Renormalization of Scalar Field Theories in Riemannian Manifolds with Boundaries

Physik Combo Talk

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Motivation

- Investigation of QFT in the presence of branes
- Description of defects or junctions, e.g. in the context of conformal field theory
- Quantifying Casimir effect in curved backgrounds
- Deriving critical boundary exponents

General Problem

- Divergence of non-linear field operators, e.g. $\langle\hat{\phi}^2(x)\rangle$

Pointsplit Renormalization (without boundary)

Let $H(x,x')$ the Hadamard parametrix (divergent part of 2pt correlator) then the regularized squared field operator is defined as:

$$\langle\hat{\phi}^2(x)\rangle_H := \lim_{x'\to x}\left[\langle\hat{\phi}(x)\hat{\phi}(x')\rangle - H(x,x')\right]$$

In 4D one has: $H(x,x') = \frac{U(x,x')}{\sigma(x,x')} + V(x,x')\log(\sigma(x,x'))$
with $\sigma(x,x')$ as the half squared geodesic distance.

- Introduction of boundary creates additional divergences
  $\Rightarrow$ find universal divergent structure on boundary: find $H_\partial(x,x')$

Our Model under Consideration

- Riemannian manifold M with boundary ∂M and field ϕ defined by:
  $\mathscr{L} = -\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{m^2}{2}\phi^2 - \frac{\zeta}{2}R\phi^2 - \frac{\lambda}{4!}\phi^4$
- Free dynamics:
  $\hat{\mathscr{K}}\phi := (\triangle_g - m^2 - \zeta R)\phi = 0$
- Use Gaussian coordinates where:
  - ξ is Riemannian normal coordinate from $\bar{x}$ to $\bar{x}'$
  - z, z' affine parameters of normal geodesics from $\bar{x}, \bar{x}'$ to x, x'
  - Metric expansion:

$$ds^2 = dz^2 + g_{ab}(x)d\xi^a d\xi^b$$

$$g_{ab}(x) = \delta_{ab} + [2\overline{K}_{ab}z] + [[\overline{K}_{ac}\overline{K}^c_b - \overline{R}_{a0b0}]z^2 + 2D_c\overline{K}_{ab}\xi^c z - \frac{1}{3}\hat{\overline{R}}_{acbd}\xi^c\xi^d] + \ldots$$

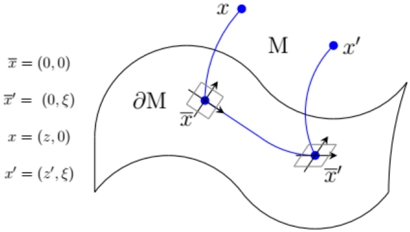

FIG.1: Sketch of the coordinates.

Goal and Approach

- Goal of this talk: find a local, covariant parametrix H_∂ for pointsplit
- First candidate: singular part of Greens function $G(x,x')$ to $\hat{\mathscr{K}}$
- From spectral calculus: $G(x,x') = \int_0^\infty K(\tau|x,x')d\tau$
- $K(\tau|x,x')$ is the heat kernel satisfying:

$$
\begin{aligned}
(\partial_\tau - \hat{\mathscr{K}})K(\tau|x,x') &= 0 \\
K(\tau|x,x')|_{(x\vee x')\in\partial M} &= 0 \\
\lim_{\tau\to 0} K(\tau|x,x') &= \delta(x,x')
\end{aligned}
$$

- Note: divergent part is encoded in lower integration boundary $\Rightarrow$ asymptotic expansion of heat kernel suffices
- Asymptotic expansion of heat kernel already investigated by Grieser (2004) ($\Rightarrow$ existence, uniqueness)

Defining Function Space Ψ_∂^α

Definition

Let $\alpha \leq 0$ then $\Psi_\partial^\alpha \subset C^\infty((0,\infty) \times M^2)$ such that $\forall\ A \in \Psi_\partial^\alpha$ holds that:

(a) $D^\gamma_{\tau,x,y} A(\tau, x, y)$ decays rapidly for $x \neq y$ at $t \to 0$

(b) $\forall p \in M/\partial M\ \exists$ local coordinate system $U \ni p$ and a function $\tilde{A}^{int} \in C^\infty([0,\infty) \times \mathbb{R}^4 \times U)$ with rapid decay for $\tilde{A}^{int}(\tau, X, Z, y)$ and its derivatives as $(|X| + |Z|) \to \infty$ such that:
$A(\tau, x, y) = \tau^{-3-\alpha} \tilde{A}^{int}(\tau, \frac{\xi-\eta}{\sqrt{\tau}}, \frac{z-w}{\sqrt{\tau}}, y)$

(c) $\forall p \in \partial M\ \exists$ local coordinate system $U \ni p$ and functions $\tilde{A}^{dir} \in C^\infty([0,\infty) \times \mathbb{R}^4 \times U)$, $\tilde{A}^{refl} \in C^\infty([0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^2_+ \times U \cap \partial M)$ with rapid decay for $\tilde{A}^{dir}$ as in (b) and for $\tilde{A}^{refl}(\tau, X, Z, W, \hat{y})$ as $(|X| + |Z| + |W|) \to \infty$ such that:

$$A(\tau, x, y) = \tau^{-3-\alpha}[\tilde{A}^{dir}(\tau, \frac{\xi-\eta}{\sqrt{\tau}}, \frac{z-w}{\sqrt{\tau}}, y) - \tilde{A}^{refl}(\tau, \frac{\xi-\eta}{\sqrt{\tau}}, \frac{z}{\sqrt{\tau}}, \frac{w}{\sqrt{\tau}}, \hat{y})]$$
$$=: \tau^{-3-\alpha} \tilde{A}^{bd}(\tau, \frac{\xi-\eta}{\sqrt{\tau}}, \frac{z}{\sqrt{\tau}}, \frac{w}{\sqrt{\tau}}, \hat{y})$$

Properties of Ψ_∂^α

- For $A \in \Psi_\partial^\alpha$ define interior and boundary leading parts of A as:
  - $\Phi_\alpha^{int}(A) := \tilde{A}^{int}(0, X, Z, y)$ and $\Phi_\alpha^{bd}(A) := \tilde{A}^{bd}(0, X, Z, W, \hat{y})$

Lemma: Properties of Ψ_∂^α

Let $A \in \Psi_\partial^\alpha$ and $B \in \Psi_\partial^\beta$ for $\alpha, \beta \leq 0$ then:

(a) $\Psi_\partial^{\alpha-\frac{1}{2}} \subset \Psi_\partial^\alpha$ and if $\Phi_\alpha^{int/bd}(A) = 0 \Rightarrow A \in \Psi_\partial^{\alpha-\frac{1}{2}}$

(b) If $\alpha \leq -1$ then $R := (\partial_\tau - \hat{\mathscr{K}})A \in \Psi_\partial^{\alpha+1}$

(c) The convolution $(A * B) \in \Psi_\partial^{\alpha+\beta}$ with

$$(A * B)(\tau|x, x') := \int_0^\tau d\tau' \int_M dV(y) A(\tau - \tau'|x, y) B(\tau'|y, x')$$

Existence Theorem by Grieser

Theorem: Existence of an Asymptotic Expansion

Assume $K_1 \in \Psi_\partial^{-1}$ satisfying:

$$\begin{aligned} (i) \quad & (\partial_\tau - \hat{\mathscr{K}})K_1 = R \in \Psi_\partial^{-\frac{1}{2}} \\ (ii) \quad & K_1(\tau, x, x') = 0 \ \text{ for } x \vee x' \in \partial M \\ (iii) \quad & \lim_{\tau \to 0^+} K_1(\tau, x, x') = \delta(x, y) \end{aligned}$$

Then we have that:

(a) Volterra series $K := K_1 - (K_1 * R) + (K_1 * (R * R)) - (K_1 * (R * (R * R))) + \dots$ converges in $C^\infty((0, \infty) \times M^2)$ and $K \in \Psi_\partial^{-1}$

(b) K is a Dirichlet heat kernel

(c) The Volterra series is an asymptotic series as $\tau \to 0$

Can we find a suitable K_1

- Yes! Let K_1 be the euclidean heat kernel satisfying our boundary conditions, meaning that (*ii*) and (*iii*) are fulfilled:

$$K_1(\tau|x,x') := \frac{e^{-\frac{\xi^a\xi^b}{4\tau}g_{ab}(x')}}{(4\pi\tau)^2}\left[e^{\frac{-(z-z')^2}{4\tau}} - e^{\frac{-(z+z')^2}{4\tau}}\right] \in \Psi_\partial^{-1}$$

- One would expect $(\partial_\tau - \hat{\mathscr{K}})K_1 = R \in \Psi_\partial^0$
- However, since K_1 is an euclidean heat kernel $\Rightarrow \Phi_0^{int/bd}(R) = 0$ and hence $R \in \Psi_\partial^{-\frac{1}{2}}$, which is precisely requirement (*i*)
- Volterra series gives rise to asymptotic expansion of a heat kernel K $\Rightarrow$ need to compute contributions up to a given order of interest

Explicit Calculation

- Calculate $R = (\partial_\tau - \hat{\mathscr{K}})K_1$ and $(K_1 * (R * (R * \ldots)))$ up to a given order of interest
- Calculating convolutions in bulk is no problem (Gaussian integrals over $\mathbb{R}^4$, and τ' integration leads to beta functions)
- Calculating convolutions at the boundary introduces some subtleties, due to the restriction of integrating over half space

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$\Rightarrow$ However, it can be shown that all integrals can be solved explicitely

$\Rightarrow$ At each order, K can be expressed as linear combinations of error functions and Gaussians

General Form of the Coefficients

Theorem: General structure of the Coefficients

Let $N, M, n, m \in \mathbb{N}_0$, $k \in \mathbb{Z}$, $\tilde{N} \in \mathbb{Z}_2$ and $\delta = N + \tilde{N} - 1$.
At order M in curvature quantities, K takes the form:

$$K^{(M)} = \left([P]_-^M e^{\frac{-(z-z')^2}{4\tau}} + [P]_+^M e^{\frac{-(z+z')^2}{4\tau}} + [P]_0^M \frac{1}{\sqrt{\tau}} \operatorname{erfc}\left[\frac{z+z'}{2\sqrt{\tau}}\right] \right) \frac{e^{\frac{-|\xi|^2}{4\tau}}}{(4\pi\tau)^2}$$

$[P]_-^M \in \mathrm{Span}\{z^n z'^m \tau^{k+\tilde{N}/2} f(\frac{\xi}{\sqrt{\tau}}) | f \in \Gamma^{(M)}_{N,\tilde{N}}, N \leq M - \tilde{N} = n + m + 2k, k \geq 0\}$

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$[P]_0^M \in \mathrm{Span}\{z^n z'^m \tau^{k+\tilde{N}/2} f(\frac{\xi}{\sqrt{\tau}}) | f \in \Gamma^{(M)}_{N,\tilde{N}}, N \leq M - \tilde{N} = n + m + 2k - 1, k \geq -\delta\}$

where $\Gamma^{(M)}_{N,\tilde{N}}$ is the set of order $2N + \tilde{N}$ polynomials at curvature quantity order M with a "regularizing effect".

Tensor Structure

Definition: Regularizing Effect

For $N, M \in \mathbb{N}_0$, $\tilde{N} \in \mathbb{Z}_2$, $s \in (0,1)$, let $f \in \Gamma^{(M)}_{N,\tilde{N}}$, then the "regularizing effect" means that:

$$I[f] := \int_{R^3} s^{\tilde{N}/2} f(Y/\sqrt{s}) \frac{e^{\frac{-|Y-sX|^2}{4s(1-s)}}}{\sqrt{4\pi s(1-s)}^3} d^3Y = s^{N+\tilde{N}} f(X)$$

- Concrete example: let's consider $g \in \Gamma^{(2)}_{2,0}$
$g(\frac{Y}{\sqrt{s}}) := \frac{Y^a Y^b Y^c Y^d}{s^2} \overline{K}_{ab}\overline{K}_{cd} - \frac{Y^a Y^b}{s}[8\overline{K}_{ac}\overline{K}^c_b + 4\overline{K}\overline{K}_{ab}] + [8\overline{K}_{ab}\overline{K}^{ab} + 4\overline{K}^2]$
- One can check: $I[g] = s^2 g(X)$ (note: changing one of the coefficients would generate additional terms linear in s and independent of s)

What are the Subtleties of the Calculation

- Difficulties can be traced back to the occurence of the following family of integrals:

$$I_{l,k}^{(a,b)} := \int_0^1 (1-s)^{\frac{l}{2}} s^{\frac{k}{2}} \operatorname{erfc}\left[\frac{a}{2}\sqrt{\frac{s}{1-s}}Z + \frac{b}{2}\sqrt{\frac{1-s}{s}}Z'\right] ds$$

  where $a, b = \pm 1$ and $l, k \in \mathbb{Z}$.

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  where $a, b = \pm 1$ and $l, k \in \mathbb{Z}$.

- Origin of error function is integral over the half space
- One can show the following:
  $\Rightarrow$ only need $l \in 2\mathbb{N}_0 \Rightarrow$ w.l.o.g. set $l = 0$
  $\Rightarrow$ k is even, e.g. $k = 2q$ with $q \in \mathbb{Z}$
  $\Rightarrow$ q is non negative, e.g. $q \in \mathbb{N}_0$ (requires the "regularizing effect")
- $I_{0,2q}^{(a,b)}$ for $q \in \mathbb{N}_0$ can be solved explicitly

Further Considerations

- We saw that there are negative powers of τ in the prefactors of
$$K = \frac{e^{\frac{-|\xi|^2}{4\tau}}}{(4\pi\tau)^2}\left([P]_- e^{\frac{-(z-z')^2}{4\tau}} + [P]_+ e^{\frac{-(z+z')^2}{4\tau}} + [P]_0 \frac{\operatorname{erfc}\left[\frac{z+z'}{2\sqrt{\tau}}\right]}{\sqrt{\tau}}\right)$$
- Can we absorb the negative τ powers into the exponentials?

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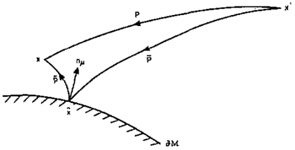

FIG.2: Sketch of $\sigma/\overline{\sigma}$

Theorem

Let $\sigma/\overline{\sigma}$ the direct/reflected half squared geodesic distence between x and x' then one can rewrite:

$$K = \frac{1}{(4\pi\tau)^2}\left[\Omega(\tau|x,x')e^{\frac{-\sigma(x,x')}{2\tau}} + \overline{\Omega}(\tau|x,x')e^{\frac{-\overline{\sigma}(x,x')}{2\tau}}\right]$$

here $\Omega/\overline{\Omega}$ contain only non negative powers of τ

- the expression above was also investigated by McAvity and Osborn

Outlook

- What was achived?
  - Showed that asymptotic coefficients can be expressed analytically at each order
  - Showed symmetry of asymptotic coefficients
  - Concrete calculation up to fourth order in curvature quantities via Mathematica

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- What was achived?
  - Showed that asymptotic coefficients can be expressed analytically at each order
  - Showed symmetry of asymptotic coefficients
  - Concrete calculation up to fourth order in curvature quantities via Mathematica
- What are the next steps?
  - Perform τ integration to obtain singular structure of $G(x, x')$
  - Check that this gives rise to a Hadamard parametrix
  - Compute physical quantities (energy densities, critical boundary exponents)

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