Mode-Sum Renormalization in Black Hole Spacetimes

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- J. Arrechea, C. Breen, A. Ottewill and PT, arXiv:2307.10307 (2023)
- PT, C. Breen and A. Ottewill, Phys. Rev. D 106, 065023 (2022)
- PT and C. Breen, Phys. Rev. D 96, 105020 (2017)
- PT and C. Breen, Phys. Rev. D 94, 125024 (2016)

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I. BACKGROUND

- Consider a physical process characterized by

$$
\begin{array}{ll}
\text{mass scale} & \mathcal{M} \\
\text{length scale} & \mathcal{L} \\
\text{time scale} & \tau
\end{array}
$$

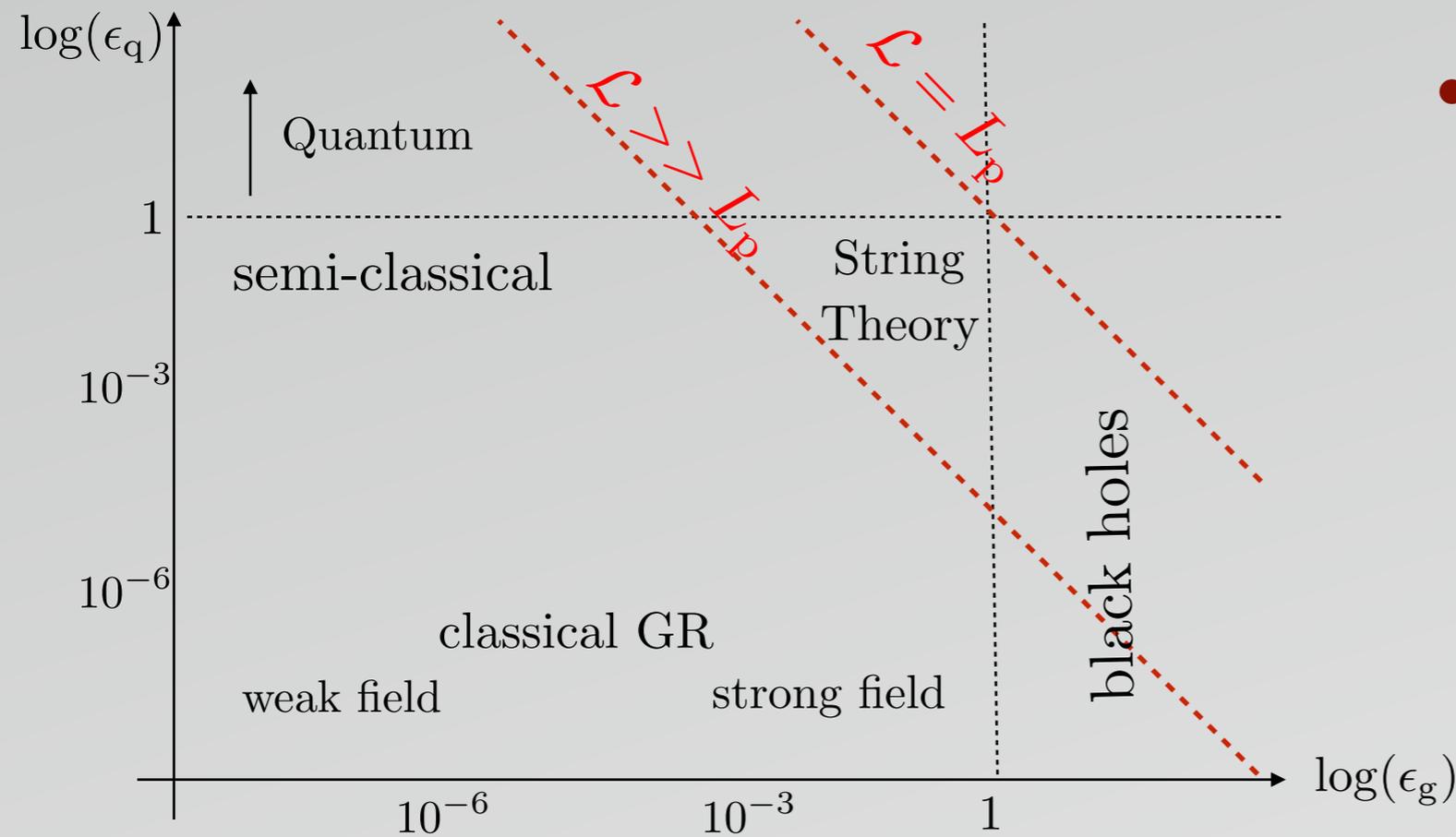- Can construct dimensionless numbers

$$\epsilon_{\mathrm{r}} = \frac{\mathcal{L}}{c\,\tau} \qquad \text{Is SR important?}$$

$$\epsilon_{\mathrm{g}} = \frac{G\mathcal{M}}{\mathcal{L}c^2} \qquad \text{Is GR important?}$$

$$\epsilon_{\mathrm{q}} = \frac{\hbar}{\mathcal{M}\mathcal{L}c} \qquad \text{Is QM important?}$$

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I. BACKGROUND

- Semi-classical gravity: Treat gravity classically and all other fields quantum mechanically. Assume (naively) metric satisfies

$$G_{ab} + \Lambda\, g_{ab} = 8\pi \langle A|\hat{T}_{ab}|A\rangle$$

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I. BACKGROUND

- Assuming for simplicity that we just have a scalar field, the action is

$$S[\phi; g_{ab}] = -\tfrac{1}{2} \int d^4x \sqrt{-g} \left\{ g^{ab} \nabla_a \phi \nabla_b \phi + \xi R \phi^2 + \mu^2 \phi^2 \right\}$$

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Varying w.r.t. the field

$$\left(\nabla^a \nabla_a - \xi R - \mu^2 \right) \phi = 0$$

Varying w.r.t. the metric

$$T_{ab} \equiv (1 - 2\xi) \nabla_a \phi \nabla_b \phi + (2\xi - \tfrac{1}{2}) g_{ab} \nabla_c \phi \nabla^c \phi - 2\xi \phi \nabla_a \nabla_b \phi$$
$$+ 2\xi g_{ab} \phi \nabla_c \nabla^c \phi + \xi G_{ab} \phi^2 - \tfrac{1}{2} \mu^2 g_{ab} \phi^2$$

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$$+ 2\xi g_{ab}\phi\nabla_c\nabla^c\phi + \xi\,G_{ab}\phi^2 - \tfrac{1}{2}\mu^2 g_{ab}\,\phi^2$$

Quantization: Field $\to$ Operator-Valued Distribution $\qquad \phi(x) \to \hat{\phi}(x)$

Commutation Relations $\quad \left[\hat{\phi}(t, \mathbf{x}), n^a\nabla_a\hat{\phi}(t, \mathbf{x}')\right] = i\dfrac{\delta^3(\mathbf{x} - \mathbf{x}')}{\sqrt{h}}$

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- $\hat{T}_{ab}$ is quadratic in an operator-valued distribution which is ill-defined.

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2. POINT-SPLITTING REGULARIZATION   (DeWitt, Physics Reports 1975, Christensen, PRD 1976)

$$\hat{T}_{ab}(x) \qquad x$$

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2. POINT-SPLITTING REGULARIZATION

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$$\lim_{x' \to x} \hat{T}_{ab}(x, x')$$

$$\hat{T}_{ab}(x, x') \equiv \hat{\Theta}_{ab} \left(\mathcal{T} \hat{\phi}(x) \hat{\phi}(x') \right)$$

where

$$\hat{\Theta}_{ab} = (1 - 2\xi) g_{b'b} \nabla_a \nabla^{b'} + (2\xi - \tfrac{1}{2}) g_{ab} g_{c'}{}^{c} \nabla_c \nabla^{c'}$$
$$- 2\xi \nabla_a \nabla_b + 2\xi \, g_{ab} \nabla_c \nabla^c + \xi \, G_{ab} - \tfrac{1}{2} m^2 g_{ab}$$

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$$\langle A| \hat{T}_{ab}(x) |A\rangle = \lim_{x' \to x} \langle A| \hat{T}_{ab}(x, x') |A\rangle = \lim_{x' \to x} \hat{\Theta}_{ab} \langle A| \mathcal{T} \hat{\phi}(x) \hat{\phi}(x') |A\rangle$$

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$$x'$$

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- Can we make physical sense of this limit, i.e., can we remove the divergent terms in a controlled way?

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3. HADAMARD RENORMALIZATION   (e.g. Wald, Comm. math. Phys. 1977; Wald, PRD 1978)

- Restrict to a class of quantum states for which the Green's function has the Hadamard property, i.e., for x sufficiently close to x', the Green's function has the form

$$G_A(x, x') = \frac{i}{8\pi^2} \left\{ \frac{\Delta^{\frac{1}{2}}(x, x')}{\sigma(x, x') + i\epsilon} + V(x, x') \log\left(\frac{2\sigma(x, x')}{\ell^2} + i\epsilon \right) + W_A(x, x') \right\}$$

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$$K(x, x')$$

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$$G_A(x,x') = [G_A(x,x') - K(x,x')] + K(x,x')$$

$$\underline{W_A(x,x')}$$

regular as $x \to x'$

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$$\underbrace{}_{W_A(x,x')}$$

regular as $x \to x'$

- $K(x,x')$ depends only on the local geometry.

$$\hat{\Theta}_{ab}K(x,x') \sim \gamma(x,x')G_{ab}(x) + \Lambda(x,x')g_{ab}(x) + \alpha(x,x')A_{ab}(x) + \beta(x,x')B_{ab}(x) + \cdots$$

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4. THE COMPUTATIONAL PROBLEM

$$\langle A|\hat{T}_{ab}|A\rangle_{\mathrm{ren}} = \left[\hat{\Theta}_{ab}W_A(x,x')\right] = \left[\hat{\Theta}_{ab}\left(G_A(x,x') - K(x,x')\right)\right]$$

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mode-sum (global)

$$G_A(x, x') = \frac{1}{8\pi^2} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos\gamma) \int d\omega \, e^{-i\omega\Delta t} g_{\omega l}(r, r')$$

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expansion in σ (local)

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expansion in σ (local)

Computational Problem: How do we subtract $K(x,x')$ from $G_A(x,x')$ in such a way that the coincidence limit can be meaningfully taken?

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5. THE CANDELAS-HOWARD APPROACH

- Work with Euclideanized metric: $ds^2 = f(r)d\tau^2 + dr^2/f(r) + r^2 d\Omega^2$

- Point-split only in temporal direction, i.e., take $\mathbf{x}' \to \mathbf{x}$.

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$$G(\Delta\tau, r) = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \cos n\kappa\Delta\tau \left(\sum_{l=0}^{\infty} (2l+1)\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - \frac{\kappa}{r\,f^{1/2}} \right)$$

$$K(\Delta\tau, r) = \frac{1}{2\pi^2} \frac{1}{f(r)\Delta\tau^2} + \mathcal{O}(1) \qquad \left(2\sigma = f(r)\Delta\tau^2 + \mathcal{O}(\Delta\tau^4) \right)$$

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$$\frac{1}{\Delta\tau^2} = -\kappa^2 \sum_{n=1}^{\infty} n\, \cos n\kappa\Delta\tau + \mathrm{O}(1).$$

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$$\frac{1}{\Delta\tau^2} = -\kappa^2 \sum_{n=1}^{\infty} n \, \cos n\kappa\Delta\tau + \mathrm{O}(1).$$

$$\langle \hat{\phi}^2 \rangle_{\mathrm{ren}} = [W(x, x')] = \lim_{\Delta\tau \to 0} \left(G(\Delta\tau, r) - K(\Delta\tau, r) \right)$$

$$= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_{l=0}^{\infty} (2l+1) \left(\frac{p_{nl}(r) q_{nl}(r)}{N_{nl}} - \frac{\kappa}{r\, f^{1/2}} \right) + \frac{2n\kappa^2}{f} \right\} + \mathcal{O}(1)$$

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Not a complete mode-by-mode subtraction.
Not amenable to numerical computation

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5. THE CANDELAS-HOWARD APPROACH

- Treat the *l*-sum quasi-analytically.

$$\sum_{l=0}^{\infty}(2l+1)\left(\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}}-\frac{\kappa}{r\,f^{1/2}}\right)=\sum_{l=0}^{\infty}(2l+1)\left(\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}}-WKB\right.$$

$$\left.+WKB-\frac{\kappa}{r\,f^{1/2}}\right)$$

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5. THE CANDELAS-HOWARD APPROACH
(Candelas-Howard PRD 1984, PRL 1984,
Anderson-Hiscock-Samuel, PRL 1993, PRD 1995)

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$$\left.+WKB-\frac{\kappa}{r\,f^{1/2}}\right)$$

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rapidly convergent

$$
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$$

$$
+\boxed{WKB - \frac{\kappa}{r\,f^{1/2}}}\Big)
$$

$$
\sum_{l=0}^{\infty}(2l+1)\Big(WKB - \frac{\kappa}{r\,f^{1/2}}\Big) \to \int_{\mathcal{C}} dl
$$

Watson-Somerfeld
Transform

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5. THE CANDELAS-HOWARD APPROACH (Candelas-Howard PRD 1984, PRL 1984, Anderson-Hiscock-Samuel, PRL 1993, PRD 1995)

- Treat the *l*-sum quasi-analytically.

rapidly convergent

$$\sum_{l=0}^{\infty}(2l+1)\left(\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - \frac{\kappa}{r\,f^{1/2}}\right) = \sum_{l=0}^{\infty}(2l+1)\left(\boxed{\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - WKB} \right.$$

$$\left. + \boxed{WKB - \frac{\kappa}{r\,f^{1/2}}}\right)$$

$$\sum_{l=0}^{\infty}(2l+1)\left(WKB - \frac{\kappa}{r\,f^{1/2}}\right) \to \int_{\mathcal{C}} dl$$

Watson-Somerfeld
Transform

- Parts of the integral can be done explicitly, singular and slowly converging terms cancel.

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5. THE CANDELAS-HOWARD APPROACH

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$$\sum_{l=0}^{\infty}(2l+1)\left(\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}}-\frac{\kappa}{r\,f^{1/2}}\right)=\sum_{l=0}^{\infty}(2l+1)\left(\boxed{\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}}-WKB}\right.$$

rapidly convergent

$$\left.+\boxed{WKB-\frac{\kappa}{r\,f^{1/2}}}\right)$$

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- **Drawbacks:**   — Inefficient and cumbersome to implement
  — Convergence is not explicit, requires WS transform
  — WKB not uniform

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Is there a more direct approach?

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6. THE "EXTENDED COORDINATE" METHOD

- Two key considerations in obtaining a mode-by-mode expression for $\langle \hat{\phi}^2 \rangle_{\mathrm{ren}}$.

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6. THE "EXTENDED COORDINATE" METHOD

(Taylor & Breen, PRD 2016, PRD 2017;
Taylor, Breen & Ottewill, PRD 2022)

- Two key considerations in obtaining a mode-by-mode expression for $\langle \hat{\phi}^2 \rangle_{\mathrm{ren}}$.

  1. Point-split in multiple directions.

  2. Expand in "extended coordinates".

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$$\varpi^2 = \frac{2}{\kappa^2}(1 - \cos \kappa \Delta \tau) \qquad s^2 = f(r)\,\varpi^2 + 2r^2(1 - \cos \gamma)$$

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$$= 2\sigma + \mathcal{O}(\Delta x^3)$$

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$$= 2\sigma + \mathcal{O}(\Delta x^3)$$

- Expand Hadamard parametrix in these variables.

$$K(x, x') \sim \sum_{i=0}^{m} \sum_{j=0}^{i} \mathcal{D}_{ij}^{(R)}(r)\frac{\varpi^{2i+2j}}{s^{2j+2}} + \sum_{i=1}^{m} \sum_{j=1}^{i} \mathcal{D}_{ij}^{(P)}(r)\varpi^{2i-2j}s^{2j-2}$$

$$+ \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} \mathcal{T}_{ij}^{(R)}(r)\frac{\varpi^{2i+2j+2}}{s^{2j+2}} + \sum_{i=0}^{m-1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(L)}(r)\varpi^{2j}s^{2i-2j} \log\left(s^2/\ell^2\right) + \sum_{i=1}^{m-1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(P)}(r)\varpi^{2j}s^{2i-2j}$$

$$+ \mathcal{O}(\epsilon^{2m} \log \epsilon)$$

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Can we express these as mode-sums?

$$+ \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} \mathcal{T}_{ij}^{(R)}(r)\frac{\varpi^{2i+2j+2}}{s^{2j+2}} + \sum_{i=0}^{m-1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(L)}(r)\varpi^{2j}s^{2i-2j} \log\left(s^2/\ell^2\right) + \sum_{i=1}^{m-1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(P)}(r)\varpi^{2j}s^{2i-2j}$$

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$$\frac{\varpi^{2i+2j}}{s^{2j+2}} = \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\Psi_{nl}(i,j|r)$$

$$\varpi^{2j}s^{2i-2j}\log(s^2/\ell^2) = \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\tau}\chi_{nl}(i,j|r)$$

- We can invert these to obtain $\Psi_{nl}(i,j|r)$ and $\chi_{nl}(i,j|r)$ in closed form (associated Legendre functions).

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$$\Psi_{nl}(i,j|r) = \frac{2^{2i-j-1}(-1)^{n+j}i!\,\Gamma(i+\frac{1}{2})}{\sqrt{\pi}\,\kappa^{2i+2j}\,j!\,r^{2j+2}}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)^j\sum_{k=n-i}^{n+i}\frac{P_l^{-|k|}(\eta)Q_l^{|k|}(\eta)}{(i-n+k)!(i+n-k)!}$$

$$\eta = \sqrt{1+\frac{f}{\kappa^2 r^2}}$$

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$$\eta = \sqrt{1+\frac{f}{\kappa^2 r^2}}$$

- Similarly we can derive expressions for $\chi_{nl}(i,j|r)$, though they're more complicated to write down.

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$$K(x, x') = \frac{1}{8\pi^2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} k_{nl}^{(m)}(r) + \left[\text{terms polynomial in } \varpi^2, s^2\right]$$

$$k_{nl}^{(m)}(r) = \sum_{i=0}^{m} \sum_{j=0}^{i} \mathcal{D}_{ij}^{(R)}(r) \Psi_{nl}(i,j|r) + \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} \mathcal{T}_{ij}^{(R)}(r) \Psi_{nl}(i+1,j|r) + \sum_{i=0}^{m-1} \sum_{j=0}^{i} \mathcal{T}_{ij}^{(L)}(r) \chi_{nl}(i,j|r)$$

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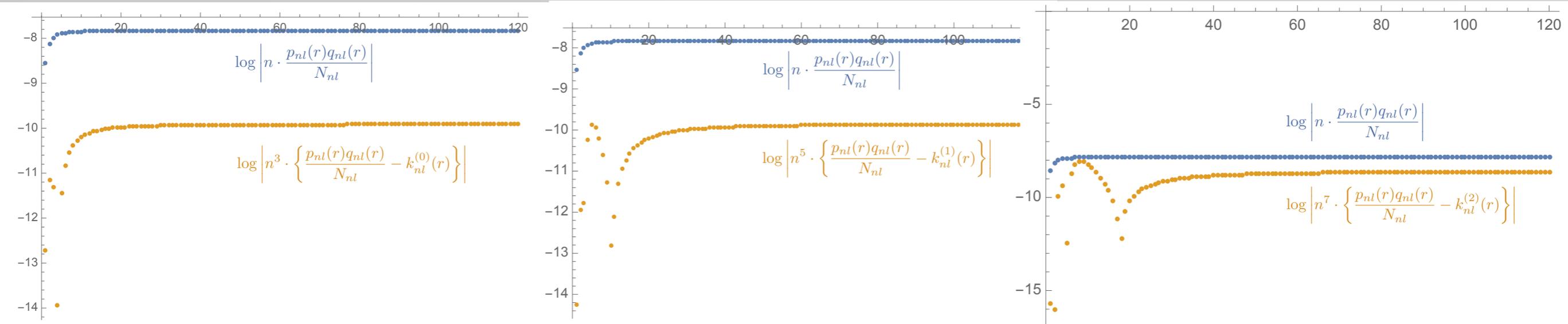$$\langle\hat{\phi}^2\rangle_{\text{ren}} = [G(x,x') - K(x,x')]$$

$$= \frac{1}{8\pi^2}\sum_{l=0}^{\infty}\sum_{n=-\infty}^{\infty}(2l+1)\left\{\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - k_{nl}^{(m)}(r)\right\} + \frac{f'(r)}{48\pi^2 r}$$

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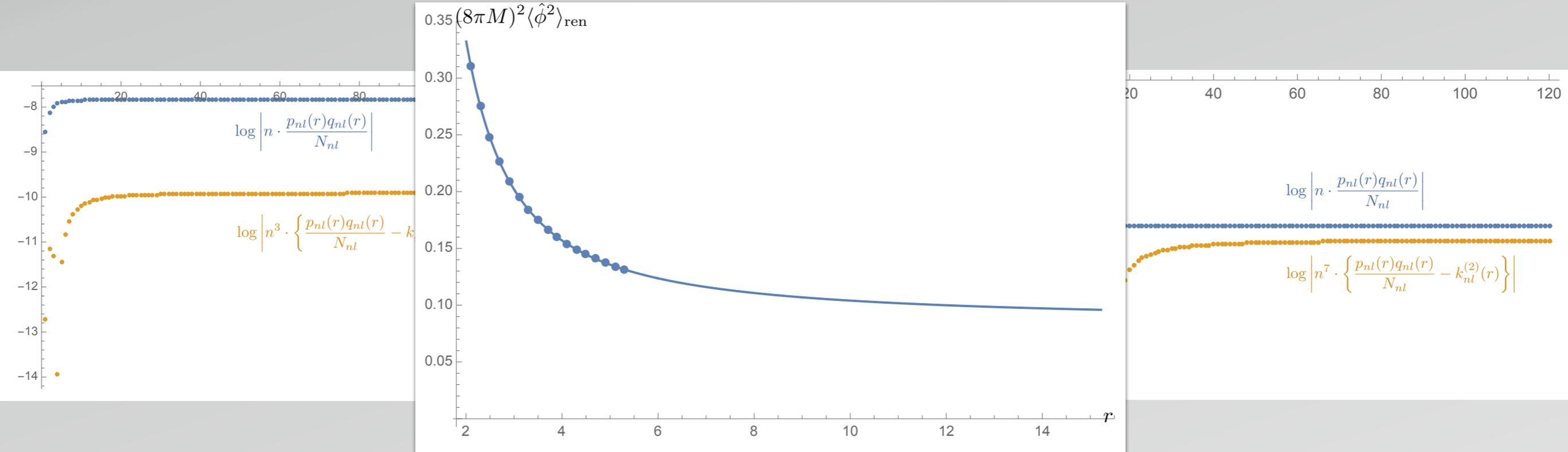
$$= \frac{1}{8\pi^2} \sum_{l=0}^{\infty} \sum_{n=-\infty}^{\infty} (2l+1) \left\{ \frac{p_{nl}(r) q_{nl}(r)}{N_{nl}} - k_{nl}^{(m)}(r) \right\} + \frac{f'(r)}{48\pi^2 r}$$

$$\sim n^{-2m-3} \quad \text{large } n, \text{ fixed } l$$

$$\sim l^{-2m-3} \quad \text{large } l, \text{ fixed } n$$

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- Remarkably the RSET can be computed with very little additional complexity.

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$$\langle \hat{T}^a{}_b \rangle_{\mathrm{ren}} = -\mathsf{w}^a{}_b - (\xi - \tfrac{1}{2})\mathsf{w}^{;a}{}_b + (\xi - \tfrac{1}{4})\Box\mathsf{w}\,\delta^a{}_b + \xi\,R^a{}_b\mathsf{w} - \frac{1}{8\pi^2}v_1\delta^a{}_b$$

$$\mathsf{w} = [W] = \langle \hat{\phi}^2 \rangle_{\mathrm{ren}} \qquad \mathsf{w}_{ab} = [W_{;ab}] = [W_{;a'b'}]$$

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$$[W_{;\tau'\tau}] = -\frac{1}{8\pi^2}\sum_{l=0}^{\infty}\sum_{n=-\infty}^{\infty}(2l+1)n^2\kappa^2\left\{\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - k_{nl}^{(m)}(r)\right\} + \mathcal{O}(1)$$

$$[W_{;\phi'\phi}] = \frac{1}{16\pi^2}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)l(l+1)(2 - \delta_0^n)\left\{\frac{p_{nl}(r)q_{nl}(r)}{N_{nl}} - k_{nl}^{(m)}(r)\right\} + \mathcal{O}(1)$$

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$$ {\sf w} = [W] = \langle \hat{\phi}^2 \rangle_{\rm ren} \qquad {\sf w}_{ab} = [W_{;ab}] = [W_{;a'b'}] = \tfrac{1}{2}{\sf w}_{;ab} - [W_{;a'b}] $$

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- ${\sf w}^r{}_r$ can be computed from the wave equation

$$ {\sf w}^a{}_a - \xi\,R\,{\sf w} - \mu^2{\sf w} = -\tfrac{3}{4\pi^2}v_1 $$

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$$\mathsf{w}^a{}_a - \xi\,R\,\mathsf{w} - \mu^2\mathsf{w} = -\tfrac{3}{4\pi^2}v_1$$

- We can essentially use our calculation for $\langle \hat{\phi}^2 \rangle_{\text{ren}}$ to compute $\langle \hat{T}^a{}_b \rangle_{\text{ren}}$

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7. THE RSET IN OTHER QUANTUM STATES   (Arrechea, Breen, Ottewill & PT, 2023)

- Method presented applies only to the Hartle-Hawking state. What about other quantum states?

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- Method presented applies only to the Hartle-Hawking state. What about other quantum states?

- Let $\langle \hat{T}^a{}_b \rangle_{\mathrm{R}}$ be the RSET in some Hadamard reference state, and $\langle \hat{T}^a{}_b \rangle_{\mathrm{Q}}$ be the RSET in any other Hadamard state. Then

$$\langle \hat{T}^a{}_b \rangle_{\mathrm{Q}} = \langle \hat{T}^a{}_b \rangle_{\mathrm{R}} - \left[\delta G^{;a}_{\mathrm{Q}\,b} \right] - (\xi - \tfrac{1}{2}) \left[\delta G_{\mathrm{Q}} \right]^{;a}{}_b + (\xi - \tfrac{1}{4}) \delta^a{}_b \Box \left[\delta G_{\mathrm{Q}} \right] + \xi\, R^a{}_b \left[\delta G_{\mathrm{Q}} \right]$$

where
$$\delta G_{\mathrm{Q}} = G_{\mathrm{Q}}(x, x') - G_{\mathrm{R}}(x, x')$$

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- Let $\langle \hat{T}^a{}_b \rangle_{\mathrm{R}}$ be the RSET in some Hadamard reference state, and $\langle \hat{T}^a{}_b \rangle_{\mathrm{Q}}$ be the RSET in any other Hadamard state. Then

$$\langle \hat{T}^a{}_b \rangle_{\mathrm{Q}} = \langle \hat{T}^a{}_b \rangle_{\mathrm{R}} - \left[\delta G^{;a}_{\mathrm{Q}\,b} \right] - (\xi - \tfrac{1}{2}) \left[\delta G_{\mathrm{Q}} \right]^{;a}{}_b + (\xi - \tfrac{1}{4}) \delta^a{}_b \Box \left[\delta G_{\mathrm{Q}} \right] + \xi\, R^a{}_b \left[\delta G_{\mathrm{Q}} \right]$$

where $\qquad\qquad \delta G_{\mathrm{Q}} = G_{\mathrm{Q}}(x, x') - G_{\mathrm{R}}(x, x')$

**compute in HH state
from Euclidean methods**

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7. THE RSET IN OTHER QUANTUM STATES   (Arrechea, Breen, Ottewill & PT, 2023)

- Method presented applies only to the Hartle-Hawking state. What about other quantum states?

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**compute in HH state
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**smooth since both are
Hadamard states**

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7. THE RSET IN OTHER QUANTUM STATES (Arrechea, Breen, Ottewill & PT, 2023)

- In particular, for static spherically symmetric spacetimes:

$$\left\{ \frac{d^2}{dr_*^2} + \omega^2 - V_l(r) \right\}(r\,\Phi) = 0 \qquad V_l \to \begin{cases} 0 & r \to r_+ \\ \mu^2 & r \to \infty. \end{cases}$$
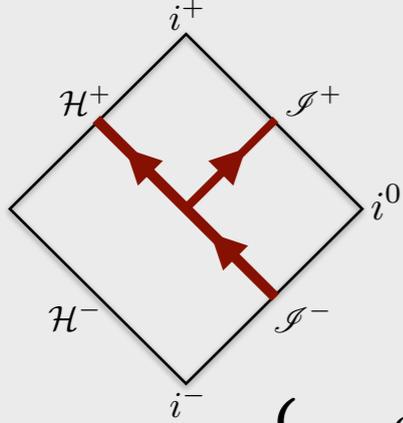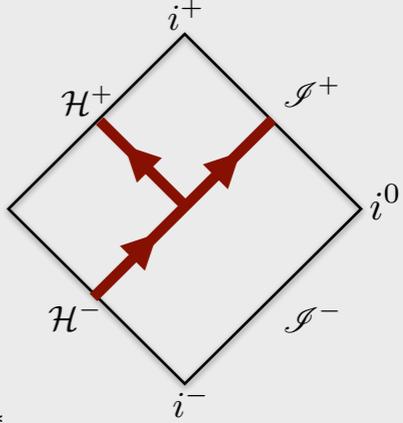
$$\tilde{\omega} = \sqrt{\omega^2 - \mu^2}$$

$$\Phi_{\omega l}^{\text{in}}(r) \sim \begin{cases} B_{\omega l}^{\text{in}} \dfrac{e^{-i\omega\, r_*}}{r_+}, & r \to r_+, \\[2em] \dfrac{e^{-i\tilde{\omega} r_*}}{r} + A_{\omega l}^{\text{in}} \dfrac{e^{i\tilde{\omega} r_*}}{r}, & r \to \infty, \end{cases}$$
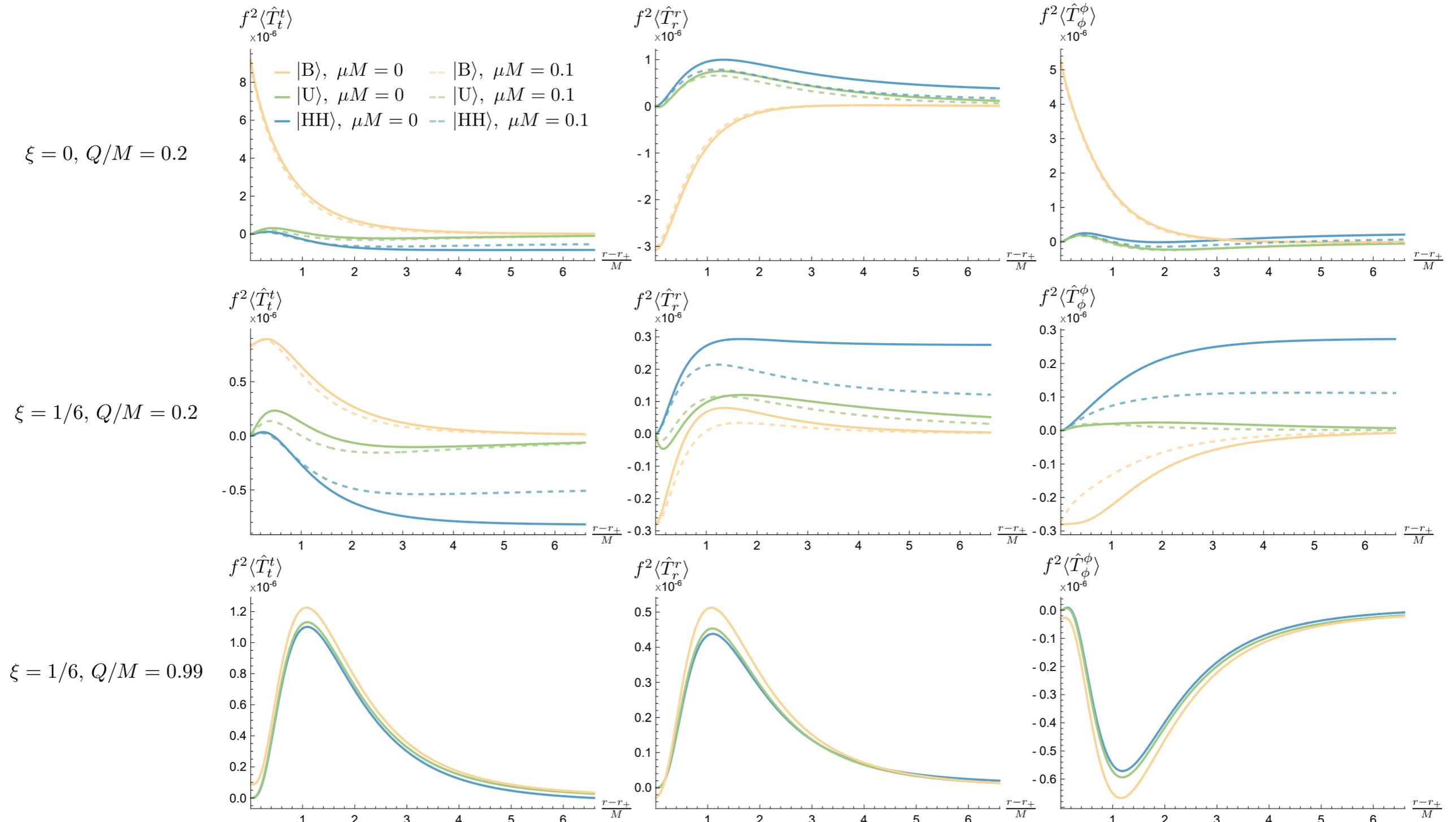
$$\Phi_{\omega l}^{\text{up}}(r) \sim \begin{cases} \dfrac{e^{i\omega\, r_*}}{r_+} + A_{\omega l}^{\text{up}} \dfrac{e^{-i\omega\, r_*}}{r_+}, & r \to r_+, \\[2em] B_{\omega l}^{\text{up}} \dfrac{e^{i\tilde{\omega} r_*}}{r}, & r \to \infty, \end{cases}$$

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$$\delta G_{\text{U}} = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\int_{\mu}^{\infty}\frac{d\omega}{2\pi\tilde{\omega}}\frac{e^{-i\omega\,\Delta t}}{(1-e^{2\pi\omega/\kappa_+})}\Phi_{\omega l}^{\text{in}}(r)\Phi_{\omega l}^{\text{in}\,\dagger}(r')\right]$$

$$\delta G_{\text{B}} = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\int_{\mu}^{\infty}\frac{d\omega}{2\pi\tilde{\omega}}\frac{e^{-i\omega\,\Delta t}}{(1-e^{2\pi\omega/\kappa_+})}\Phi_{\omega l}^{\text{in}}(r)\Phi_{\omega l}^{\text{in}\,\dagger}(r') + \int_{0}^{\infty}\frac{d\omega}{2\pi\omega}\frac{e^{-i\omega\Delta t}}{(1-e^{2\pi\omega/\kappa_+})}\Phi_{\omega l}^{\text{up}}(r)\Phi_{\omega l}^{\text{up}\,\dagger}(r')\right]$$
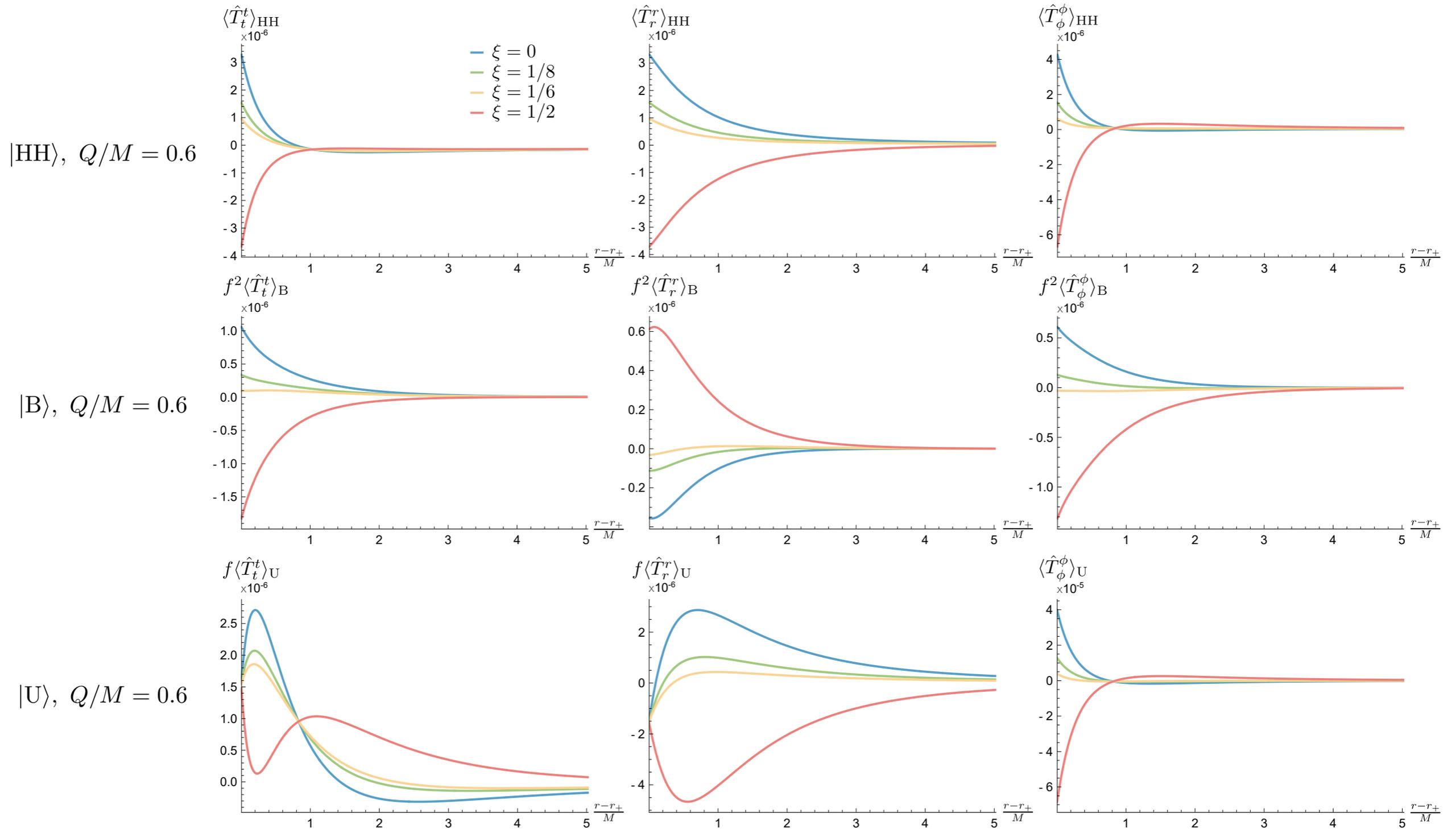
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8. RESULTS

(Arrechea, Breen, Ottewill & PT, 2023)

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CONCLUSIONS

broad perspective

our method

outlook

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CONCLUSIONS

broad perspective

- After decades of stagnation, there now exists a suite of methods to accurately and efficiently compute the RSET (see Noa Zilberman's talk for another approach).

- We can focus our efforts on physics and not on the intricacies of the RSET calculation.

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our method

- Splitting in multiple directions guarantees a mode-by-mode subtraction.

- A judicious choice of expansion "coordinates" permits us to compute all regularization parameters in closed form.

- Resultant mode-sums are rapidly convergent, even close to the horizon.

- Renormalization in BH spacetimes is surprisingly agnostic to number of dimensions and horizon topology. (*PT and C. Breen (PRD 2016+PRD 2017); T. Morley, E. Winstanley & PT (PRD 2021)*)

outlook

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outlook

- Work ongoing to extend to semi-classical stars (*J. Arrechea, A. Ottewill & C. Breen*), higher dimensions (*E. Scanlon & C. Breen*), charged fields (*E. Winstanley & G. Montagnon*), BH interiors (*L. Pisani*), cosmological spacetimes (*A. Ferreiro*) and Kerr (*A. Heffernan et al.*).

Quantum Effects in Gravitational Fields

28 Aug-01 Sep 2023, Leipzig University

Thank you for your attention.

- J. Arrechea, C. Breen, A. Ottewill and PT, arXiv:2307.10307 (2023)
- PT, C. Breen and A. Ottewill, Phys. Rev. D 106, 065023 (2022)
- PT and C. Breen, Phys. Rev. D 96, 105020 (2017)
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