Global Fixed Functions in Nonlinear Electrodynamics

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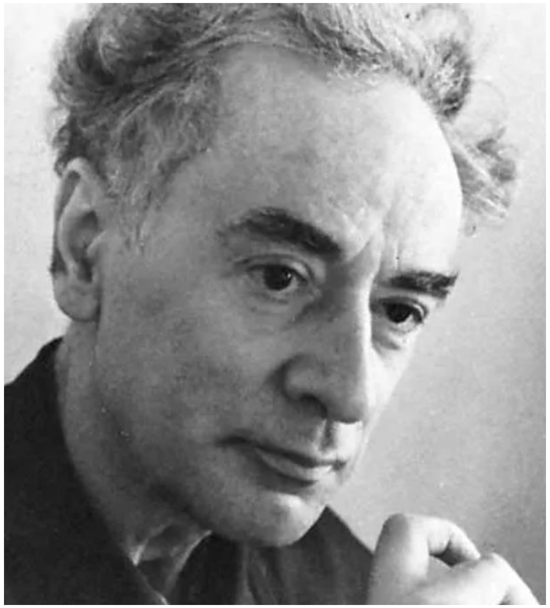
INVITATION

WHY
GLOBAL
EXISTENCE
IN
NONLINEAR
ELECTRODYNAMICS

?

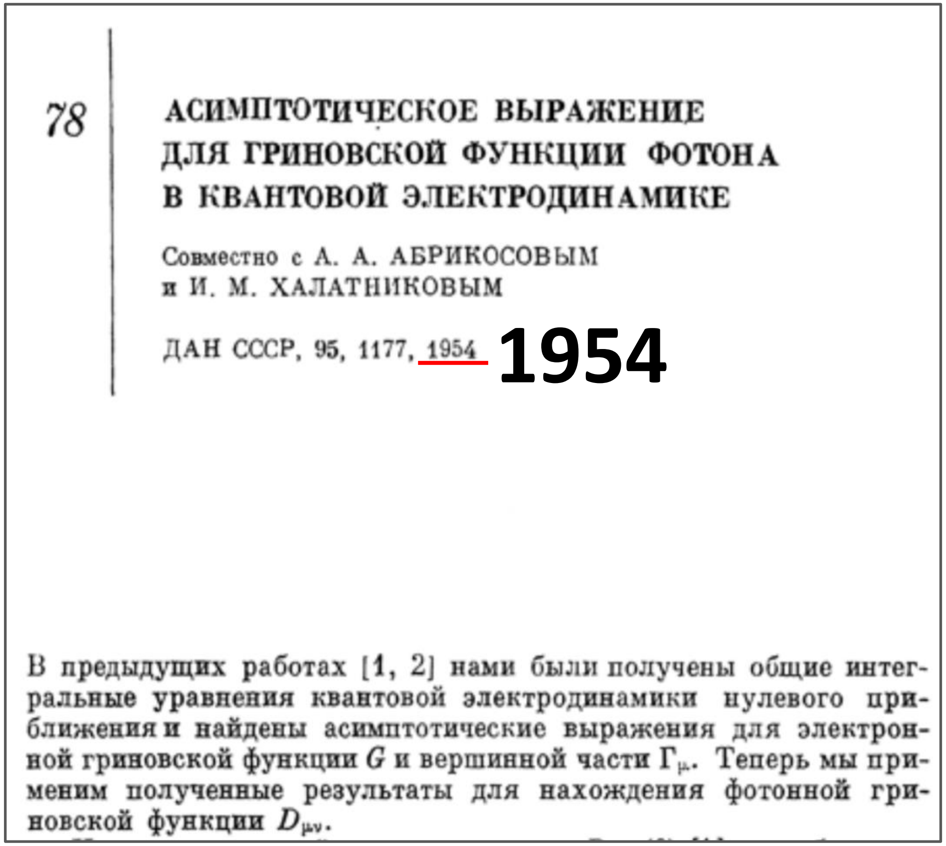
INVITATION



Lew. D. Landau (*1908, † 1968)

$$D(k) = \frac{1}{1 - \frac{1}{2}\beta_1 \ln\left(\frac{k^2}{m_R^2}\right)}$$

Gauge invariant photon propagator



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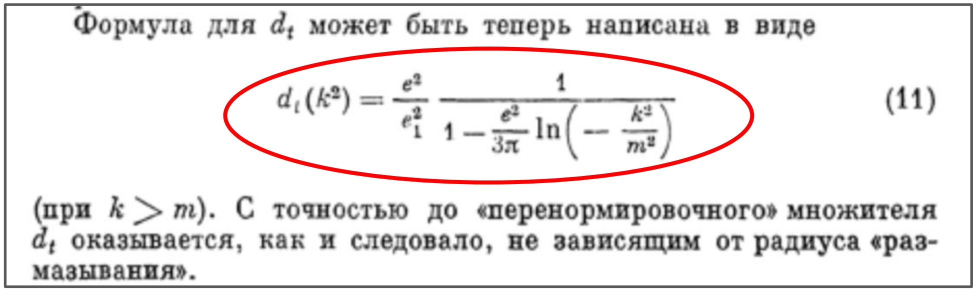
АСИМПТОТИЧЕСКОЕ ВЫРАЖЕНИЕ ДЛЯ ГРИНОВСКОЙ ФУНКЦИИ ФОТОНА В КВАНТОВОЙ ЭЛЕКТРОДИНАМИКЕ

Совместно с А. А. АБРИКОСОВЫМ
и И. М. ХАЛАТНИКОВЫМ

ДАН СССР, 95, 1177, 1954

1954

В предыдущих работах [1, 2] нами были получены общие интегральные уравнения квантовой электродинамики нулевого приближения и найдены асимптотические выражения для электронной грииновской функции G и вершинной части Γ_μ. Теперь мы применим полученные результаты для нахождения фотонной гриновской функции $D_{\mu\nu}$.



Формула для d_t может быть теперь написана в виде

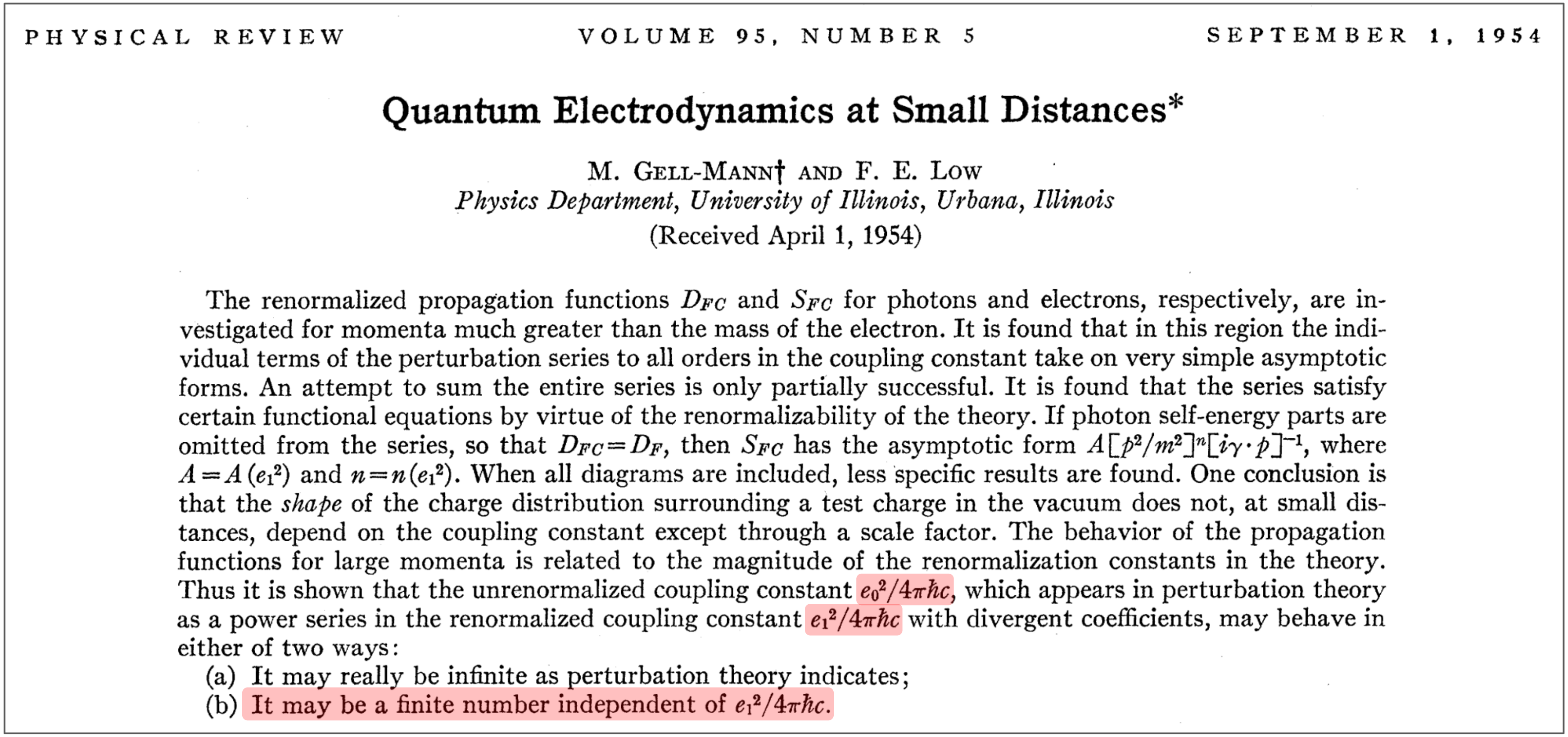
$$d_t(k^2) = \frac{e^2}{e_1^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(-\frac{k^2}{m^2}\right)} \qquad (11)$$

(при $k > m$). С точностью до «перенормировочного» множителя d_t оказывается, как и следовало, не зависящим от радиуса «размазывания».

INVITATION

$$\boxed{k = \Lambda_{\mathrm{L}} = m_R \mathrm{e}^{1/\beta_1} < \infty}$$

perturbative Landau pole



PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Quantum Electrodynamics at Small Distances*

M. GELL-MANN† AND F. E. LOW
Physics Department, University of Illinois, Urbana, Illinois

(Received April 1, 1954)

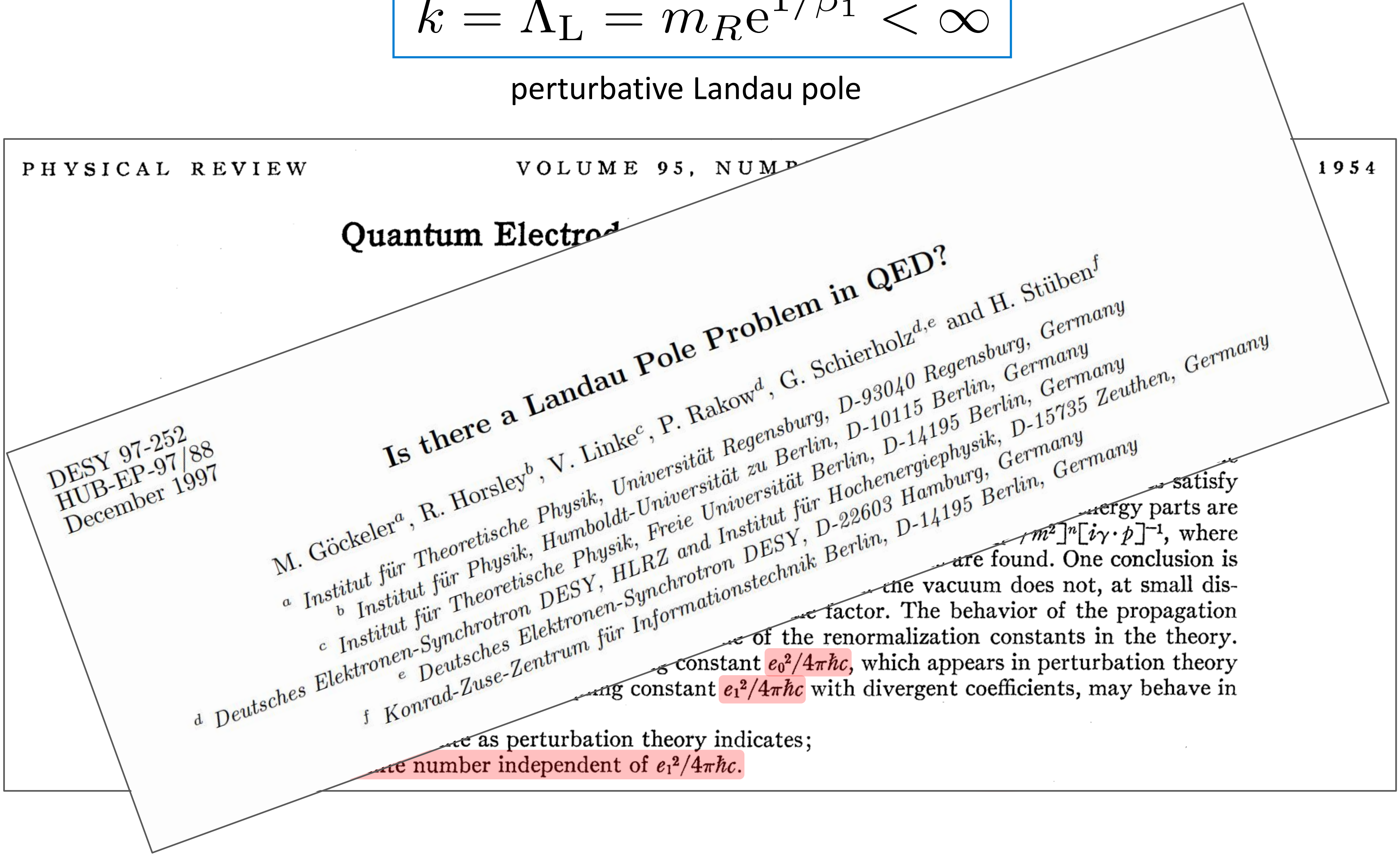
The renormalized propagation functions D_{FC} and S_{FC} for photons and electrons, respectively, are investigated for momenta much greater than the mass of the electron. It is found that in this region the individual terms of the perturbation series to all orders in the coupling constant take on very simple asymptotic forms. An attempt to sum the entire series is only partially successful. It is found that the series satisfy certain functional equations by virtue of the renormalizability of the theory. If photon self-energy parts are omitted from the series, so that $D_{FC}=D_F$, then S_{FC} has the asymptotic form $A[p^2/m^2]^n[i\gamma\cdot p]^{-1}$, where $A=A(e_1^2)$ and $n=n(e_1^2)$. When all diagrams are included, less specific results are found. One conclusion is that the *shape* of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through a scale factor. The behavior of the propagation functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e_1^2/4\pi\hbar c$ with divergent coefficients, may behave in either of two ways:

(a) It may really be infinite as perturbation theory indicates;
(b) It may be a finite number independent of $e_1^2/4\pi\hbar c$.

INVITATION

$$\boxed{k = \Lambda_{\mathrm{L}} = m_R \mathrm{e}^{1/\beta_1} < \infty}$$

perturbative Landau pole



PHYSICAL REVIEW VOLUME 95, NUM 1954

Quantum Electro

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DESY 97-252
HUB-EP-97/88
December 1997

Is there a Landau Pole Problem in QED?

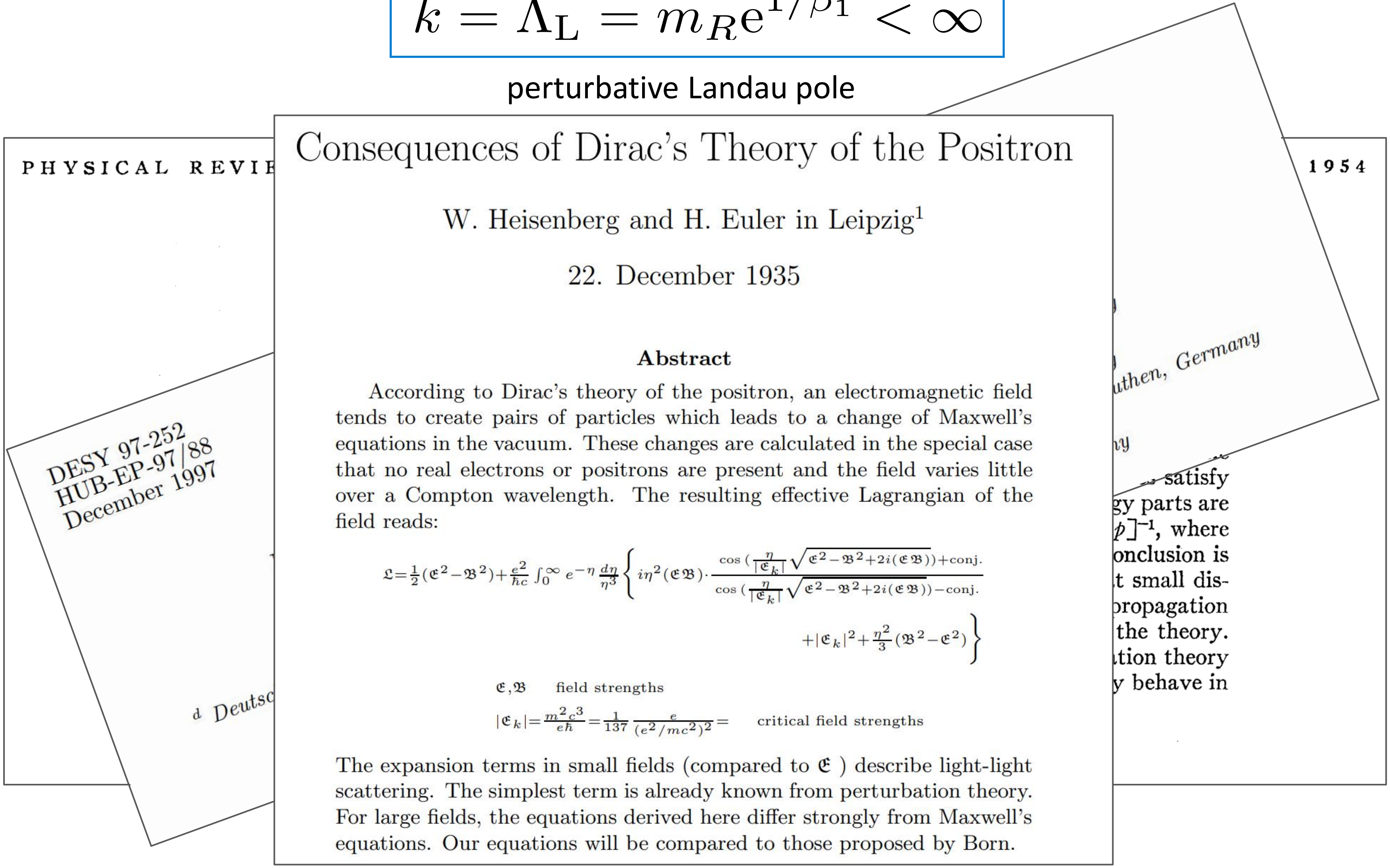
M. Göckeler[a], R. Horsley[b], V. Linke[c], P. Rakow[d], G. Schierholz[d,e] and H. Stüben[f]

[a] *Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*
[b] *Institut für Physik, Humboldt-Universität zu Berlin, D-10115 Berlin, Germany*
[c] *Institut für Theoretische Physik, Freie Universität Berlin, D-14195 Berlin, Germany*
[d] *Deutsches Elektronen-Synchrotron DESY, HLRZ and Institut für Hochenergiephysik, D-15735 Zeuthen, Germany*
[e] *Deutsches Elektronen-Synchrotron DESY, D-22603 Hamburg, Germany*
[f] *Konrad-Zuse-Zentrum für Informationstechnik Berlin, D-14195 Berlin, Germany*

INVITATION

$$k = \Lambda_{\mathrm{L}} = m_R \mathrm{e}^{1/\beta_1} < \infty$$

perturbative Landau pole



PHYSICAL REVI

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Consequences of Dirac's Theory of the Positron

W. Heisenberg and H. Euler in Leipzig[1]

22. December 1935

Abstract

According to Dirac's theory of the positron, an electromagnetic field tends to create pairs of particles which leads to a change of Maxwell's equations in the vacuum. These changes are calculated in the special case that no real electrons or positrons are present and the field varies little over a Compton wavelength. The resulting effective Lagrangian of the field reads:

$$\mathfrak{L}=\frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2)+\frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{conj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{conj.}} + |\mathfrak{E}_k|^2+\frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}$$

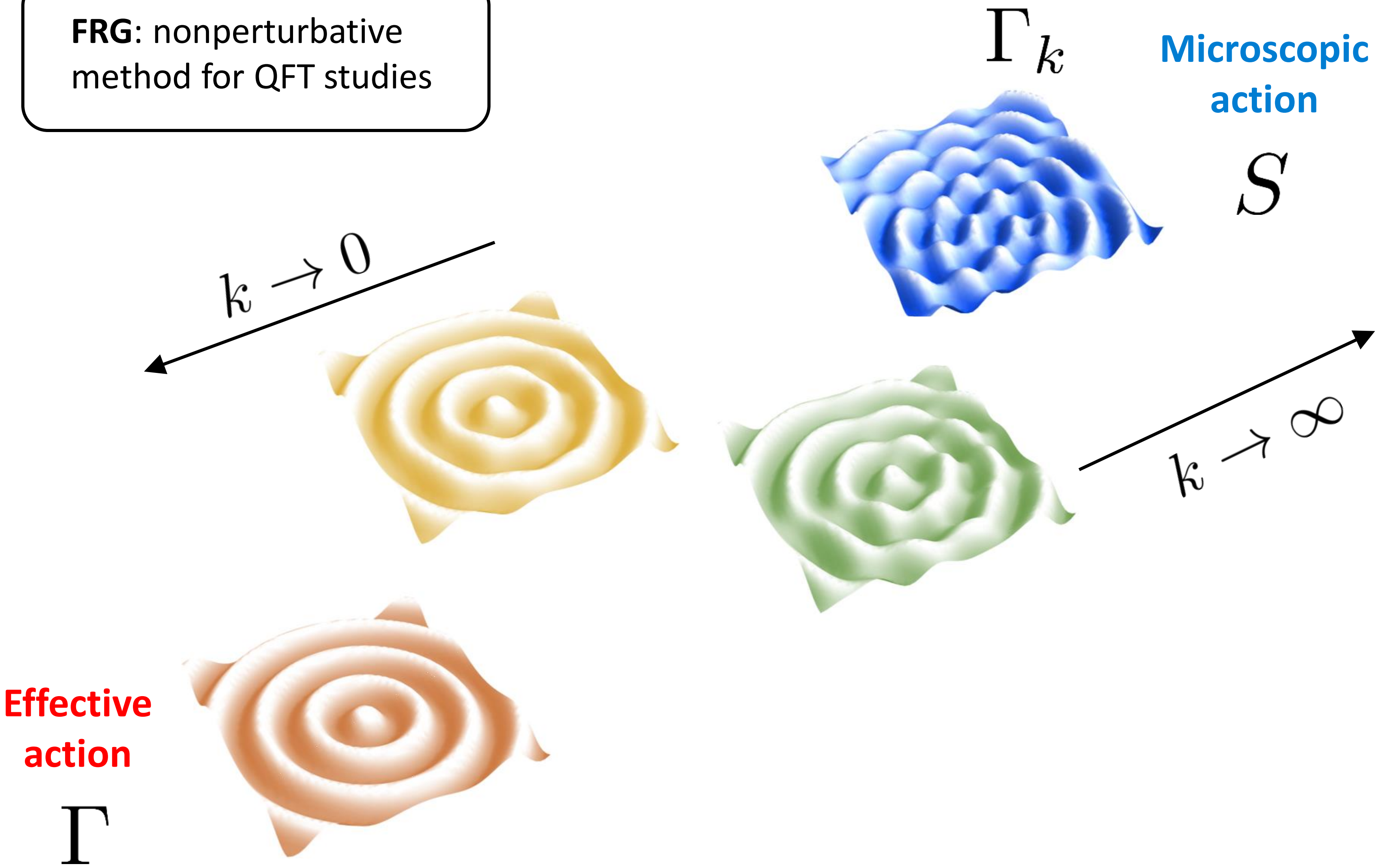
$\mathfrak{E},\mathfrak{B}$ field strengths

$|\mathfrak{E}_k|=\frac{m^2c^3}{e\hbar}=\frac{1}{137}\frac{e}{(e^2/mc^2)^2}=$ critical field strengths

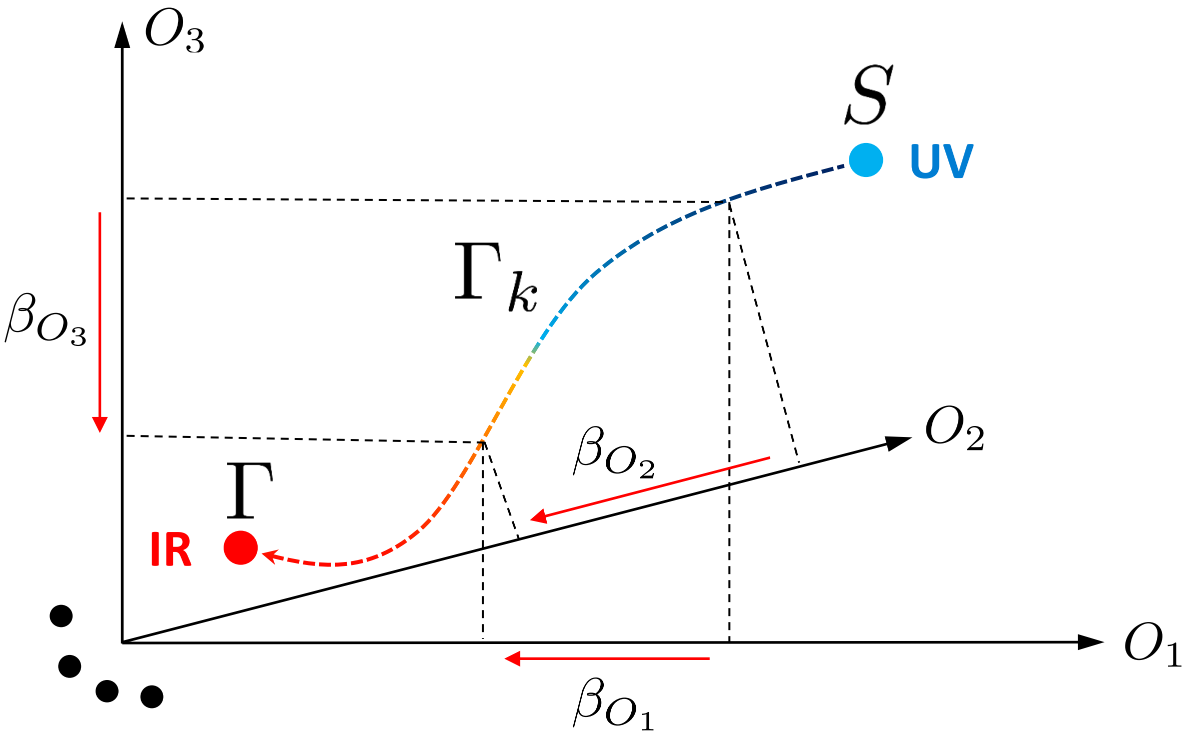
The expansion terms in small fields (compared to $\mathfrak{E}$) describe light-light scattering. The simplest term is already known from perturbation theory. For large fields, the equations derived here differ strongly from Maxwell's equations. Our equations will be compared to those proposed by Born.

FUNCTIONAL RENORMALIZATION GROUP

FRG: nonperturbative method for QFT studies



FUNCTIONAL RENORMALIZATION GROUP

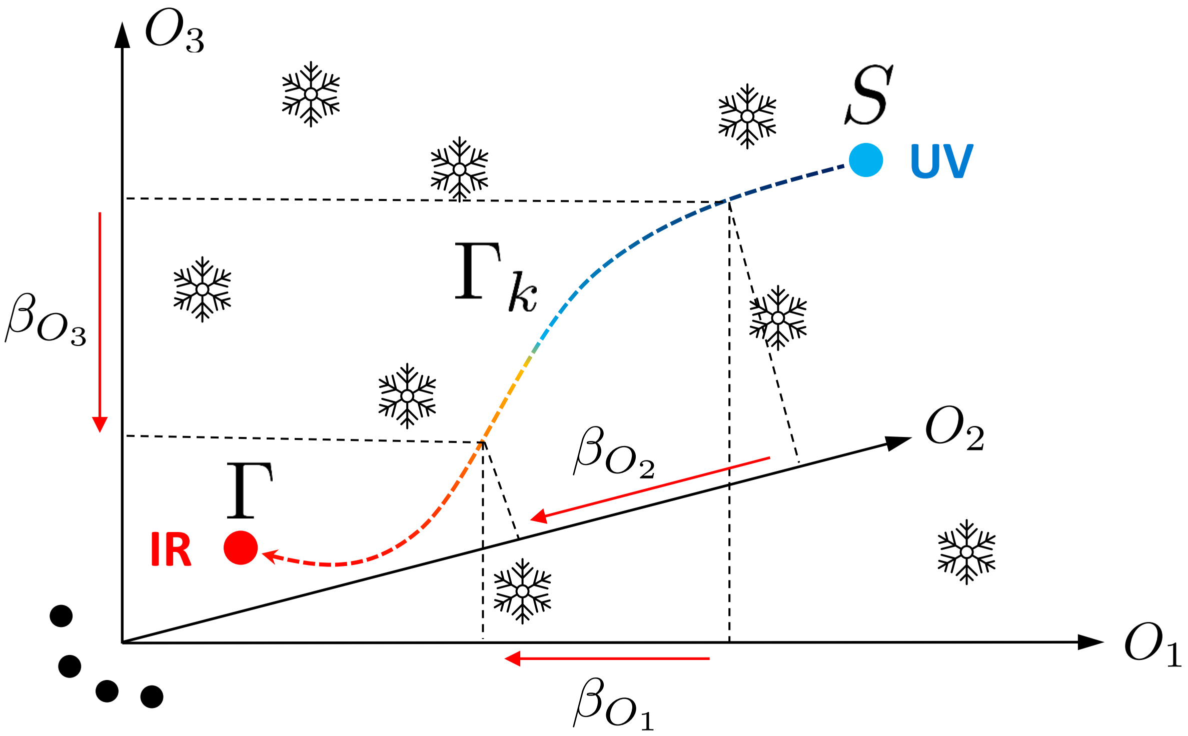


$$k\partial_k\Gamma_k = \frac{1}{2}\mathbf{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right]$$

WETTERICH EQUATION

(WETTERICH `92, MORRIS `94)

FUNCTIONAL RENORMALIZATION GROUP



$$\mathbf{Tr}\left[\left(\Gamma_k^{(2)}+\mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right]\Bigg|_{\mathrm{FP}} = 0$$

FIXED POINT EQUATION

FRG SETUP & EFFECTIVE ACTION

Maxwell action in vacuum:

$$\mathsf{S}\left[\bar{\mathsf{A}}\right] = \int\limits_{\mathbb{R}^{3,1}} \left(-\frac{1}{4}\bar{\mathsf{F}}_{\mu\nu}\bar{\mathsf{F}}^{\mu\nu}\right) \mathrm{d}^4 x$$

$$\bar{\mathsf{F}}_{\mu\nu} = \partial_\mu \bar{\mathsf{A}}_\nu - \partial_\nu \bar{\mathsf{A}}_\mu$$

field strength

FRG SETUP & EFFECTIVE ACTION

Maxwell action in vacuum:

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$$\xrightarrow{\subseteq}$$

Nonlinear electrodynamics:

$$\Gamma\left[\bar{\mathsf{A}}\right] = \int_{\mathbb{R}^{3,1}} \underbrace{\mathcal{L}\left(\bar{\mathsf{F}}, \partial\bar{\mathsf{F}}, \partial^2\bar{\mathsf{F}}, \dots\right)} \mathrm{d}^4x$$

derivative expansion scheme:

$$= \bar{\mathcal{W}}\left(\bar{\mathcal{F}}, \bar{\mathcal{G}}\right) + \dots$$

$$\bar{\mathcal{F}} = \frac{1}{4}\bar{\mathsf{F}}_{\mu\nu}\bar{\mathsf{F}}^{\mu\nu} \qquad \bar{\mathcal{G}} = \frac{1}{4}\bar{\mathsf{F}}_{\mu\nu}\left(\star\bar{\mathsf{F}}\right)^{\mu\nu}$$

fundamental local U(1) invariants

W-Rot.

$$\xleftarrow{}$$

P-Inv.

Euclidean action:

$$\Gamma\left[\bar{A}\right] = \int_{\mathbb{R}^4} \bar{\mathscr{W}}\left(\bar{\mathscr{F}}, \bar{\mathscr{G}}^2\right) \mathrm{d}^4x$$

FRG SETUP & EFFECTIVE ACTION

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fundamental local U(1) invariants

W-Rot.

$$\longleftarrow$$

P-Inv.

Euclidean action:

$$\Gamma_{k}\left[\bar{A}\right] = \int\limits_{\mathbb{R}^4} \bar{\mathscr{W}}_{k}\left(\bar{\mathscr{F}}, \bar{\mathscr{G}}^2\right) \mathrm{d}^4x$$

continuous scale-dependence

re-definitions:

$w_k := k^{-4}\bar{\mathscr{W}}_k$

$\mathscr{F} := Z_k k^{-4}\bar{\mathscr{F}}$

etc. ...

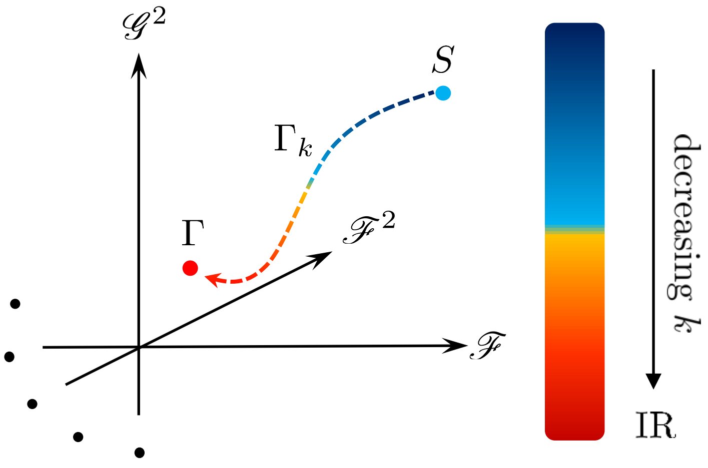
scale-dependent & gauge-fixed effective action:

$$\Gamma_k\left[A\right] = k^4 \int\limits_{\mathbb{R}^4} \left(w_k\left(\mathscr{F}, \mathscr{G}^2\right) + \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2\right) \mathrm{d}^4x$$

FRG Flow

scale-dependent & gauge-fixed effective action:

$$\Gamma_k\left[A\right] = k^4 \int\limits_{\mathbb{R}^4} \left(w_k\left(\mathscr{F}, \mathscr{G}^2\right) + \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \right) \mathrm{d}^4 x$$



FRG: assume the existence of a flow equation for Γ_k.

$$k\partial_k \Gamma_k = \frac{1}{2}\mathbf{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]$$

Regulator:

$$\mathcal{R}_k(p) = Z_k p^2 r\left(\frac{p^2}{k^2}\right)\left[\mathbf{P}_T + \frac{1}{\alpha}\mathbf{P}_L\right]$$

Field space projectors

FRG Flow

$$\Gamma_k\left[A\right] = k^4 \int\limits_{\mathbb{R}^4} \left(w_k\left(\mathscr{F}, \mathscr{G}^2\right) + \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \right) \mathrm{d}^4 x$$

$$\mathcal{R}_k(p) = Z_k p^2 r\left(\frac{p^2}{k^2}\right)\left[\mathbf{P}_T + \frac{1}{\alpha}\mathbf{P}_L\right]$$

$$\boxed{k\partial_k \Gamma_k = \frac{1}{2}\mathbf{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]}$$

Projection on w_k (field strength homogeneity)

$$\boxed{\underbrace{k\partial_k w_k}_{\text{RG time derivative}} + \underbrace{4w_k - \left(\eta_k + 4\right)\left(w_k'\mathscr{F} + 2\dot{w}_k\mathscr{G}^2\right)}_{\text{Scaling contributions}} = -\frac{1}{32\pi^2}\underbrace{\int\limits_{\mathbb{R}^4} y^2\left(\eta_k r + 2y^2 r'\right)Y_k \,\mathrm{d}^4 y}_{\text{Loop level contributions}}}$$

GLOBAL FIXED FUNCTIONS

Q: What are global fixed functions?

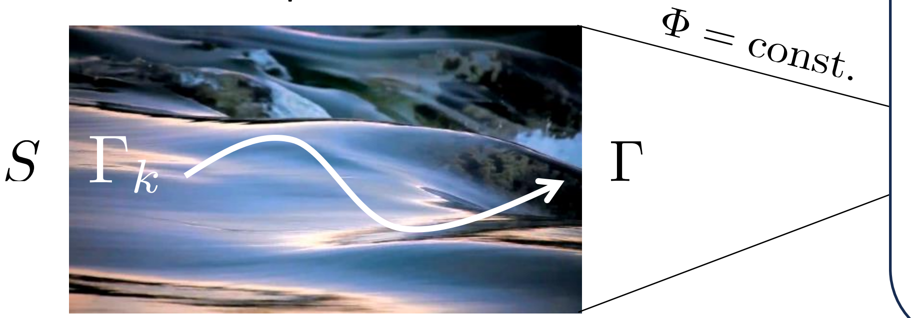
EAA of any theory:

$$\Gamma_k\left[\Phi\right] = k^4 \int_{\mathbb{R}^4} \mathcal{L}_k\left(\Phi, \partial\Phi, \partial^2\Phi, \ldots\right) \, \mathrm{d}^4x$$

collection of d.o.f.

dimensionless Lagrangian density

Assume existence of a flow equation



Define RG time: $t := \ln\left(\frac{k}{\Lambda}\right)$

$$\partial_t \mathcal{L}_k = \frac{1}{2k^4}\mathbf{Tr}\left[G_k \partial_t \mathcal{R}_k\right] - 4\mathcal{L}_k$$

RG stationarity condition: $\partial_t \mathcal{L}_k|_* = 0$

$$\mathcal{L}_* = \frac{1}{8k^4}\mathbf{Tr}\left[G_k \partial_t \mathcal{R}_k\right]\Big|_*$$

Fixed Function Equation (FFE)

FFE: PDE for **fixed function**; global solution in the mathematical sense defines a global fixed function.

TRUNCATIONS

Flow equation for nonlinear electrodynamics:

$$w_* = \left(1+\frac{\eta_*}{4}\right)\left(w'_*\mathscr{F} + 2\dot{w}_*\mathscr{G}^2\right) - \frac{1}{32\pi^2}\int_{\mathbb{R}^4} y^2\left(\eta_* r + 2y^2 r'\right) Y_*\, \mathrm{d}^4 y$$

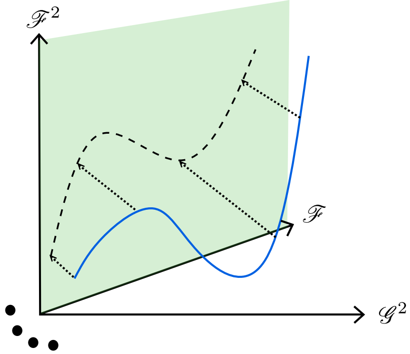
Partial differential eq.

- contains derivatives of fixed function,
- highly nonlinear

I. Theory Space Truncation

Discard dependence on the pseudo-scalar invariant:

$$w_*\left(\mathscr{F}, \mathscr{G}^2\right) \to w_*\left(\mathscr{F}\right)$$



II. Field Space Projection

Restrict on self-dual field configurations:

$$F = \star F$$

$$\Downarrow$$

$$(Fy)^2 = (\star F y)^2 = \mathscr{F} y^2$$

WILSON-FISHER PROCEDURE

Inspired from the Wilson-Fisher fixed point solution;
O(1,$\mathbb{R}$) Ising model in 1+2 dimensions

EAA ansatz:

$$\Gamma_k\left[\phi\right] = \int\limits_{\mathbb{R}^3} \left(\frac{k}{2}(\partial_\mu \phi)^2 + k^3 V_k(\phi) \right) \mathrm{d}^3 x$$

scale-dependent,
dimensionless
effective potential

FFE:

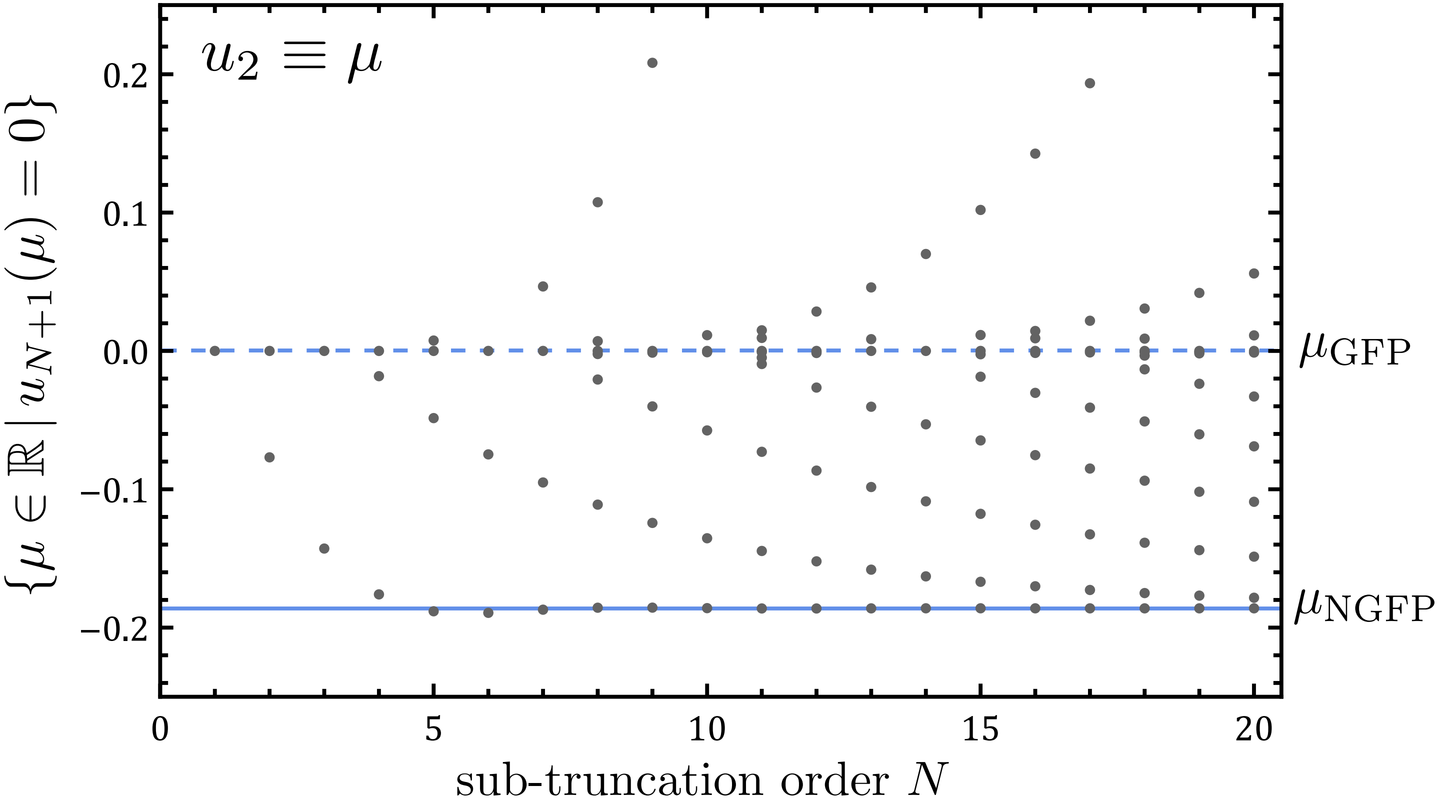
$$V_* = \frac{1}{18\pi^2} \frac{1}{1+V_*''} + \frac{1}{6}\phi V_*'$$

WILSON-FISHER PROCEDURE

Inspired from the Wilson-Fisher fixed point solution;
O(1,ℝ) Ising model in 1+2 dimensions

Expansion for small fields:

$$V_*(\phi) = \sum_{i=0}^{N} \frac{u_i}{i!}\phi^i$$



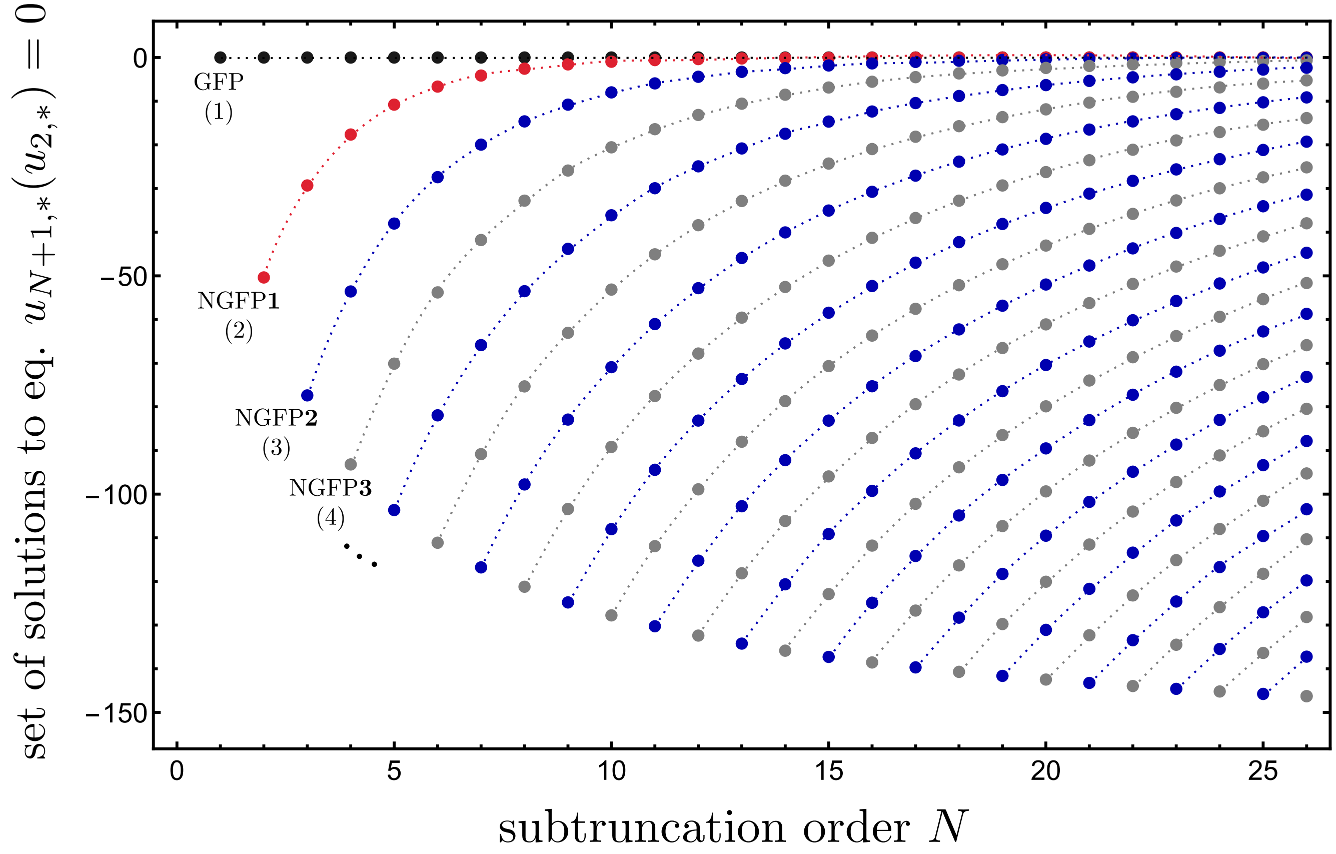
WILSON-FISHER PROCEDURE

Inspired from the Wilson-Fisher fixed point solution;
O(1,ℝ) Ising model in 1+2 dimensions

Expansion for small fields:

$$w_*(\mathscr{F}) = \sum_{i=0}^{N} u_i \mathscr{F}^i$$

NLED



Very similar to shift-symmetric scalar fields

de Brito, Knorr & Schiffer (23):
no legitimate solutions to flow equation

η-PERSPECTIVE

$$\mu(\eta_*) = 96\pi^2 \frac{\eta_*}{8-\eta_*}$$

e.g. 1-loop pert. theory of photon field:

$$\eta_{\mathrm{ph}} \simeq \frac{2\alpha}{3\pi}$$

Instead of tracing successive sub-truncation orders (Wilson-Fisher), consider the anomalous dimension as an external parameter.

→ Fixes all coefficients of small field expansion

$$w_*(\mathscr{F}) = \sum_{i=0}^{N} u_i \mathscr{F}^i$$

✓

Expansion for large values of invariant

$$w_*(\mathscr{F}) = c + \lambda \mathscr{F}^{\Delta} + \sum_{I=1}^{\infty}\sum_{a=1}^{I} \lambda_I^a \mathscr{F}^{a\Delta - I}$$

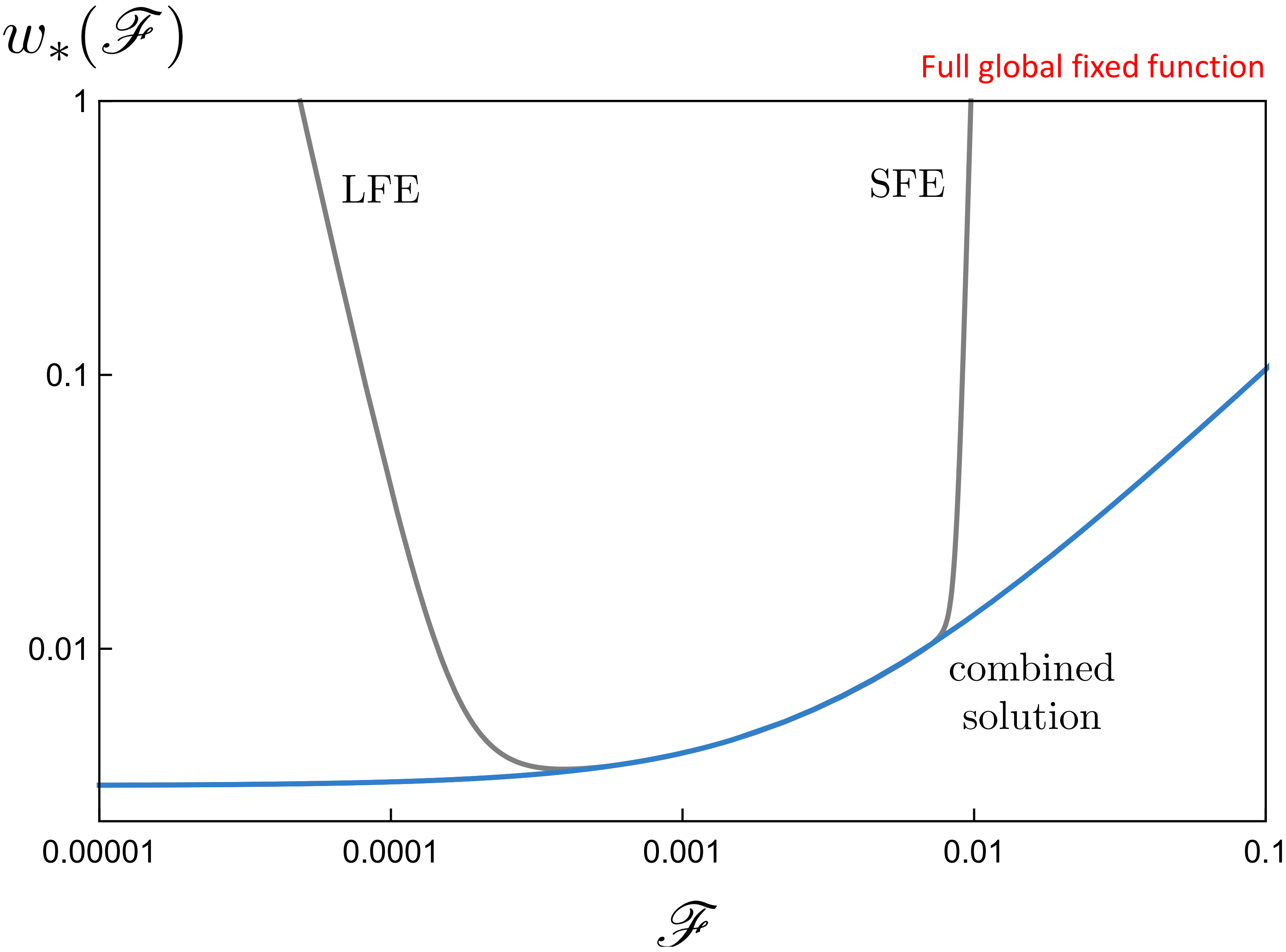
constant

free parameter

$$= \frac{4}{4+\eta_*}$$

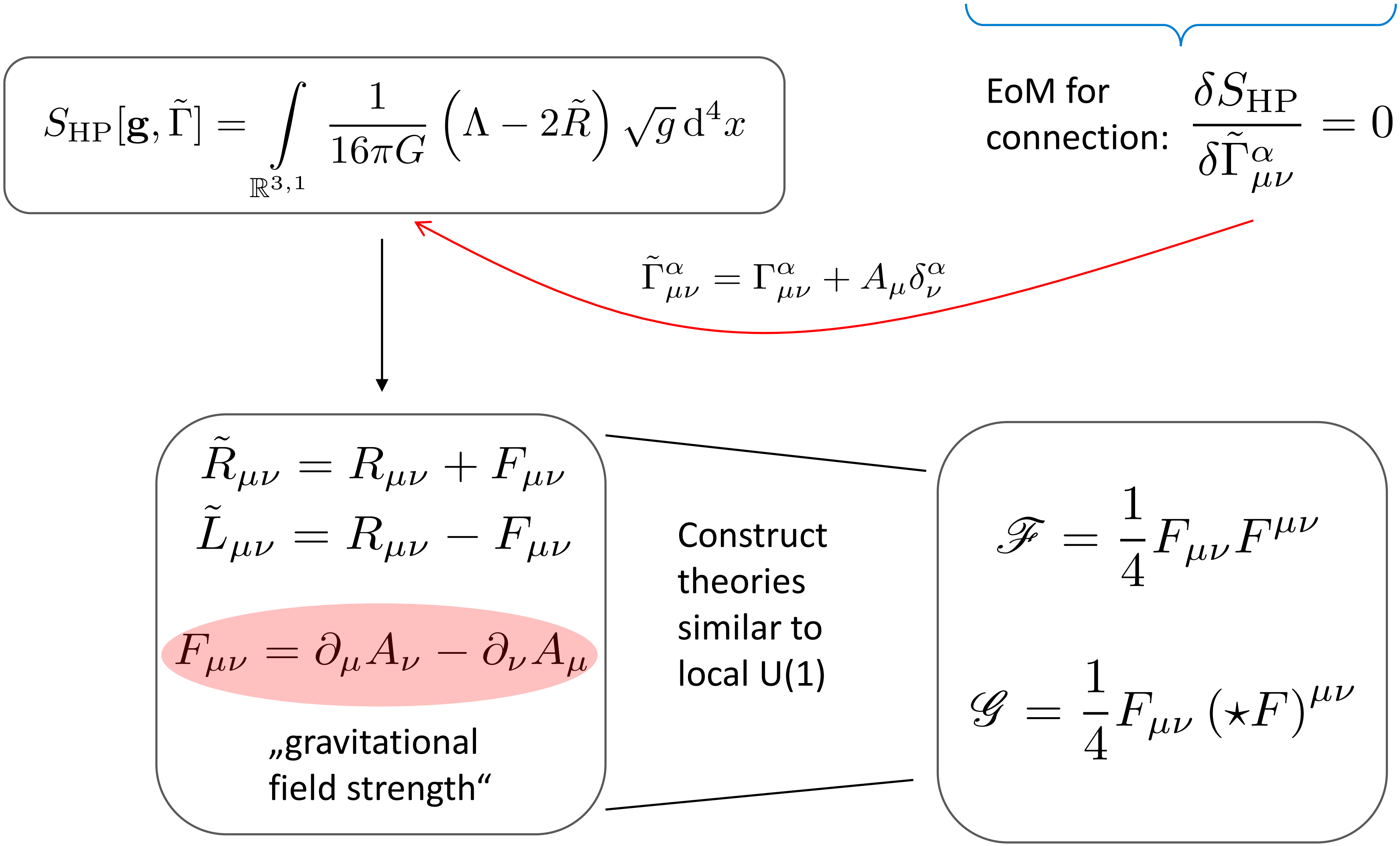
unknown coefficients

η-PERSPECTIVE



LINK TO GRAVITY

Gies, Salek ('22): Asymptotically Safe Hilbert-Palatini Gravity in an On-Shell Reduction Scheme



$$S_{\mathrm{HP}}[\mathbf{g},\tilde{\Gamma}] = \int\limits_{\mathbb{R}^{3,1}} \frac{1}{16\pi G}\left(\Lambda - 2\tilde{R}\right)\sqrt{g}\,\mathrm{d}^4x$$

EoM for connection: $\dfrac{\delta S_{\mathrm{HP}}}{\delta \tilde{\Gamma}^{\alpha}_{\mu\nu}} = 0$

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + A_\mu \delta^{\alpha}_{\nu}$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + F_{\mu\nu}$$

$$\tilde{L}_{\mu\nu} = R_{\mu\nu} - F_{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

„gravitational field strength“

Construct theories similar to local U(1)

$$\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

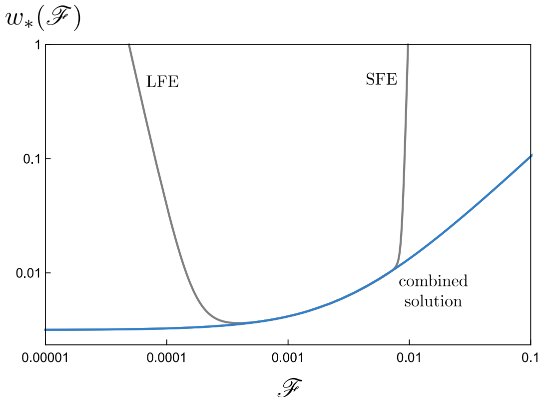
$$\mathscr{G} = \frac{1}{4}F_{\mu\nu}\left(\star F\right)^{\mu\nu}$$

SUMMARY

- First derivation of full functional RG flow for NLED.

$$k\partial_k w_k + 4w_k - (\eta_k + 4)\left(w_k'\mathscr{F} + 2\dot{w}_k\mathscr{G}^2\right) = -\frac{1}{32\pi^2}\int\limits_{\mathbb{R}^4} y^2(\eta_k r + 2y^2 r')Y_k\,\mathrm{d}^4 y$$

- Globally existing fixed function for pure $\mathscr{F}$-dependencies.
  ⇒ No Landau singularities in strong field regime of NLED.



➡ How does this result extend to more general/complete systems? ?

THANK YOU FOR LISTENING!

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