Canonical Momenta in Digitized $SU(2)$ Lattice Gauge Theory

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  - Tensor Networks and Quantum Computing
  - study dynamical phenomena
    - real-time dynamics, string breaking, phase structure of gauge theories at finite fermionic densities
    - avoiding the sign problem

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    - common approach: choose basis of your Hilbert space that diagonalizes the electric part of the Hamiltonian $\Rightarrow$ character expansion/loop-string formulation
- our approach: diagonal gauge field operators
  - natural generalization $U(1) \rightsquigarrow SU(N)$
  - gauge links remain unitary $\Rightarrow$ implementable as gates on quantum devices

Discrete Quantum Mechanics

- Usual idea: discretize phase space and replace derivatives with finite difference operators

$$H(x,p) \rightsquigarrow H_{ij} = \langle x_i | H | x_j \rangle$$

- Example: non-relativistic particle in 1D box, discretized with N points at lattice spacing a

$$x \rightsquigarrow \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Na \end{pmatrix} \qquad p = -i\frac{d}{dx} \rightsquigarrow -i\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

Discrete Quantum Mechanics Problems

- Continuum results often require multiple extrapolations
- Requires numerical testing of when N is "sufficiently large"
- Canonical commutation relations are broken for all N:

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \Rightarrow \mathrm{tr}([A,B]) = 0$$

- $[x,p] = i$ is only recoverable in a functional sense by its action on test functions

$$([x,p] - i)\psi \to 0$$

Discretized Gauge Theories?

- Canonical momenta are Lie derivatives

$$L_a\psi(U) = -i\partial_\omega\psi\left(e^{i\omega\tau_a}U\right)\Big|_{\omega=0}$$

- Be careful about gauge invariance!
- Need to impose Gauss' Law.

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- Advantages:
  - conceptually simple
  - Gauge links remain unitary operators $\rightsquigarrow$ implementable as gates on quantum computer

Hamiltonian of Lattice Gauge Theories

$$H = \frac{g_0^2}{4} \sum_{x,c,k} \left(L_{c,k}^2(x) + R_{c,k}^2(x) \right) - \frac{1}{2g_0^2} \sum_{x,k<l} \operatorname{tr} \mathfrak{R} P_{kl}(x)$$

- g_0 bare gauge coupling
- x spatial lattice coordinate, k direction, c color index
- Plaquette operator

$$P_{kl}(x) = U_k(x) U_l(x+k) U_k^\dagger(x+l) U_l^\dagger(x)$$

- Suited for tensor networks and possibly quantum simulations

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- Electric and magnetic part
- common choice: diagonalize electric part
- we investigate: diagonal magnetic part

Group Manifold

- Gauge links: $U_\mu(x) \in SU(2)$
- canonical momenta: $L_a, R_a \in \mathfrak{su}(2)$

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- Construction of link operators:
  - choose finite set of N elements in $S_3 \cong SU(2)$
  - for each point, define a state $|\mathcal{U}\rangle \in \mathcal{H}$ where $\mathcal{U} \in SU(2)$:

$$U = \begin{pmatrix} \mathcal{U}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{U}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{U}_N \end{pmatrix}$$

  - Equivalently: consider $U = e^{i\alpha_a \tau_a}$ in terms of the manifold coordinate operator α where each element of the spectrum of α identifies a point on S_3.

Commutation Relations

$$[L_c, U_{mn}] = -(\tau_c)_{mj} U_{jn} \qquad [R_c, U_{mn}] = U_{mj} (\tau_c)_{jn}$$

- τ_c is one of the generators of $SU(2)$
- Lie algebra structure:

$$[L_a, L_b] = i f_{abc} L_c \qquad [R_a, R_b] = i f_{abc} R_c$$

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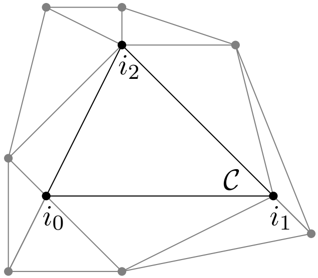
- In the continuum manifold $SU(2)$ these are solved by

$$L_c \psi(U) = -i \partial_\omega \psi\left(e^{i\omega\tau_c} U\right)\Big|_{\omega=0}$$
$$R_c \psi(U) = -i \partial_\omega \psi\left(U e^{i\omega\tau_c}\right)\Big|_{\omega=0}$$

- How to define on finite subsets of $SU(2)$?

L_a from Delaunay triangulation https://inspirehep.net/literature/2649261

- Delaunay triangulation of point ins $SU(2)$ yields a set of simplices $\mathcal{C} = \{i_0, i_1, i_2, i_3\}$



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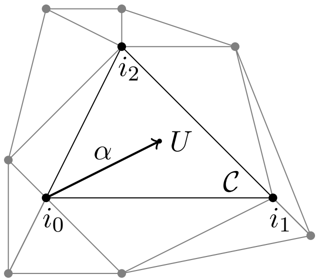
- write arbitrary U as $U = e^{i\langle\alpha,\tau\rangle}U_{i_0}$
- approximate functions ψ as

$$\psi(U) = \psi(U_{i_0}) + \langle\nabla\psi_{i_0}, \alpha\rangle + \mathcal{O}(\alpha^2)$$



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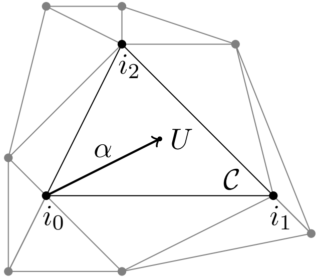
$$\psi(U) = \psi(U_{i_0}) + \langle\nabla\psi_{i_0}, \alpha\rangle + \mathcal{O}(\alpha^2)$$

- $L = -\nabla$ is obtained from imposing vertex conditions (linear interpolation of $\psi(i_0)$, $\psi(i_1)$, $\psi(i_2)$, $\psi(i_3)$)

$$\begin{pmatrix}\alpha_1^T\\ \alpha_2^T\\ \alpha_3^T\end{pmatrix}\nabla\psi_{i_0} = \begin{pmatrix}\psi(i_1)-\psi(i_0)\\ \psi(i_2)-\psi(i_0)\\ \psi(i_3)-\psi(i_0)\end{pmatrix}$$



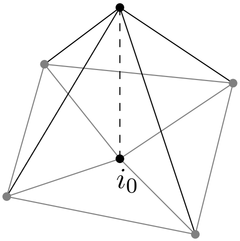
Remarks

- averaging over all simplices containing the point i gives better estimates
- similar construction for R using $U = U_{i_0} e^{i\langle \alpha, \tau \rangle}$
- discretizing L^2 directly in the Hamiltonian converges faster than taking product $L_a \cdot L_a$ of discretized L_a
- in the continuum limit L^2 becomes the S_3-Laplace-Beltrami operator

$$L^2 = -\cot\vartheta \ \partial_\vartheta - \partial_\vartheta^2 - \frac{1}{\sin^2\vartheta}\partial_\varphi^2 + 2\frac{\cos\vartheta}{\sin^2\vartheta}\partial_\psi\partial_\varphi - \frac{1}{\sin^2\vartheta}\partial_\psi^2$$

L^2 from Delaunay triangulation

- hat functions on triangulated lattice: $\varphi_{i_0}(U_i) = \delta_{i_0,i}$ and piece-wise linear interpolation
- test Laplace equation against these distributions
  $L^2 u = -\Delta u = f \quad \Rightarrow \quad \forall i: \ -\langle \Delta u, \varphi_i \rangle = \langle f, \varphi_i \rangle$



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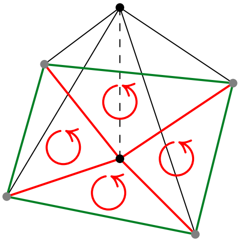

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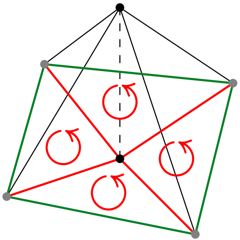
- expand $u = \sum_j u_j \varphi_j$:

$$\langle \Delta u, \varphi_i \rangle = -\sum_{\mathcal{C}} \sum_j u_j \int_{\mathcal{C}} \langle \nabla \varphi_j, \nabla \varphi_i \rangle = \sum_j S_{ij} u_j$$



L^2 from Delaunay triangulation

- approximate right-hand side

$$\langle f, \varphi_i \rangle = \sum_{\mathcal{C}} \int_{\mathcal{C}} f \varphi_i \approx v_i f_i \quad \text{with} \quad v_i = \sum_{\mathcal{C} \ni i} \frac{\text{vol}(\mathcal{C})}{4}$$

and the Laplace equation $L^2 u = -\Delta u = f$ becomes

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- discrete version of L^2 is given by

$$L^2_{ij} = -\frac{S_{ij}}{v_i}$$

- if i and j are not connected by a simplex, then $S_{ij} = 0$, so $L^2 = R^2$ are local operators

Triangulation recap

- What we do:
  - select N points on sphere S_3
  - map points to eigenstates of U
  - asymptotically dense in the group manifold for $N \to \infty$

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  - generalizable to $SU(n)$ and $U(n)$
- To check:
  - spectral convergence of L^2
  - convergence of commutation relations
  - impact of choice of partitionings (choice of points in S_3)

Convergence of Discretized Gradient ∇_P and Laplace Δ_P

- $\nabla_P \to \nabla$ holds w.r.t strong operator topology using net of discretizations with $P' \leq P$ if and only if every vertex of the triangulation P' is also a vertex in P

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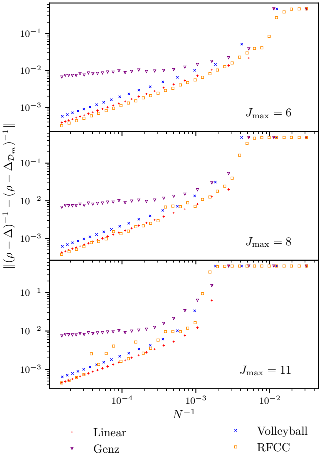
- implies convergence of low energy spectrum with arbitrarily large cutoffs

Gap convergence of different partitioning schemes

- test different partitioning schemes for rate of convergence (only 4 shown as an example here)
- Genz points: offer polynomially exact integration
- Linear: variation on Genz points ensuring uniform scaling of simplex volumes
- Volleyball: generalization of Volleyball stitchings, i.e., uniformity of simplices
- RFCC: based on rotated face centered cubical lattice, i.e., uniformity of chosen points



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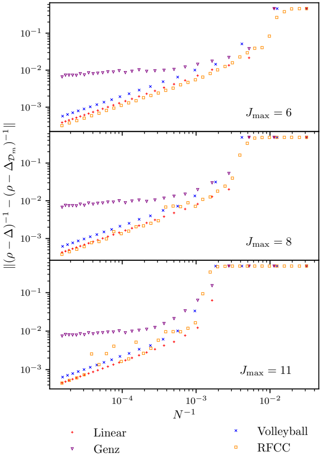
- RFCC: based on rotated face centered cubical lattice, i.e., uniformity of chosen points
- uniformity seems important for rate of spectral convergence

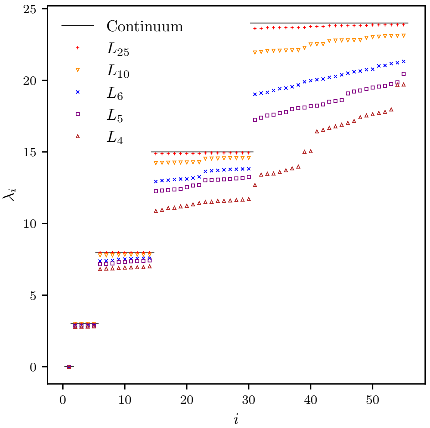


Spectral Convergence of $-\Delta$ with Linear Partitionings L_m

- Continuum eigenvalues are

$$\lambda = J(J+2) \quad \text{for} \quad J \in \mathbb{N}_0$$

  with multiplicity $(J+1)^2$
- low energy spectrum approaches the continuum spectrum with $m \to \infty$

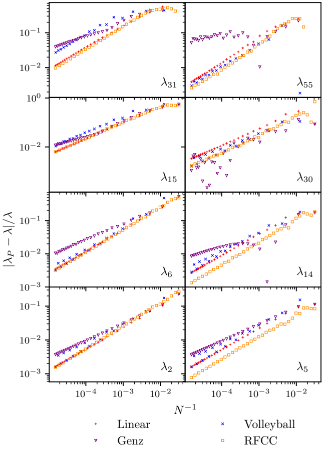


Convergence of Eigenvalues

- RFCC best performing
- relative error of eigenvalues fit

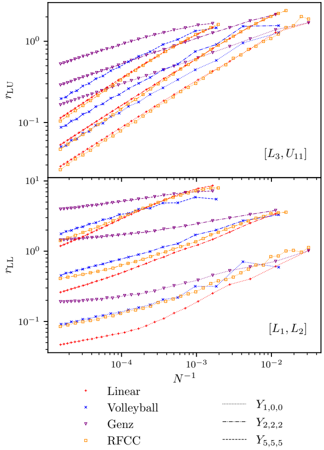
$$\frac{|\lambda_P - \lambda|}{\lambda} \approx cN^{-\alpha}$$

with $\alpha \approx 0.6$



Convergence of Commutators

- Y_{J,l_1,l_2} 4d-spherical harmonics (Eigenfunctions of $-\Delta$)
- r_{LU} mean deviation of $([L_a, U_{jl}] + (\tau_a)_{ji} U_{il}) Y_{J,l_1,l_2}$ weighted by barycentric cell volume v_i
- r_{LL} mean deviation of $([L_a, L_b] + 2i f_{abc} L_c) Y_{J,l_1,l_2}$ weighted by barycentric cell volume v_i
- RFCC best performing on $[L, U]$
- Linear best performing on $[L_a, L_b]$



Derivatives on S_3 S. Romiti, C. Urbach https://inspirehep.net/literature/2724218

- Eigenfunctions are Wigner D-functions $D^j_{m\mu} \Rightarrow \mathfrak{su}(2)$ irreducible representations
- $\sum_a R_a^2 |j,m,\mu\rangle = \sum_a L_a^2 |j,m,\mu\rangle = j(j+1)|j,m,\mu\rangle$
- $L_3 |j,m,\mu\rangle = m|j,m,\mu\rangle$, $R_3 |j,m,mu\rangle = -\mu|j,m,\mu\rangle$
- $(L_1 \pm iL_2)|j,m,\mu\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1,\mu\rangle$
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- Fix truncation $j \le q$ and we get N_q states with

$$N_q = \sum_{j\le q}(2j+1)^2 = \frac{(4q+3)(2q+2)(2q+1)}{6} \in \mathcal{O}(q^3)$$

- How many eigenstates of U can be reproduced in discretized S_3?

Frequencies on S_3

- Non-abelian manifold + Shannon-Nyquist Theorem = N_α points cannot sample N_α Fourier modes
- we need $N_\alpha > N_q$ or more precisely

$$N_\alpha \geq \begin{cases} (q+1/2)(4q+1)^2 & , \ q \text{ half-integer} \\ (q+1)(4q+1)^2 & , \ q \text{ integer} \end{cases}$$

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- Physical consequences
  - U unitary $\Rightarrow \nexists$ change of basis V between electric and magnetic basis
  - V at best embeds into a larger space of the first N_q $\mathfrak{su}(2)$ irreps
  - presence of unwanted states

Discrete Jacobi Transform (DJT)

- V satisfies

$$f(\vec{\vartheta}) = f(\vartheta, \varphi, \psi) = \sum_{j=0}^{q} \sum_{m,\mu=-j}^{j} V^j_{m,\mu}(\vec{\vartheta}) \hat{f}(j, m, \mu)$$

$$V^j_{m,\mu}(\vec{\vartheta}) = \sqrt{\frac{(j + 1/2) w_s}{N_\varphi N_\psi}} D^j_{m,\mu}(\vec{\vartheta})$$

- w_s Gaussian weights of Legendre polynomials
- V is of size $N_\alpha \times N_q$
- $V^\dagger V = 1_{N_q \times N_q}$
- $\dim \ker V^\dagger = N_\alpha - N_q$, so $VV^\dagger \neq 1_{N_\alpha \times N_\alpha}$

Properties of Discrete Momenta

- $L_a = V \hat{L}_a V^\dagger$, $R_a = V \hat{R}_a V^\dagger$
- exact Lie algebra: $i f_{abc}$
- first N_q eigenstates $|j, m, \mu\rangle$ are reproduced exactly
- Commutation relations fulfilled for the first $N_{q'} = N_{q-1/2}$ irreps

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- dense matrices for the momenta (local for $q \to \infty$)
- $N_\alpha - N_q$ states degenerate with the electric vacuum $\Rightarrow$ decouple by lifting with projector $P_{j>q}$
- Gauss law G^a: $[G^a, H] \neq 0$ on $(N_\alpha - N_{q'})$-dim subspace

1 + 1D $SU(2)$ with Fermions https://inspirehep.net/literature/2726571

- Hamiltonian

$$H = \mu \sum_x \sum_{c=1}^{2} (-1)^x \chi_x^{c\dagger} \chi_x^c + \frac{1}{2} \sum_{a,x} \left(\chi_x^{c\dagger} U_x^{cc'} \chi_x^{c'} + h.c. \right) + \frac{g^2}{2} \sum_x L_x^2$$

- Gauss Law

$$G_x^a = L_x^a - R_x^a - \frac{1}{2} \chi_x^\dagger \tau^a \chi_x$$

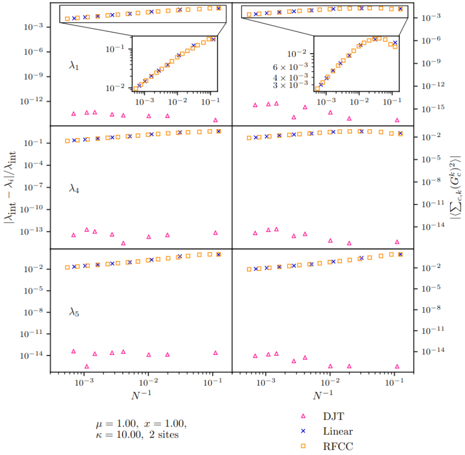
- physical states are states with $G^a |\psi\rangle = 0$
- add Gauss law penalty term for non-physical states

$$H_{Penalty} = \kappa \sum_x G_x^2$$

- no plaquette in 1D: magnetic Hamiltonian = 0 and Gauss law can be enforced exactly by analytically integrating out the gauge fields

Spectrum with DJT momenta

- exact results recovered for large N for both “normal” and integrated Hamiltonian
- for small N Gauss law operator G^a shows discretization effects



Single Plaquette System https://inspirehep.net/literature/2726571

- Hamiltonian

$$H = \frac{g^2}{2} \sum_{c=1}^{3} \sum_{i=0}^{3} (L_i^c)^2 - \frac{2}{g^2} \mathrm{tr}(U_0 U_1 U_2 U_3)$$

- we compare against anaytic solutions: Bauer et al. (2023) https://arxiv.org/abs/2307.11829
- Gauss Law

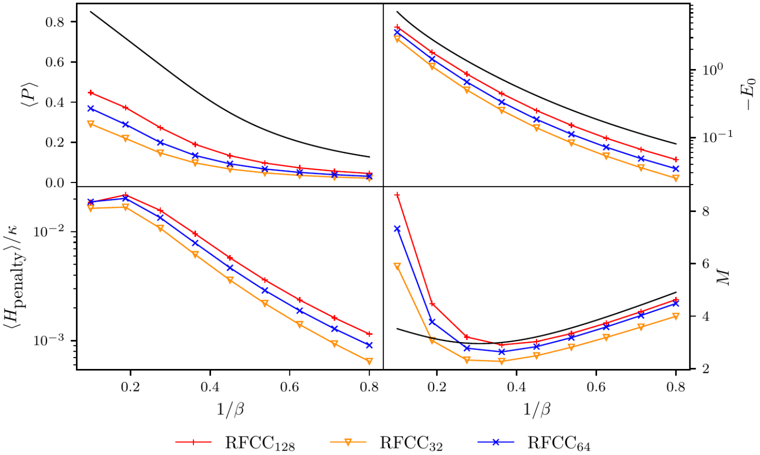
$$G_a(x) = \sum_{\mu=1}^{d} (L_a)_\mu(x) + (R_a)_\mu(x-\mu) - \frac{1}{2}\chi^\dagger(x)\tau^a\chi(x)$$

- $G_a |\psi\rangle = 0 \iff |\psi\rangle$ physical: Gauss Law penalty

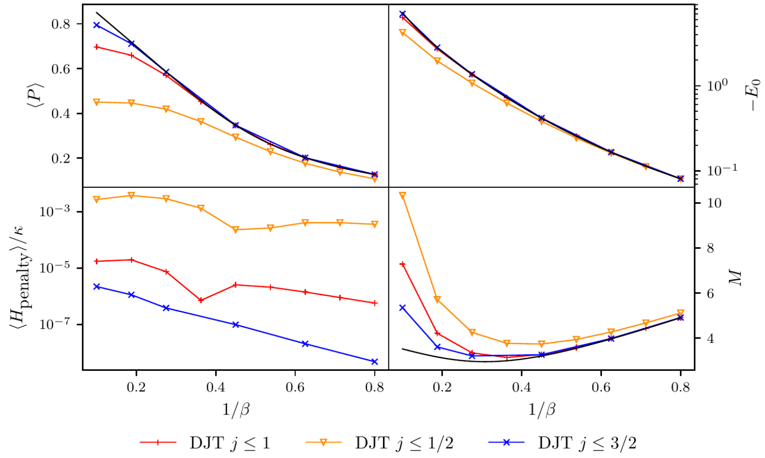
$$H_{penalty} = \kappa \sum_{x,\mu} \sum_{a} (G_a)_\mu(x)^2$$

RFCC

- coupling $\beta = \frac{4}{g^2}$
- plaquette $\langle P \rangle$
- vacuum energy E_0
- residual Gauss law violation $\langle H_{penalty} \rangle / \kappa$
- mass gap M

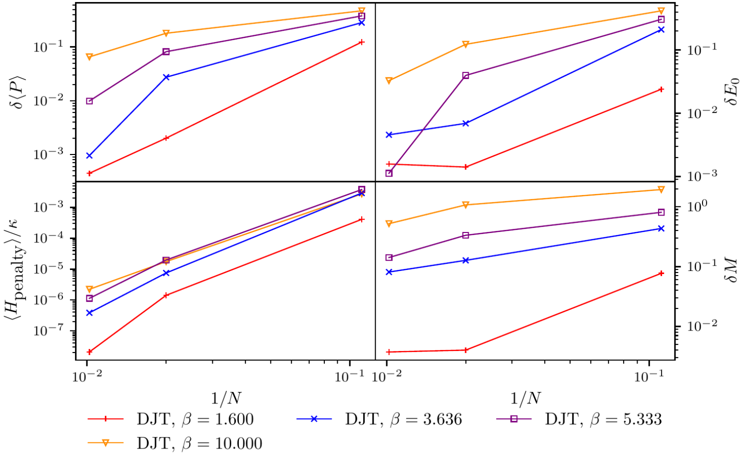


DJT



$\langle P \rangle$
0.8
0.6
0.4
0.2
$-E_0$
10^0
10^{-1}
$\langle H_{\text{penalty}} \rangle / \kappa$
10^{-3}
10^{-5}
10^{-7}
M
10
8
6
4
0.2
0.4
0.6
0.8
$1/\beta$
$1/\beta$
DJT $j \leq 1$
DJT $j \leq 1/2$
DJT $j \leq 3/2$

DJT convergence



What have we got?

- Digitized $SU(2)$ lattice Hamiltonian with discrete subsets of S_3
- 2 approaches: Delaunay Triangulation and Discrete Jacobi Transform
- well-understood applicability in terms of N and/or cutoff of the theory
- Convergence to exact results of Schwinger-like model and single plaquette
- DJT very good, but Triangulations can be improved

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Outlook

- larger systems
- going beyond exact diagonalization (QC, tensor networks)
- generalization to $SU(3)$ and beyond