, "The quantum detection of projectors in finite-dimensional algebras and holography," JHEP-2023

This allows the exploitation of exponential improvements provided by quantum algorithms (compared to classical algorithms) for certain computational tasks in linear algebra, along the lines of

Harrow, Hassidim, Lloyd, "Quantum algorithm for solving linear systems of equations," PRL-2009, https://arxiv.org/abs/0811.3171

Part 3: Quantum detection of projectors P_R

The algorithm uses black box unitaries $U_k = e^{\frac{2\pi i}{\chi_k^{\text{max}}} T_k}$ and

$$U_k P_R = e^{\frac{2\pi i}{\chi_k^{\text{max}}} \widehat{\chi}^R(T_k)} P_R$$

χ_k^{max} is the maximum of the eigenvalues $\frac{\chi^R(T_k)}{d_R}$ as R ranges over Young diagrams with n boxes.

Quantum phase estimation involves applications of powers of U_k - assumed to be available as black boxes - to the initial state P_R, which is assumed to be given as a quantum state in the Hilbert space $\mathcal{Z}(\mathbb{C}(S_n))$.

Query complexity counts the number of uses of these black boxes in the quantum circuit. Known results for QPE are used from standard texts e.g. Nielsen and Chuang. Gate complexity counts the total number of other boxes in the circuit.

Part 3: Quantum detection of projectors P_R

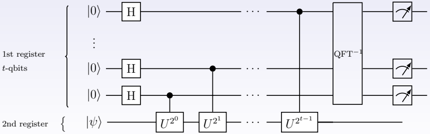

Figure 1: Quantum phase estimation by a quantum circuit acting on the initial state $|0\rangle^{\otimes t} \otimes |\psi\rangle$: H-boxes are Hadamard gates, U^{2^i}-boxes stand for CU-operators, $i = 0, \ldots, t-1$, QFT^{-1} for the inverse quantum Fourier transform, and the last stage involves a measurement on the first register.

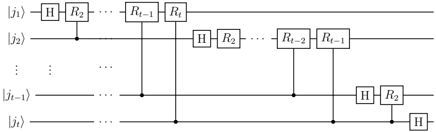

Figure 2: A circuit for quantum Fourier transform $|j_1 j_2 \ldots j_t\rangle$: H-boxes are Hadamard gates, R_k-boxes stand for C-R_k-operators, $k = 2, \ldots, t$.

Part 3: Quantum detection of projectors P_R

Standard result in QPE - for one unitary - gives query complexity $\mathcal{O}(t)$ and gate complexity $\mathcal{O}(t^2)$, where t is the number of bits needed to code the eigenvalues of interest.

S_n group theory input – about values of the normalised characters $\frac{\chi^R(T_k)}{d_R}$ - relates this to n.

In our application we have a range of unitaries $\{U_2, \cdots , U_{k_*(n)}\}$. We argued for a heuristic approximate lower bound $k_*(n) \gtrapprox n^{1/4}$ in the large n limit (JBG-SR-2023), and we assumed $k_*(n) \sim n^\alpha$ with $1/4 \leq \alpha < 1/2$.

$k_*(n) \leq n^{1/2}$ has been argued more recently in

Kemp, " A generalized dominance ordering for 1/2-BPS states" - https://arxiv.org/abs/2305.06768

Part 3: Quantum detection of projectors P_R

Based on the input from Quantum information, and the S_n rep theory estimates, we arrived at the complexity estimate for the detection of the projectors P_R :

Query complexity : $\mathcal{O}(n^{2\alpha} \log n)$

Gate complexity : $\mathcal{O}(n^{3\alpha} (\log n)^2)$

These are both bounded by $n^{3/2+\epsilon}$ with $\epsilon > 0$.

Hence the quantum projector detection is polynomially bounded in n - although $p(n) \sim e^{\sqrt{n}}$ and $n! \sim e^{n \log n - n}$.

Part 4 : Detecting LLM geometries

Lin-Lunin-Maldacena (2004) classified the half-BPS super-gravity solutions with $AdS_5 \times S^5$ asymptotics, which take the form:

$$ds^2 = -h^{-2}(dt + \sum_{i=1}^{2} V_i dx_i)^2 + h^2(dy^2 + \sum_{i=1}^{2} dx_i dx_i) + R^2 d\Omega^2 + R^2 d\tilde{\Omega}^2$$

The functions V_1, V_2, h, R appearing above are all functions of (x_1, x_2, y), and are all determined by one function $u(x_1, x_2, y)$. The function obeys a harmonic equation in y and is determined by its value on the $y = 0$ plane.

The function $u(x_1, x_2)$ on the LLM plane is determined by using a Wigner phase space distribution associated to the quantum many-body fermion state (associated with Young diagram R).

Vijay Balasubramanian, Bartlomiej Czech, Klaus Larjo, Joan Simon, "Integrability vs. Information Loss: A Simple Example," -2006, https://arxiv.org/abs/hep-th/0602263 (BGLS-2006)

Part 4 : Detecting LLM geometries

The upshot of this discussion gives an expression for u that allows the determination of the conserved charges from the semi-classical geometry :

$$u(\rho,\theta) = 2\cos^2\theta \sum_{l=0}^{\infty} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1)\ {}_2F_1(-l, l+1; 1; \sin^2\theta)$$

Here $\rho \in [0,\infty]$, $\theta \in [0,\frac{\pi}{2}]$, $\mathcal{F} = \{f_1, f_2, \ldots, f_N\}$ a set of increasing integers related to the eigenvalues of individual Fermion $E_i = \hbar(f_i + \frac{1}{2})$, $i = 1, 2, \ldots, N$. In such an expansion, $A^l(f)$ is a polynomial of order l in f (its explicit form can be found in BGLS-2006).

Part 4 : Detecting LLM geometries

These polynomials

$$\sum_{f\in\mathcal{F}} A^l(f) = C_l(R)$$

are Casimirs of $U(N)$.
These are in turn related to the eigenvalues of $\frac{\chi_R(T_p)}{d_R}$ - for example there are relations of the form

$$C_2(R) = Nn + \frac{\chi_R(T_2)}{d_R}$$

Such relations follow from Schur-Weyl duality which relates $U(N)$ rep theory to rep theory of

$$\bigoplus_{n=0}^{\infty} \mathbb{C}(S_n)$$

Knowledge of Casimirs $C_2(R), \cdots, C_k(R)$ is equivalent to knowing the normalised characters $\frac{\chi_R(T_2)}{d_R}, \cdots, \frac{\chi_R(T_k)}{d_R}$.

Part 4 : Detecting LLM geometries

The point made in BGLS-2006 was that determining a general Young diagram and corresponding LLM geometry requires knowing N Casimirs, while the Planck scale cutoff means we have access to far fewer Casimirs/multi-pole moments – order $N^{1/4}$ - an interpretation of information loss as a toy model for black holes.

We can give a more careful discussion of this argument by defining a $k_*(n, N)$ - number of cycle central elements (Casimirs) needed to distinguish all Young diagrams with n boxes and no more than N rows.

When $n < N$, $k_*(n, N) = k_*(n)$ which is the case we stick with for now.

Part 4: Holographic classical detection of projectors P_R

Remarkably - Because of AdS/CFT, the same task of detecting P_R has a classical counterpart. Use the long-distance behaviour of the metric/form-fields, which are determined by one function $u(\rho, \theta)$, to identify the Casimirs C_l up to a cut-off $l \sim k_*(n) \sim n^\alpha$ and estimate the complexity of the task.

In JBG-SR-2023, we described how to use the standard "Fast Fourier Transform algorithm" - along with known analytic properties of $u(\rho, \theta)$ from LLM and BGLS-2006, to reconstruct the Casimirs from the values of $u(\rho, \theta)$ at different values of θ.

And we used standard results on the complexity of FFT, along with the $k_*(n) \sim n^\alpha$ with $1/4 \leq \alpha < 1/2$, to obtain complexity estimates.

Part 4: Holographic classical detection of projectors P_R

Formula for $u(\rho,\theta)$:

$$u(\rho,\theta) = 2\cos^2\theta \sum_{l=0}^{\infty} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1)\ {}_2F_1(-l, l+1; 1; \sin^2\theta)$$

Introduce a cut-off $\Lambda \sim n^\alpha$ and expressing the $2F1$ in terms of Jacobi polynomials

$$u(\rho,\theta) = 2\cos^2\theta \sum_{l=0}^{\Lambda} \frac{\sum_{f\in\mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1) P_l^{0,0}(\cos 2\theta)$$

A rescaled form of u :

$$\widetilde{u}(\rho, X, \Lambda) = \frac{u(\rho, X, \Lambda)}{(1+X)} = \sum_{l=0}^{\Lambda} U(l,\rho) P_l^{0,0}(X)$$

where the Casimirs of interest are given by

$$\sum_{f\in\mathcal{F}} A^l(f) = (-1)^l \rho^{2l+2} \frac{U(l,\rho)}{l+1}$$

Part 4: Holographic classical detection of projectors P_R

With a little re-organisation

$$\widetilde{u}(\rho,\theta,\Lambda) = \sum_{l=0}^{\Lambda} U(l,\rho) \sum_{m=-l}^{l} \widetilde{p}_{l,m}\, e^{2i\theta m}$$

$$= \sum_{m=-\Lambda}^{\Lambda} \left[\sum_{l=|m|}^{\Lambda} U(l,\rho)\, \widetilde{p}_{l,m} \right] e^{2i\theta m} = \sum_{m=-\Lambda}^{\Lambda} \widetilde{C}_m(\rho,\Lambda)\, e^{2i\theta m}$$

$$\widetilde{C}_m(\rho,\Lambda) := \sum_{l=|m|}^{\Lambda} U(l,\rho)\, \widetilde{p}_{l,m}$$

We have a Fourier expansion, with coefficients that know about the Casimirs- and needs solution of a linear system involving Jacobi polynomial coeffs. - to go from Fourier coeffs. to Casimirs.

Part 4: Holographic classical detection of projectors P_R

The physical input into the algorithm (FFT and linear-system inversion) is a set of values of $u(\rho, \theta)$ at Λ discrete values of θ.

We assume that the computational complexity of measuring the $\widetilde{u}(\rho, \theta_l = \frac{\pi l}{\Lambda+1}, \Lambda)$, $l = 0, \ldots, \Lambda$, is bounded from above by a certain function $c_{\widetilde{u}}(\Lambda)$, the complexity of measuring $\widetilde{u}$ at separations of $2\pi/(\Lambda+1)$. The estimation of $c_{\widetilde{u}}(\Lambda)$ will require a complexity analysis of measurements in classical gravity, which we leave for future discussion and calculation.

Putting everything together, we arrive at the complexity of detecting the P_R/corresponding-LLM-geometry :

$$f(\Lambda) \leq c_0 \, \Lambda \, c_{\widetilde{u}}(\Lambda) + c_1 \Lambda \log \Lambda + c_2 \Lambda^2$$

where c_0, c_1 and c_2 are n-independent constants.

Part 4: Holographic classical detection of projectors P_R

Recalling $\Lambda \sim k_*(n) \sim n^\alpha$ with $1/4 \leq \alpha < 1/2$, and assuming $c_{\tilde{u}}(\Lambda)$ grows at most polynomially with Λ, we find that this holographic classical detection of R has a complexity which is polynomial in n - like the quantum phase estimation algorithm we described before.

Quantum Linear algebra algorithms (of HHL type) have been compared to randomised classical algorithms. In concrete real world-applications (recommendation systems) where the data is classical - and has to be converted to a quantum state vector - the (quantum-inspired randomised classical algorithms were found to be competitive with the quantum algorithm.

Ewin Tang "A quantum-inspired classical algorithm for recommendation systems"

For the projector detection task at hand - intrinsically a more quantum problem - the corresponding “quantum-inspired classical algorithms” we came up with in JBG-SR-2023 were exponentially worse than the quantum algorithm.

Part 5 : Generalisations - 2-matrix invariants

The algebra $\mathcal{Z}(\mathbb{C}(G))$ has the basis of conjugacy classes, defined by imposing on the set of $\sigma \in G$ the equivalence

$$\sigma \sim \gamma\sigma\gamma^{-1} \quad \text{for all } \gamma \in G$$

The algebra $\mathbb{C}_H(G))$, imposes on the set of $g \in G$ the equivalence

$$g \sim hgh^{-1} \quad \text{for all } h \in H \subset G$$

Part 5 : 2-matrix invariants

The case $G = S_{m+n}$ with $H = S_m \times S_n$ comes up when considering composite operators made of two matrices X, Y (quarter-BPS in $\mathcal{N} = 4$ SYM) with m copies of X and n copies of Y.

$$\mathcal{O}_\sigma(X, Y) = \sum_{i_1 \cdots i_{m+n}=1}^{N} X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_m}_{i_{\sigma(m)}} Y^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Y^{i_{m+n}}_{i_{\sigma(m+n)}}$$

$$\mathcal{O}_\sigma(X, Y) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(X, Y)$$

γ is restricted to $S_m \times S_n$ because of the bosonic symmetry of X's and Y's.

Part 5 : 2-matrix invariants - non-commutative algebra $\mathcal{A}(m, n)$

Taking the subspace of $\mathbb{C}(S_{m+n})$ which commutes with $S_m \times S_n$ gives a *non-commutative algebra* $\mathcal{A}(m, n)$ with projector-like basis

$$Q^{R}_{R_1, R_2; \mu, \nu}$$

where $R_1 \vdash m, R_2 \vdash n, R \vdash (m+n)$, and

$$1 \leq \mu, \nu \leq g(R_1, R_2, R)$$

Part 5 : Detecting the triple

Given a quantum state in this basis, we have the task of detecting with the conjugacy class operators, $T_k^{S_m}, T_k^{S_n}, T_k^{S_{m+n}}$, the triple R_1, R_2, R_3.

This can be done in polynomial time complexity for similar reasons as the $\mathcal{Z}(\mathbb{C}(S_n)$ problem.

Part 5 : Detecting the triple in the centre $\mathcal{Z}(\mathcal{A}(m, n))$

The center has a projector basis

$$P^R_{R_1,R_2} = \sum_\mu Q^R_{R_1,R_2;\mu\mu}$$

These quantum states P, Q exist if the LR coefficient is non-zero, and the detection can be done in polynomial time.

This can be viewed as a quantum version of the NP-question for LR coefficients ... (which has a positive answer).

Part 5 : Tensor invariants

3-index tensor invariants leads to an algebra $\mathcal{K}(n)$ - the subspace of

$$\mathbb{C}(S_n \times S_n)$$

which commutes with the diagonal S_n subgroup.

Projectors labelled by a triple of Young diagrams of S_n, which exist only if the Kronecker coefficient for the triple is non-zero, and the protocol given here gives polynomial time complexity.

Another broadly related discussion (with open questions on the precise relation) of quantum complexity associated with Kronecker coefficients is given in

"Quantum complexity of Kronecker coefficients" , arxiv.org[2302.11454v2], Bravyi, Chowdhury, Gosset, Havlicek, Zhu

Also finds polynomial time detection of a projector which exists for non-vanishing Kronecker ... framed in the context of QMA (Quantum Melin Arthur) ...

Summary and Outlook

We defined a "quantum detection of projectors in algebras" task inspired by AdS/CFT.

In the simplest case considered, motivated by half-BPS states, the algebra is $\mathcal{Z}(\mathbb{C}(S_n))$.

Other algebras considered in JBG-SR-2023 are related to Littlewood-Richardson coefficients and Kronecker coefficients.

Our discussion on the classical gravity side is a "proof of concept" complexity calculation – important and interesting conceptual questions remain on the intrinsic complexities of the gravitational measurements. Are there classical gravitational analogs of the query v/s gate complexities in the quantum side.

Explore similar classical/quantum comparisons in other instances of quantum-state/classical-geometry correspondences in string theory, e.g. Mathur programme, AdS3 etc.

Can we improve the bounds $n^{1/4} < k_*(n) \leq n^{1/2}$

Summary and Outlook

Are there "real-world" quantum information/computation of these projector detection tasks for some G, H pairs ? quantum communication ? quantum cryptography ?