Quasinormal modes of Schwarzschild de Sitter black holes from Liouville CFT

Paolo Arnaudo

SISSA, Trieste, Italy

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Based on
G. Aminov, P.A., G. Bonelli, A. Grassi, and A. Tanzini,
Black hole perturbation theory and multiple polylogarithms,
arXiv:2307.10141[hep-th]

Quantum Effects in Gravitational Fields

Leipzig University

Introduction

- The recent experimental verification of gravitational waves renewed the interest in theoretical studies of General Relativity and black hole perturbation theory.
- In particular, we look for exact computational techniques to produce high precision tests of General Relativity equations by computing analytical expressions for significant gravitational quantities.

The QNM frequencies

- The QNMs are quantized frequencies which can be seen as characteristic oscillations of black holes, and are responsible for the damped oscillations appearing, for example, in the ringdown phase of two colliding black holes.
- Mathematically, the QNMs arise in the analysis of a linear perturbation ψ around fixed gravitational backgrounds. The perturbation usually obeys linear 2nd order differential equations with singularities, whose symmetry properties are dictated by the symmetries of the background. The quasinormal modes are obtained by imposing suitable boundary conditions to the perturbation fields.

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$$\psi = \psi_{\rm in}^{\rm hor} = \mathcal{M}_1 \psi_1^{\rm sing} + \mathcal{M}_2 \psi_2^{\rm sing} \tag{1}$$

The connection coefficients $\mathcal{M}_1, \mathcal{M}_2$ analytically continue the (ingoing) local solution at the horizon, providing a linear combination of two independent local solutions around the singularity closest to the region where the second boundary condition is imposed.

Schwarzschild de Sitter BH in 4 dimensions

The metric describing the Schwarzschild de Sitter black hole in four dimensions (SdS_4) is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{2}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2, \tag{3}$$

where M is the mass of the black hole and Λ is the cosmological constant. In what follows, we will fix $\Lambda = 3$.
We will denote the roots of $rf(r) = 0$ by

$$R_h, \quad R_\pm, \tag{4}$$

where

$$R_\pm = \frac{-R_h \pm \sqrt{4 - 3R_h^2}}{2}. \tag{5}$$

We will deal with the *small black hole regime* $R_h \ll 1$.

Regge-Wheeler Equation

We study a class of linear perturbations with spin $s \in \{0, 1, 2\}$, encoded in the following radial equation

$$\left(\partial_r^2 + \frac{f'(r)}{f(r)}\partial_r + \frac{\omega^2 - V(r)}{f(r)^2}\right)\Phi(r) = 0, \tag{6}$$

where the potential is

$$V(r) = f(r)\left[\frac{\ell(\ell+1)}{r^2} + (1 - s^2)\left(\frac{2M}{r^3}\right)\right]. \tag{7}$$

As boundary conditions, we will require the presence of only ingoing modes near the horizon, and the presence of only outgoing modes near the cosmological horizon. In terms of the *tortoise coordinate* r_*, we ask that Φ behaves as $\exp(-i\omega r_*)$ for $r \sim R_h$ and as $\exp(i\omega r_*)$ for $r \sim R_+$.

Heun equation

Under the change of variable

$$z(r) = \frac{r(R_+ - R_-)}{R_+(r - R_-)}, \tag{8}$$

and redefinition of the wave function

$$\psi(z) = z^{-\gamma/2}(z-1)^{-\delta/2}(z-t)^{-\epsilon/2}\sqrt{f(r)}\,(r - R_-)^{-1}\,\Phi(r), \tag{9}$$

where

$$\begin{aligned} t &= \frac{R_h(R_- - R_+)}{R_+(R_- - R_h)}, \quad \delta = 1 - \frac{2i\,\omega\,R_+}{(R_+ - R_h)(R_+ - R_-)}, \\ \gamma &= 1 - 2s, \quad \epsilon = 1 + \frac{2i\omega R_h}{1 - 3R_h^2}, \end{aligned} \tag{10}$$

the equation becomes a **canonical Heun equation**

$$\left(\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}\right)\psi(z) = 0, \tag{11}$$

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon.$$

Boundary conditions

In the z coordinate, the horizon $r = R_h$ is mapped to $z = t$, and the cosmological horizon $r = R_+$ is mapped to $z = 1$.
The independent solutions of the canonical Heun equation for $z \sim t$ are

$$\psi_-^{(t)}(z) = \mathrm{Heun}\left(\frac{t}{t-1}, \frac{q - t\alpha\beta}{1-t}, \alpha, \beta, \epsilon, \delta, \frac{z-t}{1-t}\right),$$
$$\psi_+^{(t)}(z) = (z-t)^{1-\epsilon}\mathrm{Heun}\left(\frac{t}{t-1}, \frac{q - (\beta-\gamma-\delta)(\alpha-\gamma-\delta)t - \gamma(\epsilon-1)}{1-t}, -\alpha+\gamma+\delta, -\beta+\gamma+\delta, 2-\epsilon, \delta, \frac{z-t}{1-t}\right), \tag{12}$$

and the ones for $z \sim 1$ are

$$\psi_-^{(1)}(z) = \left(\frac{z-t}{1-t}\right)^{-\alpha}\mathrm{Heun}\left(t, q + \alpha(\delta-\beta), \alpha, \delta+\gamma-\beta, \delta, \gamma, t\frac{1-z}{t-z}\right),$$
$$\psi_+^{(1)}(z) = (z-1)^{1-\delta}\left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta}\mathrm{Heun}\left(t, q - (\delta-1)\gamma t - (\beta-1)(\alpha-\delta+1), -\beta+\gamma+1, \alpha-\delta+1, 2-\delta, \gamma, t\frac{1-z}{t-z}\right). \tag{13}$$

Local solutions

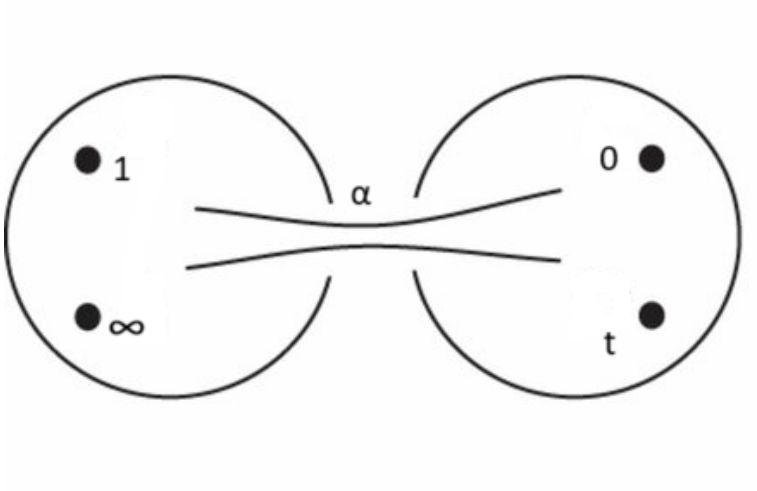
The boundary conditions imply the following behaviors for the function ψ:

$$\begin{aligned} \psi(z) &\sim 1 && \text{for} \quad z \sim 1, \\ \psi(z) &\sim (z-t)^{1-\epsilon} && \text{for} \quad z \sim t. \end{aligned} \tag{14}$$

Taking into account these boundary conditions, the connection coefficient between $\psi_+^{(t)}$ and $\psi_+^{(1)}$ has to be set equal to zero:

$$\psi_+^{(t)}(z) = \mathcal{M}_{+-}\psi_-^{(1)}(z) + \underbrace{\mathcal{M}_{++}}_{=0}\psi_+^{(1)}(z). \tag{15}$$

In the small black hole regime, we have $0 < |t| \ll 1$:



Brief intro to Liouville CFT

- Liouville CFT is an interacting CFT, with coupling b, and with central charge $c = 1 + 6Q^2$, where $Q = b + b^{-1}$.
- The spectrum of the primary fields of the theory is diagonal and continuous. The conformal dimension Δ is parametrized by $\Delta = \frac{Q^2}{4} - \alpha^2$, with $\alpha \in i\mathbb{R}$.
- The 3 point functions $C_{\alpha_1\alpha_2\alpha_3}$ are known explicitly in terms of special functions (DOZZ formula).

BPZ equation

Liouville CFT allows the existence of reducible representations of the Virasoro algebra. Any invariant submodule is generated by a *null state*, which is annihilated by all $L_{n>0}$. An example of null state is

$$\left[b^{-2}L_{-1}^2 + L_{-2}\right] V_{\alpha_{2,1}}(z), \quad \alpha_{2,1} = -b - \frac{1}{2}b^{-1}. \tag{16}$$

From

$$\langle \prod_{i=1}^{n-1} V_{\alpha_i}(z_i) \left[b^{-2}L_{-1}^2 + L_{-2}\right] V_{\alpha_{2,1}}(z)\rangle = 0, \tag{17}$$

it follows that

$$\psi^{(b)} = \langle \prod_{i=1}^{n-1} V_{\alpha_i}(z_i) V_{\alpha_{2,1}}(z)\rangle \tag{18}$$

in the semiclassical limit

$$b \to 0, \quad \alpha_i \to \infty, \quad b\alpha_i = a_i \text{ finite} \tag{19}$$

satisfies a 2nd order ODE with regular singularities at $z = z_i$.

Operator Product Expansion

The product of two operators has the following structure (OPE)

$$V_{\alpha_t}(t)V_{\alpha_0}(0) = \int_{i\mathbb{R}} \mathrm{d}\alpha\, C^{\alpha}_{\alpha_t\alpha_0}\, t^{\Delta_\alpha-\Delta_{\alpha_1}-\Delta_{\alpha_0}}\left(V_\alpha(0) + c_1\, t\, L_{-1}\, V_\alpha(0) + \mathcal{O}\left(t^2\right)\right), \tag{20}$$

where the series coefficients are determined by the Virasoro algebra.
If the OPE is performed in a correlator, one obtains the *conformal block* expansion:

$$\langle V_{\alpha_\infty}(\infty)V_{\alpha_1}(1)V_{\alpha_t}(t)V_{\alpha_0}(0)\rangle = \int_{i\mathbb{R}} \mathrm{d}\alpha\, C_{\alpha_\infty\alpha_1\alpha}\, C^{\alpha}_{\alpha_t\alpha_0}\left|\underbrace{t^{\Delta_\alpha-\Delta_{\alpha_1}-\Delta_{\alpha_0}}\left(1+\mathcal{O}(t)\right)}_{\mathfrak{F}(t)}\right|^2. \tag{21}$$

The OPE involving $V_{\alpha_{2,1}}$ is easier:

$$V_{\alpha_{2,1}}(z)V_{\alpha_i}(z_i) = \sum_{\pm} C^{\alpha_i\pm\frac{b}{2}}_{\alpha_{2,1}\alpha_i}(z-z_i)^{\frac{bQ}{2}\mp\alpha_i}\, V_{\alpha_i\mp}(0) + \mathcal{O}\left((z-z_i)^{\frac{bQ}{2}\mp\alpha_i+1}\right) \tag{22}$$

and provides local expansions of the solution of the ODE.

Crossing symmetry

Applying the OPE inside a correlator in different ways provides different expansions of the correlator. These must agree $\Rightarrow$ **crossing symmetry**.

$$\langle \prod_{i=1}^{n-1} V_{\alpha_i}(z_i) V_{\alpha_{2,1}}(z) \rangle \sim \begin{cases} \sum_{\pm} (\text{DOZZ terms}) \times \left| \mathfrak{F}_{\pm}^{(z_1)}(z - z_1) \right|^2 \\ \sum_{\pm} (\text{DOZZ terms}) \times \left| \mathfrak{F}_{\pm}^{(z_2)}(z - z_2) \right|^2 . \end{cases} \tag{23}$$

Plugging in an Ansatz of the form

$$\mathfrak{F}_{\pm}^{(z_1)}(z - z_1) = \sum_{\pm'} \mathcal{M}_{\pm\pm'} \mathfrak{F}_{\pm'}^{(z_2)}(z - z_2) \tag{24}$$

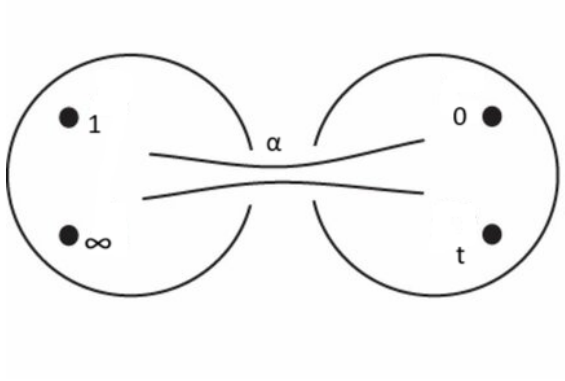
makes it possible to solve in exact form for the coefficients $\mathcal{M}_{\pm\pm'}$.

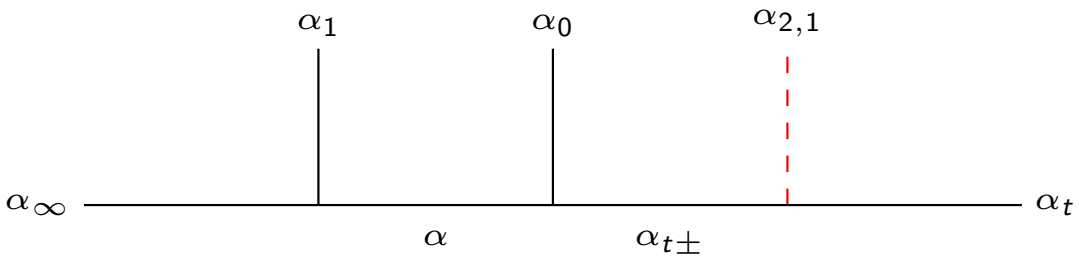
[G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, Commun.Math.Phys. 397 (2023) 2, 635-727]

Thanks to **AGT correspondence**, these have explicit combinatorial expressions.

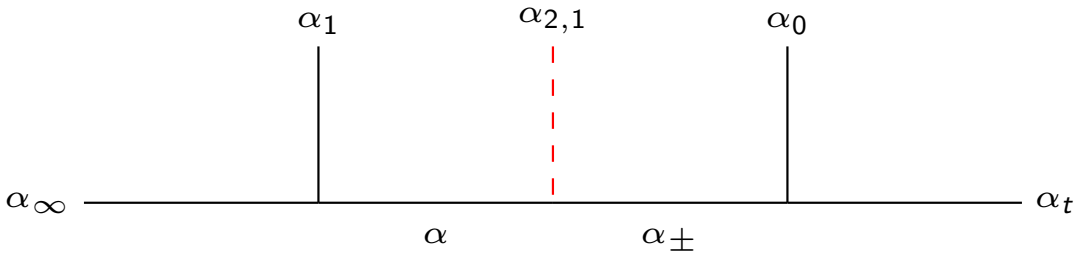
[L. F. Alday, D. Gaiotto, and Y. Tachikawa, Letters in Mathematical Physics 91.2 167–197 (2010)]

Connection problem for SdS_4

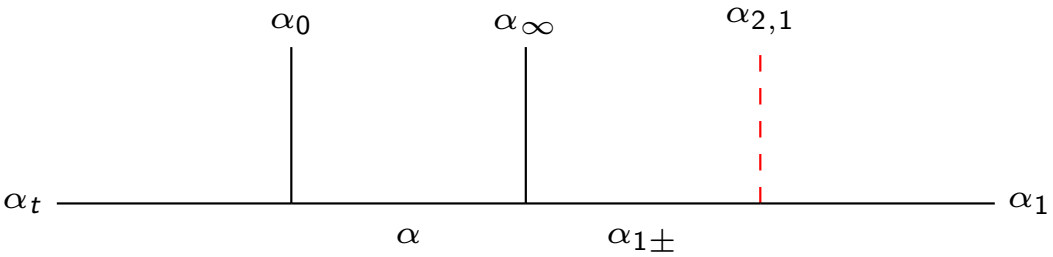




$$= \mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_0 & & \alpha_{2,1} \\ \alpha_\infty & \alpha & & \alpha_{t\pm} & \alpha_t\end{matrix}; \frac{t}{t-1}, \frac{t-z}{t}\right)$$



$$= \mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} & & \alpha_0 \\ \alpha_\infty & \alpha & & \alpha_\pm & \alpha_t\end{matrix}; \frac{z-t}{1-t}, \frac{t}{t-z}\right)$$



$$= \mathfrak{F}\left(\begin{matrix}\alpha_0 & & \alpha_\infty & & \alpha_{2,1} \\ \alpha_t & \alpha & & \alpha_{1\pm} & \alpha_1\end{matrix}; t, \frac{z-1}{z-t}\right)$$

Quantization condition for SdS_4 QNMs

The quantization condition is given by

$$\sum_{\sigma=\pm} \frac{\Gamma\left(1+2a_t\right)\Gamma\left(-2a_1\right)\Gamma\left(-2\sigma v\right)\Gamma\left(1-2\sigma v\right)}{\prod_{\pm}\Gamma\left(1/2-\sigma v+a_t\pm a_0\right)\prod_{\pm}\Gamma\left(1/2-\sigma v-a_1\pm a_\infty\right)} t^{\sigma v} e^{-\frac{\sigma}{2}\partial_v F(t)} = 0, \tag{25}$$

where

$$\begin{aligned} a_0 &= \frac{1-\gamma}{2} = s, & a_1 &= \frac{1-\delta}{2} = \frac{i\,\omega\,R_+}{(R_+-R_h)(R_+-R_-)}, \\ a_t &= \frac{1-\epsilon}{2} = -\frac{i\omega R_h}{1-3R_h^2}, & a_\infty &= \frac{\alpha-\beta}{2} = \frac{i\,\omega\,R_-}{(R_--R_h)(R_--R_+)}, \end{aligned} \tag{26}$$

$$F(t) = \frac{\left(4v^2-4a_0^2+4a_t^2-1\right)\left(4v^2+4a_1^2-4a_\infty^2-1\right)}{8-32v^2} t + \mathcal{O}\left(t^2\right),$$

$$v = \pm\left\{\sqrt{-\frac{1}{4}-u+a_t^2+a_0^2} + \frac{\left(\frac{1}{2}+u-a_t^2-a_0^2-a_1^2+a_\infty^2\right)\left(\frac{1}{2}+u-2a_t^2\right)}{2(1+2u-2a_t^2-2a_0^2)\sqrt{-\frac{1}{4}-u+a_t^2+a_0^2}} t + \mathcal{O}(t^2)\right\},$$

with $u = \dfrac{-2q+2t\alpha\beta+\gamma\epsilon-t(\gamma+\delta)\epsilon}{2(t-1)}$.

Results for QNMs

In all computed orders, we find the real part of the quasinormal modes is zero, which agrees with the earlier observations made by numerical computations. [R. A. Konoplya, and A. Zhidenko, Phys. Rev. D 106, 124004 (2022)]
The results for the imaginary part of the quasinormal mode frequencies $\omega_{n,\ell,s}$, for $n = 0$, are

$$\begin{aligned}
\mathrm{Im}\left(\omega_{0,0,0}\right) = & -1 - \frac{5}{8}R_h^2 - 3R_h^3 - \left[\frac{1287}{128} + 2\log\left(2R_h\right)\right] R_h^4 + \mathcal{O}\left(R_h^5\right), \\
\mathrm{Im}\left(\omega_{0,1,1}\right) = & -2 - \frac{7}{12}\,R_h^2 + \frac{7123}{1728}\,R_h^4 + 8\,R_h^5 + \left[\frac{2\,757\,809}{124\,416} + \frac{32}{3}\log\left(2R_h\right)\right] R_h^6 + \mathcal{O}\left(R_h^7\right), \\
\mathrm{Im}\left(\omega_{0,2,2}\right) = & -3 - \frac{27}{40}R_h^2 + \frac{51\,423}{16\,000}R_h^4 - \frac{72\,333\,747}{3\,200\,000}R_h^6 - \frac{72}{5}R_h^7 + \\
& + \left[\frac{60\,278\,884\,503}{512\,000\,000} - \frac{144}{5}\log\left(2R_h\right)\right] R_h^8 + \mathcal{O}\left(R_h^9\right).
\end{aligned} \tag{27}$$

QNMs at large ℓ

The previous quantization condition gets simplified in the large ℓ limit, neglecting non-perturbative effects in ℓ of the form R_h^ℓ:

$$\frac{1}{2} + \nu - a_1 + a_\infty = -n, \quad \text{with} \quad n \in \mathbb{Z}_{\geq 0}. \tag{28}$$

Expanding the parameters in R_h and writing ω as $\omega = \sum_{j=0}^{\infty} \omega_j R_h^j$, we find

$$\begin{aligned}
&\omega_0 = -i(\ell + n + 1);\\
&\omega_1 = 0;\\
&\omega_2 = -\frac{i}{8\ell(\ell+1)(2\ell+1)(2\ell-1)(2\ell+3)}\Big\{\ell^4\left(60n^2+60n+22\right) + \ell^3(120n^2+48ns^2+122n+\\
&\quad + 24s^2+45) + \ell^2\left[8n^2\left(3s^2+2\right) + n\left(96s^2+19\right) + 8s^4+44s^2+8\right] +\\
&\quad + \ell\left[4n^2\left(6s^2-11\right) + n\left(24s^4-43\right) + 20s^4-4s^2-15\right] + 12(n+1)^2s^2\left(s^2-2\right)\Big\};\\
&\omega_3 = 0;\\
&\quad \vdots
\end{aligned} \tag{29}$$

Notice that in this limit, all the odd orders ω_{2k+1} seem to vanish.

Generalization to Kerr-de Sitter

The **Teukolsky master equation** separates in the radial and angular equations:

$$\begin{aligned}
&\Delta_r^{-s}(r)\frac{d}{dr}\left(\Delta_r^{s+1}(r)\frac{dR(r)}{dr}\right)+\left[\frac{[\omega(r^2+a^2)-a\,m]^2\left(1+\frac{\Lambda}{3}a^2\right)^2-i\,s\,\Delta_r'(r)[\omega(r^2+a^2)-a\,m]\left(1+\frac{\Lambda}{3}a^2\right)}{\Delta_r(r)}+\right.\\
&\quad\left.+4i\,s\,\omega\left(1+\frac{\Lambda}{3}a^2\right)r-\frac{2\Lambda}{3}(s+1)(2s+1)r^2+s\left(1-\frac{\Lambda}{3}a^2\right)-A_{\ell ms}\right]R(r)=0,\\
&\frac{d}{du}\left[(a^2-u^2)\left(1+\frac{\Lambda}{3}u^2\right)\frac{dS(u)}{du}\right]+\left[-\frac{\left[\left(1+\frac{\Lambda}{3}a^2\right)[\omega(a^2-u^2)-a\,m]+s\,u\left(\frac{\Lambda}{3}a^2-\frac{2\Lambda}{3}u^2-1\right)\right]^2}{(a^2-u^2)\left(1+\frac{\Lambda}{3}u^2\right)}+\right.\\
&\quad\left.-4s\,\omega\left(1+\frac{\Lambda}{3}a^2\right)u-\frac{2\Lambda}{3}(2s^2+1)u^2+A_{\ell ms}\right]S(u)=0,\qquad\text{with } u=a\cos\theta,
\end{aligned}\tag{30}$$

where with $A_{\ell ms}$ we denote the separation constant.
The separation constant can be expanded in the small rotation regime as

$$A_{\ell ms}=\ell(\ell+1)-s^2-\frac{2m\left[\ell(\ell+1)+s^2\right]}{\ell(\ell+1)}a\,\omega+\mathcal{O}(a^2).\tag{31}$$

Conclusions

- In the SdS case, we found a branch of purely imaginary modes, thereby providing analytical confirmation of the results obtained through previous numerical studies.
- The method is less effective in the anti-de Sitter case. We deal with it using another method: the *multi polylog method*, that has a wide range of applicability, being effective for several types of differential equations and boundary conditions.
- The method can be applied also in different BH geometries and can be used to compute other relevant quantities such as the greybody factor, or Love numbers. [G. Bonelli, C. Iossa, D. P. Lichtig and A. Tanzini, Phys. Rev. D 105 (2022) 044047]

THANK YOU FOR YOUR ATTENTION!

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Multi polylog method

The problems considered in the work are two-point boundary value problems associated with differential equations on the sphere with 4 or 5 regular singularities.
In the SdS_4 case, we divide the 4-punctured sphere in two local regions (the minimal subdivision would be the patch decomposition), and we expand in each region the differential equation and its wave solution in series in the small parameter R_h,

$$\psi(z) = f_0(z) + \sum_{K\geq 1} f_K(z) R_h^K \ . \tag{32}$$

At each order in R_h, $\psi(z)$ is determined by a second-order equation

$$(f_K(z))'' + \varphi(z)\,(f_K(z))' + \nu(z) f_K(z) + \eta_K(z) = 0, \tag{33}$$

which we solve by using the method of variation of parameters.

Multi polylog method II

Let f_0, g_0 be the two solutions of the homogeneous part of (33).Then we write the generic solution to (33) as

$$f_K(z) = b_K g_0(z) + c_K f_0(z) - g_0(z) \int^z f_0(z') \frac{\eta_K(z')}{W_0(z')} \mathrm{d}z' + f_0(z) \int^z g_0(z') \frac{\eta_K(z')}{W_0(z')} \mathrm{d}z', \tag{34}$$

where W_0 is the Wronskian of the two leading order solutions

$$W_0 \equiv f_0 (g_0)' - (f_0)' g_0. \tag{35}$$

In each region, the integration constants c_K can be absorbed into a normalization of the solution.
Imposing the two boundary conditions and gluing the local solutions fixes the integration constants b_K and gives the quantization of the frequency of the perturbation.
In our case, the leading order solutions are described in terms of rational or logarithmic functions, and their Wronskian is a rational function. Hence the wave function at order R_h^K is described in terms of multiple polylogarithms of weight K and lower.

Multiple Polylogarithms in a single variable

The integrals in (34) are described in terms of the multiple polylogarithms in a single variable

$$\mathrm{Li}_{s_1,\ldots,s_k}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1}^{\infty} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}}. \tag{36}$$

The latter satisfies the following relation for $s_1 \geq 2$:

$$z\,\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{s_1,\ldots,s_k}(z) = \mathrm{Li}_{s_1-1,\ldots,s_k}(z) \tag{37}$$

and the following relation for $s_1 = 1$, $k \geq 2$:

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{1,s_2,\ldots,s_k}(z) = \mathrm{Li}_{s_2,\ldots,s_k}(z). \tag{38}$$

SdS_4 perturbation equation in Heun's form

The full dictionary is given by

$$
\begin{aligned}
t &= \frac{R_h(R_- - R_+)}{R_+(R_- - R_h)}, \\
\gamma &= 1 - 2s, \\
\delta &= 1 - \frac{2i\,\omega\,R_+}{(R_+ - R_h)(R_+ - R_-)}, \\
\epsilon &= 1 + \frac{2i\omega R_h}{1 - 3R_h^2}, \\
\alpha &= 1 - s + \frac{2i\,\omega\,R_-}{(R_- - R_h)(R_- - R_+)}, \\
\beta &= 1 - s, \\
q &= \frac{\ell(\ell+1)}{R_+(R_- - R_h)} + \frac{(1-s)^2 R_h}{R_h - R_-} - \frac{s(1-s)R_-^2}{R_+(R_h - R_-)}.
\end{aligned}
\tag{39}
$$

DOZZ formula

The Liouville three-point function is given by the DOZZ formula

$$C_{\alpha_1\alpha_2\alpha_3} = \frac{\Upsilon_b'(0)\Upsilon_b(Q+2\alpha_1)\Upsilon_b(Q+2\alpha_2)\Upsilon_b(Q+2\alpha_3)}{\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1-\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}-\alpha_1+\alpha_2+\alpha_3)}, \tag{40}$$

where

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, \quad \Gamma_b(x) = \Gamma_2\left(x|b, b^{-1}\right), \tag{41}$$

with the Barnes Double Gamma function being

$$\log\Gamma_2(s,\omega_1,\omega_2) = \left(\frac{\partial}{\partial t}\sum_{n_1,n_2=0}^{\infty}(s+n_1\omega_1+n_2\omega_2)^{-t}\right)_{t=0}. \tag{42}$$

Young diagrams

If Y is a Young diagram, we denote with $(Y_1 \geq Y_2 \geq \dots)$ the heights of its columns and with $(Y'_1 \geq Y'_2, \dots)$ the lengths of its rows. For every Young diagram Y and for every box $s = (i, j)$, we denote the arm length and the leg length of s with respect to the diagram Y as

$$A_Y(i,j) = Y_j - i, \quad L_Y(i,j) = Y'_i - j. \tag{43}$$

Note that we do not require s to be in Y: if this is the case, the arm length and the leg length are non-negative quantities, but this is not true in general.

Hypermultiplet and vector contributions

We introduce the main contributions coming into play for the definition of the instanton partition function of $\mathcal{N}=2$ $SU(2)$ gauge theory with fundamental matter. Let us denote with $\vec{Y}=(Y_1, Y_2)$ a pair of Young diagrams and with $|\vec{Y}|=|Y_1|+|Y_2|$ the total number of boxes. We denote with $\vec{a}=(a_1, a_2)$ the v.e.v. of the scalar in the vector multiplet and with ϵ_1, ϵ_2 the parameters characterizing the Ω-background. We define the hypermultiplet and vector contribution as

$$z_{\text{hyp}}\left(\vec{a}, \vec{Y}, m\right)=\prod_{k=1,2}\prod_{(i,j)\in Y_k}\left[a_k+m+\epsilon_1\left(i-\frac{1}{2}\right)+\epsilon_2\left(j-\frac{1}{2}\right)\right],$$

$$z_{\text{vec}}\left(\vec{a}, \vec{Y}\right)=\prod_{i,j=1}^{2}\prod_{s\in Y_i}\frac{1}{a_i-a_j-\epsilon_1 L_{Y_j}(s)+\epsilon_2(A_{Y_i}(s)+1)}\prod_{t\in Y_j}\frac{1}{-a_j+a_i+\epsilon_1(L_{Y_i}(t)+1)-\epsilon_2 A_{Y_j}(s)}. \tag{44}$$

We will always take $\epsilon_1=1$ and $\vec{a}=(a,-a)$.

Instanton part of NS fee energy

Let us denote with m_1, m_2, m_3, m_4 the masses of the four hypermultiplets and let us introduce the gauge parameters a_0, a_t, a_1, a_∞ satisfying

$$\begin{aligned} m_1 &= -a_t - a_0, \quad & m_2 &= -a_t + a_0, \\ m_3 &= a_\infty + a_1, \quad & m_4 &= -a_\infty + a_1. \end{aligned} \tag{45}$$

Moreover, we denote with t the instanton counting parameter $t = e^{2\pi i \tau}$, where τ is related to the gauge coupling by

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{\mathrm{YM}}^2}. \tag{46}$$

The instanton part of the NS free energy is then given as a power series in t by

$$F(t) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log \left[(1-t)^{-2\epsilon_2^{-1}\left(\frac{1}{2}+a_1\right)\left(\frac{1}{2}+a_t\right)} \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\mathrm{vec}}\left(\vec{a}, \vec{Y}\right) \prod_{i=1}^{4} z_{\mathrm{hyp}}\left(\vec{a}, \vec{Y}, m_i\right) \right]. \tag{47}$$