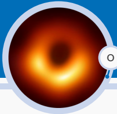
Supervised by prof. dr. Dejan Gajic, Institute for Theoretical Physics, Leipzig University

Late-time asymptotics for dynamical black hole spacetimes

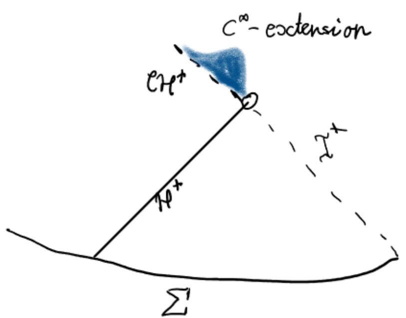
Quinten Rutgers
Physik-Combo, September 19, 2023

Content



Cosmic censorship

- Problem: the globally hyperbolic Kerr and Reissner-Nordström spacetimes can be extended non-uniquely (and smoothly) across an inner **Cauchy horizon**.
- This signals a breakdown of determinism.
- The celebrated strong cosmic censorship conjecture proposes that this unwanted behaviour can be overcome by perturbing the spacetime at the level of initial data.



Initial value formulation

Theorem [Choquet-Bruhat, Geroch ('69)]

Given any Riemannian manifold $(\Sigma, \overline{g})$ equipped with a 2-tensor field K such that $(\overline{g}, K)$ solve the constraint equations, there exists a unique maximal spacetime (M, g) such that

1. (M, g) is a solution to the Einstein field equations in vacuum;
2. Σ embeds into M as a Cauchy surface such that $\overline{g}$ and K are the induced metric and second fundamental form respectively.

- The spacetime (M, g) arising from the initial data $(\Sigma, \overline{g}, K)$ is called the **maximal Cauchy evolution** of the data.
- Idea: in the right gauge, the Einstein vacuum equations become a system of quasilinear wave equations for the metric which can be solved given the initial data $(\overline{g}, K)$.

Cosmic censorship

Strong cosmic censorship conjecture [Penrose]

For ‘generic’ initial data for the Einstein field equations, the maximal Cauchy development is inextendible as a (suitably regular) Lorentzian manifold.

- Proving this conjecture is hard because the meaning of the words ‘*generic*’ and ‘*suitably regular*’ is not so clear a priori.

Cosmic censorship

- Example: the (extended) Schwarzschild spacetime is inextendible as a C^0-manifold because of the singularity at $r = 0$. It was thought for a long time that this should be the generic scenario.

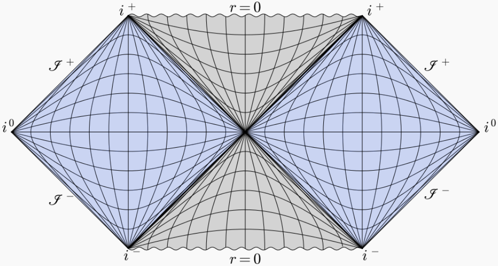
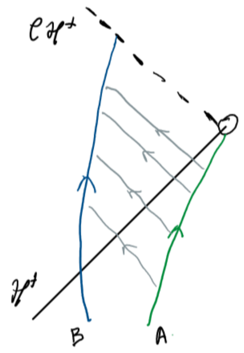


Figure: The extended Schwarzschild spacetime

Blueshift effect

- A possible resolution of this conjecture is provided by the **blueshift effect**: signals sent by observer *A* at constant proper time intervals are infinitely blueshifted as *B* approaches the Cauchy horizon.
- This indicates that perturbations might exhibit some instability at $\mathcal{CH}^+$ preventing extendibility.



The Kerr Cauchy horizon

Theorem [Dafermos, Luk ('17)]

Assuming the nonlinear stability of the Kerr exterior, small perturbations of Kerr initial data will give rise to spacetimes with a (null) Cauchy horizon, across which the metric is C^0-extendible.

- This rules out the C^0-formulation of strong cosmic censorship, and means that the Schwarzschild picture is quite special.
- Current state of the conjecture: the perturbed spacetime metric will have a null Cauchy horizon across which the metric extends in a C^0-fashion. However, the metric has non-square integrable Christoffel symbols near $\mathcal{CH}^+$, making it a so-called **weak null singularity**.
- This is the lowest regularity class in which we can speak of a '*weak solution*' to the Einstein equations.

Dynamical toy model

- To study *instability* of Cauchy horizons we will use a simpler toy model for gravitational perturbations: the spherically symmetric Einstein-Maxwell-scalar field (EMSF) system.
- Maxwell field F and a massless scalar field ϕ coupled to metric

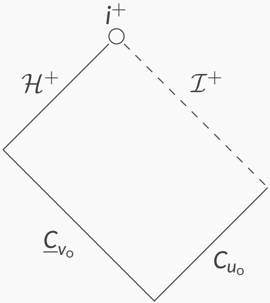
$$g = -\Omega^2(u, v)dudv + r^2 \slashed{g}_{S^2}$$

where (u, v) are double null coordinates.

Dynamical toy model

- Einstein equations reduce to a system of null transport equations and wave equations for ϕ and the area radius r. We pose initial data on two intersection null hypersurfaces.
- Hawking mass m is characterized by

$$1-\mu = 1-\frac{2m}{r} = g(\nabla r, \nabla r) = -4\frac{\partial_u r \partial_v r}{\Omega^2}$$



Previous results

Theorem [Luk, Oh ('21)]

Let (M, g, ϕ) be a solution to the EMSF system for which an L^2-averaged lower bound of the type

$$\int_{\mathcal{H}^+} v^\alpha (\partial_v \phi)^2 dv = \infty$$

holds. Then M is not C^2-extendible across $\mathcal{CH}^+$. Furthermore, for generic solutions this averaged lower bound holds.

Previous results

Theorem [Dafermos ('05)]

Let (M, g, ϕ) be a solution to the EMSF system for which a *polynomial* lower bound for the scalar field ϕ of the type

$$|\partial_v \phi|_{\mathcal{H}^+}| \gtrsim v^{-q},$$

holds along the event horizon $\mathcal{H}^+$. Then M is not C^1-extendible across $\mathcal{CH}^+$. In fact, the Hawking mass blows up identically along $\mathcal{CH}^+$, a phenomenon called *mass inflation*.

- What is still missing to complete Dafermos' result, and hence to establish the C^1-version of SCC for this system, is a proof that such a polynomial lower bound actually holds generically for this system.

Price's law

- Simplest toy model for gravitational perturbations: linear wave equation

$$\Box_g \phi = 0$$

  on a (subextremal) Reissner-Nordström background.
- The polynomial lower bound can be obtained for the linear wave equation through exact late-time asymptotics in the full exterior (including $\mathcal{I}^+$ and $\mathcal{H}^+$) known in this context as **Price's law**.

Price's Law

Theorem [Angelopoulos, Aretakis, Gajic ('18)]

Let ϕ be a spherically symmetric solution to $\Box_g \phi = 0$ with nonvanishing Newman-Penrose constant $I_0[\phi]$. Along a hyperboloidal foliation Σ_τ we have

$$\phi_{\Sigma_\tau}(\tau, \cdot) \sim_{\text{asym}} 4I_0[\phi]\frac{1}{\tau^2}$$

in the region $\{r \leq R\}$, while in the near-infinity region $\{r \geq R\}$ we have

$$\phi_{\Sigma_\tau}(\tau, \cdot) \sim_{\text{asym}} 4I_0[\phi]\left(1 + \frac{u}{v}\right)\frac{1}{uv}.$$

NP-constant is given along a hypersurface terminating at $\mathcal{I}^+$ by the limit

$$I_0[\phi] = \lim_{r\to\infty} r^2 \partial_v (r\phi)$$

and is a conserved quantity along $\mathcal{I}^+$ which is generically nonvanishing.

Price’s Law

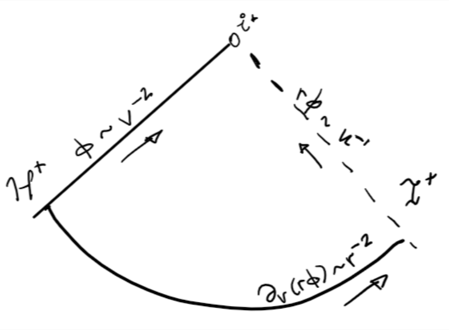


Figure: Price’s Law for the linear wave equation

Conjecture

Conjecture

Let (M, g, ϕ) be a solution to the spherically symmetric EMSF system with nonvanishing Newman-Penrose constant $I_0[\phi]$ and sufficiently small initial data (in some higher Sobolev norm).

Then along a characteristic foliation Σ_τ of M, ϕ decays inverse polynomially with decay rate and leading-order asymptotics the same as those for the linear wave equation on Reissner-Nordström.

Difficulties

- **Lack of Killing vector fields**: no timelike Killing field T, however in spherical symmetry we have the Kodama vector field which in this setting is given by
$$T = \frac{1-\mu}{\partial_v r}\partial_v + \frac{1-\mu}{-\partial_u r}\partial_u.$$
It is tangent to constant-r hypersurfaces and still gives a conserved energy, however T does **not** commute with the d'Alembertian $\Box_g$.
- **Difficulties related to choice of gauge**: a different choice of u and v coordinates allows one to normalize certain quantities at different points in the spacetime. For example, one could choose to set $\partial_u r = 1$ either on the initial surface or on $\mathcal{I}^+$. However, it is not always clear which choice of gauge is the right one for proving estimates.

Difficulties

- **Nonlinear structure of the equations**: to deal with the nonlinearities in the system we need to setup a bootstrap argument. This requires weak decay assumptions for ϕ as input, which are then improved in the course of the argument. These assumptions are consistent with decay rates one would need to prove the stability of Reissner-Nordström as a solution to the EMSF system.
- **Decay on the event horizon**: In the Reissner-Nordström spacetime the event horizon $\mathcal{H}^+$ is foliated by marginally outer-trapped surfaces, i.e. $\partial_v r = 0$ on $\mathcal{H}^+$. In our setting this is not the case and we require the Raychaudhuri equation

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = -\frac{r(\partial_v \phi)^2}{\Omega^2}$$

to prove decay for $\partial_v r$ along $\mathcal{H}^+$.

Outlook

- The next step would be to consider **compactly supported initial data**, which in particular has vanishing Newman-Penrose constant. For the linear wave equation this requires the use of **time-inversion** where the leading-order tail is now determined by the time-inverted Newman-Penrose constant

$$I_0^{(1)}[\phi] := I_0[\phi^{(1)}],$$

where $\phi^{(1)}$ is a solution to the linear wave equation such that $T\phi^{(1)} = \phi$. This time-inversion theory would have to be extended to this dynamical setting.

Summary

- The strong cosmic censorship conjecture poses that spacetimes are generically not extendible.
- We presented a toy model in which inextendibility can be shown assuming polynomial lower bounds along $\mathcal{H}^+$.
- We introduced Price's Law in a linear setting and discussed the difficulties of extending these asymptotics to the dynamical toy model.

Thank you for your attention!