Entanglement Structure of Geometric States in Holography

Veronika Hubeny

UC DAVIS UNIVERSITY OF CALIFORNIA

Physics Department & center for Quantum Mathematics and Physics

In part supported by

Quantum Effects in Gravitational Fields

Leipzig, August 29, 2023

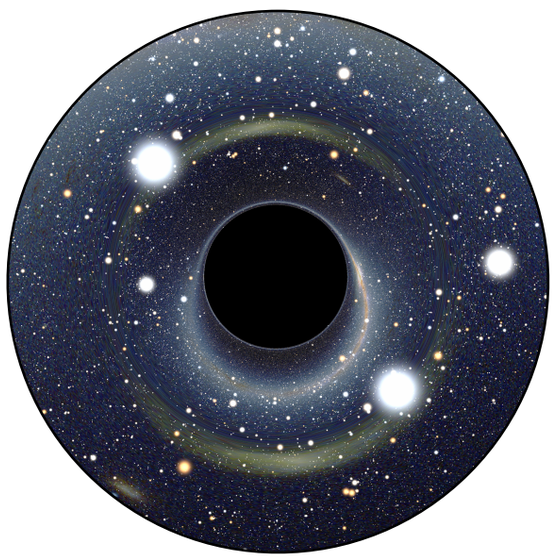
[based on 1808.07871, 1812.08133, 1912.01041, **2204.00075,** 2211.11858, 2307.10137
w/ T. He, S. Hernández-Cuenca, M. Rangamani, & M. **Rota**
+ WIP w/ T. He & M. Rota + WIP w/ S. Hernández-Cuenca, F. Jia]

AdS/CFT after 25 years

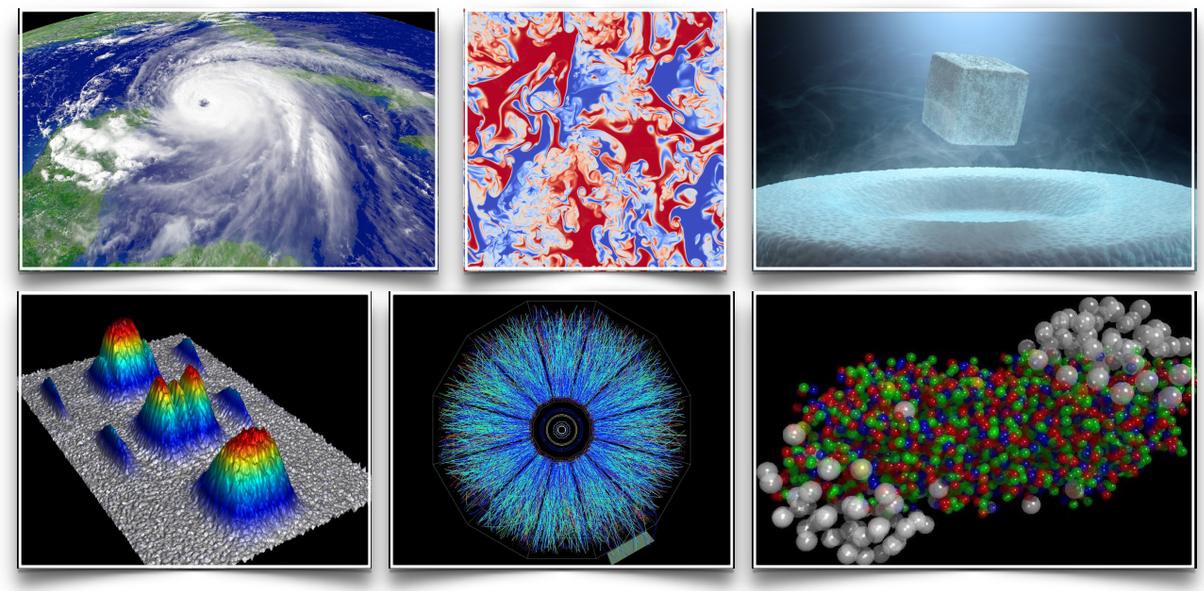
String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

"in bulk" = higher dimensions | *"on boundary" = lower dimensions*

describes gravitating systems, e.g. black holes



describes experimentally accessible systems



Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

We need to understand the AdS/CFT dictionary…

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - (How) does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - (How) does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Recent hints / expectations: entanglement plays a crucial role…

Motivation & Context

- Understand the emergence of spacetime
  - ~ Use holography
  - ~ Focus on classical bulk geometry (i.e. $N = \infty, \lambda = \infty$ regime)

Motivation & Context

- Understand the emergence of spacetime
  - ~ Use holography
  - ~ Focus on classical bulk geometry (i.e. $N = \infty, \lambda = \infty$ regime)

- Hints / expectation: "spacetime built from entanglement"
  - ~ Understand entanglement structure of 'geometric' states
  - ~ Focus on entanglement entropy (EE) for spatially-delimited subsystems

Motivation & Context

- Understand the emergence of spacetime
  - ~ Use holography
  - ~ Focus on classical bulk geometry (i.e. $N = \infty, \lambda = \infty$ regime)
- Hints / expectation: "spacetime built from entanglement"
  - ~ Understand entanglement structure of 'geometric' states
  - ~ Focus on entanglement entropy (EE) for spatially-delimited subsystems
- Characterize {EEs} relevant for holography
  - ~ Region in entropy space dubbed *holographic entropy cone* (HEC)
  - ~ Focus on its boundary (delimited by *holographic entropy inequalities*)

Motivation & Context

- Understand the emergence of spacetime
  - ~ Use holography
  - ~ Focus on classical bulk geometry (i.e. $N = \infty, \lambda = \infty$ regime)
- Hints / expectation: "spacetime built from entanglement"
  - ~ Understand entanglement structure of 'geometric' states
  - ~ Focus on entanglement entropy (EE) for spatially-delimited subsystems
- Characterize {EEs} relevant for holography
  - ~ Region in entropy space dubbed *holographic entropy cone* (HEC)
  - ~ Focus on its boundary (delimited by *holographic entropy inequalities*)
- Seek lessons independent of # of subsystems (≡N)
  - ~ Bootstrap from low N
  - ~ Focus on structural relations

Entanglement entropy

For CFT state $|\psi\rangle$ and bi-partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\rightsquigarrow$ reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

EE = $S(A) \equiv -\mathrm{Tr}\, \rho_A \, \log \rho_A$

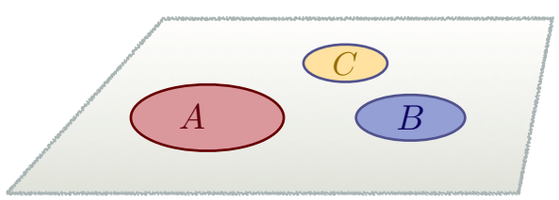
Entanglement entropy

For CFT state $|\psi\rangle$ and bi-partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\rightsquigarrow$ reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

EE = $S(A) \equiv -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- Decompose CFT into N elementary subsystems



$$\mathcal{H} = \underbrace{\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots}_{\text{N}} \otimes \underbrace{\mathcal{H}_{\overline{ABC\cdots}}}_{\text{"purifier" } O}$$

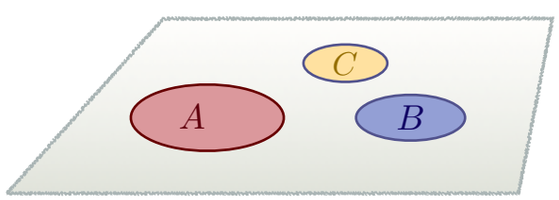
Entanglement entropy

For CFT state $|\psi\rangle$ and bi-partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\rightsquigarrow$ reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

EE = $S(A) \equiv -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- Decompose CFT into N elementary subsystems



$$\mathcal{H} = \underbrace{\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots}_{\text{N}} \otimes \underbrace{\mathcal{H}_{\overline{ABC\cdots}}}_{\text{"purifier" } 0}$$

- $\rightsquigarrow$ entropy vector in D = 2^N - 1 dimensional *entropy space*

e.g. for N=3, $\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\}$

conceptually useful to consider large N…

Entropy relations

- Physically realizable entropy vectors are restricted

- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$

Entropy relations

- Physically realizable entropy vectors are restricted

- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$

Entropy relations

- Physically realizable entropy vectors are restricted

- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
  - … (expect more relations with increasing N)
  - always permutation & purification symmetric

in any pure state, $S(A) = S(A^c)$

⇒ e.g. transforms SA into Araki-Lieb: $S(A) + S(AB) \geq S(B)$

Entropy relations

- Physically realizable entropy vectors are restricted

- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
  - … (expect more relations with increasing N)
  - always permutation & purification symmetric

- Further restrictions, depending on the system

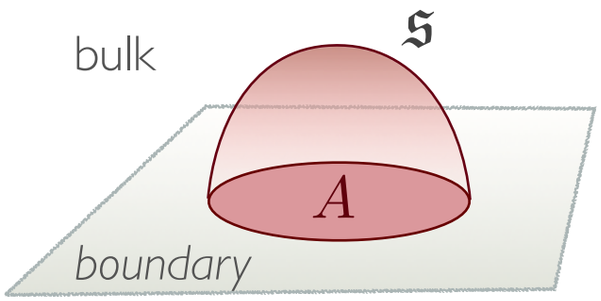
- Our task: understand the full set in holography

Holographic entanglement entropy

Proposal [RT=Ryu & Takayanagi, '06] for static configurations,
covariantized by [HRT=VH, Rangamani, Takayanagi, '07] for time-dependent situations:

Entanglement entropy $S(A)$ for a boundary region A is captured by the area of a bulk extremal surface $\mathfrak{s}$ homologous to A;

for multiple candidates, choose least area one.



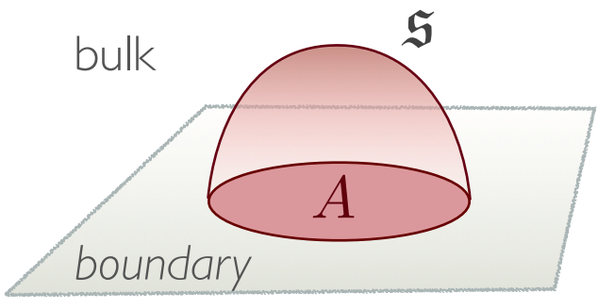
$$S(A) = \min_{\mathfrak{s} \sim A} \frac{\mathrm{Area}(\mathfrak{s})}{4\,G_N}$$

Holographic entanglement entropy

Proposal [RT=Ryu & Takayanagi, '06] for static configurations,
covariantized by [HRT=VH, Rangamani, Takayanagi, '07] for time-dependent situations:

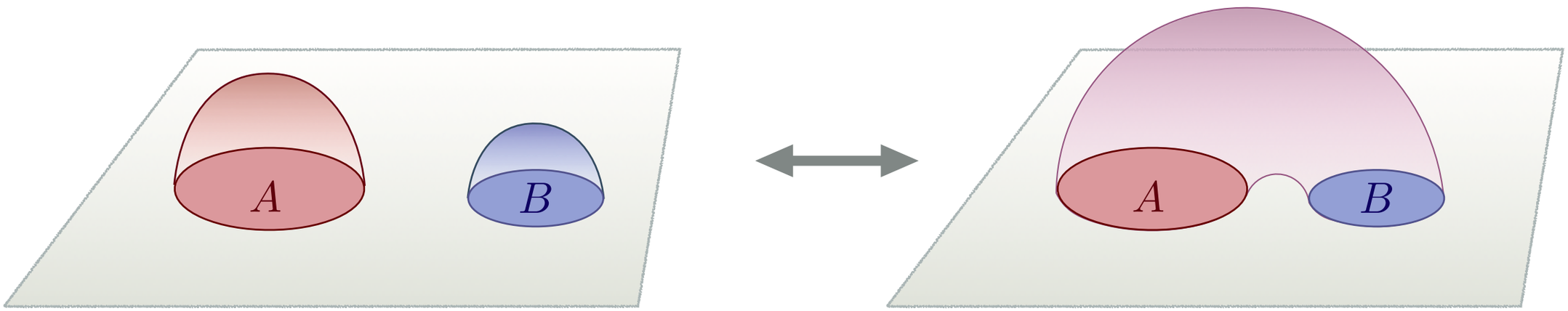
Entanglement entropy $S(A)$ for a boundary region A is captured by the area of a bulk extremal surface $\mathfrak{s}$ homologous to A;

for multiple candidates, choose least area one.



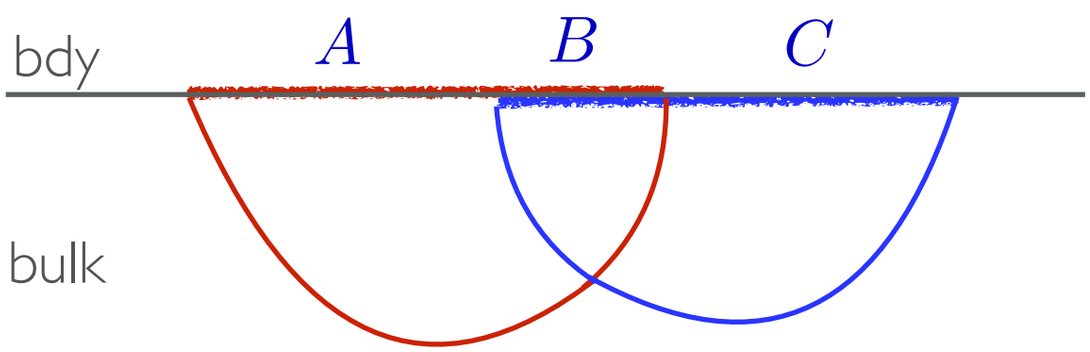
$$S(A) = \min_{\mathfrak{s} \sim A} \frac{\mathrm{Area}(\mathfrak{s})}{4\,G_N}$$

Allows for phase transitions, e.g. jump in surface for S(AB):



Proof of Strong Subadditivity

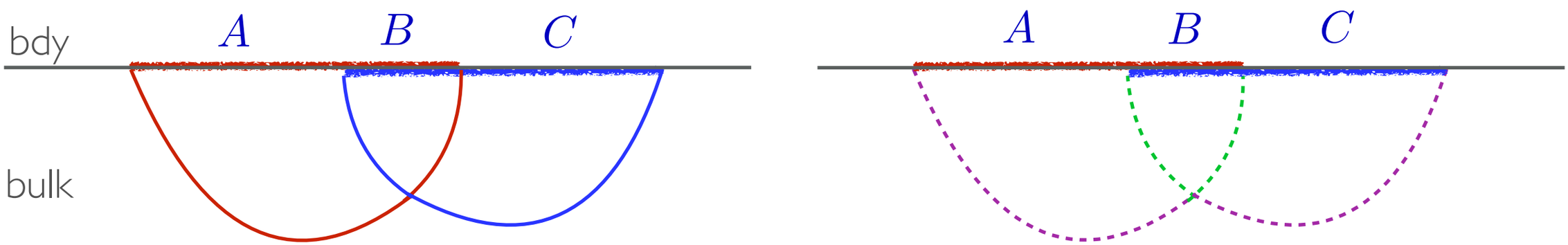
- strong subadditivity: $S(AB) + S(BC) \geq S(B) + S(ABC)$
- proof in static configurations [Headrick&Takayanagi]



$$S(AB) + S(BC)$$

Proof of Strong Subadditivity

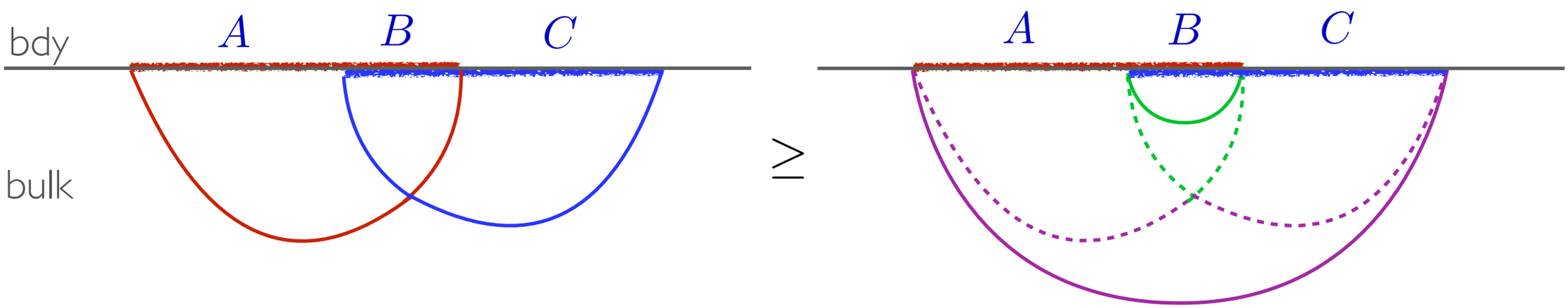
- strong subadditivity: $S(AB) + S(BC) \geq S(B) + S(ABC)$
- proof in static configurations [Headrick&Takayanagi]



$$S(AB) + S(BC) = \alpha + \beta$$

Proof of Strong Subadditivity

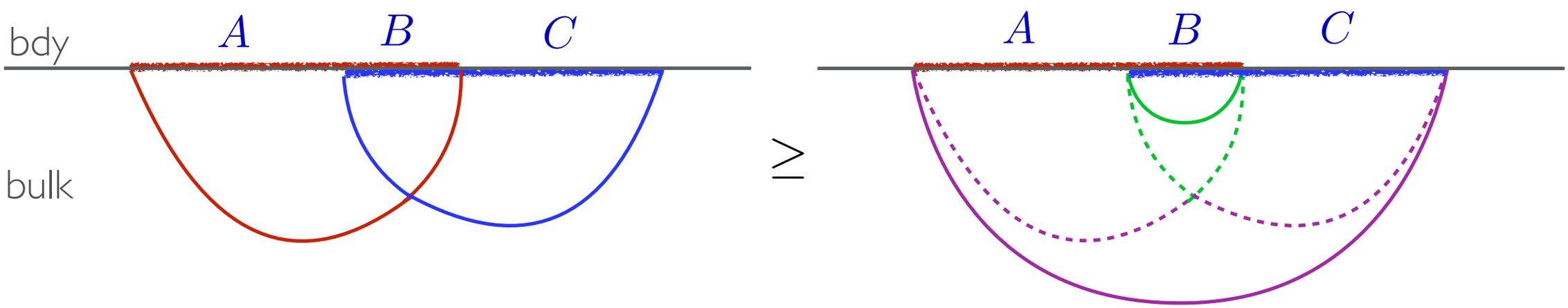
- strong subadditivity: $S(AB) + S(BC) \geq S(B) + S(ABC)$
- proof in static configurations [Headrick&Takayanagi]



$$S(AB) + S(BC) = \alpha + \beta \geq S(B) + S(ABC)$$

Proof of Strong Subadditivity

- strong subadditivity: $S(AB) + S(BC) \geq S(B) + S(ABC)$
- proof in static configurations [Headrick&Takayanagi]



$$S(AB) + S(BC) = \alpha + \beta \geq S(B) + S(ABC)$$

- proof in time-dependent setting uses maximin prescription [Wall]

Entropy cone

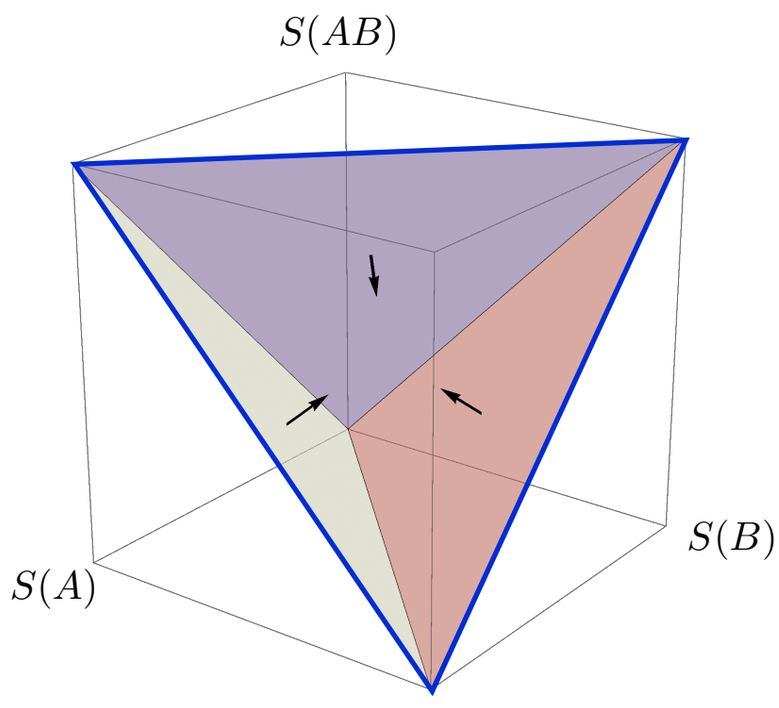
{All physically allowed entropy vectors} = convex cone in entropy space

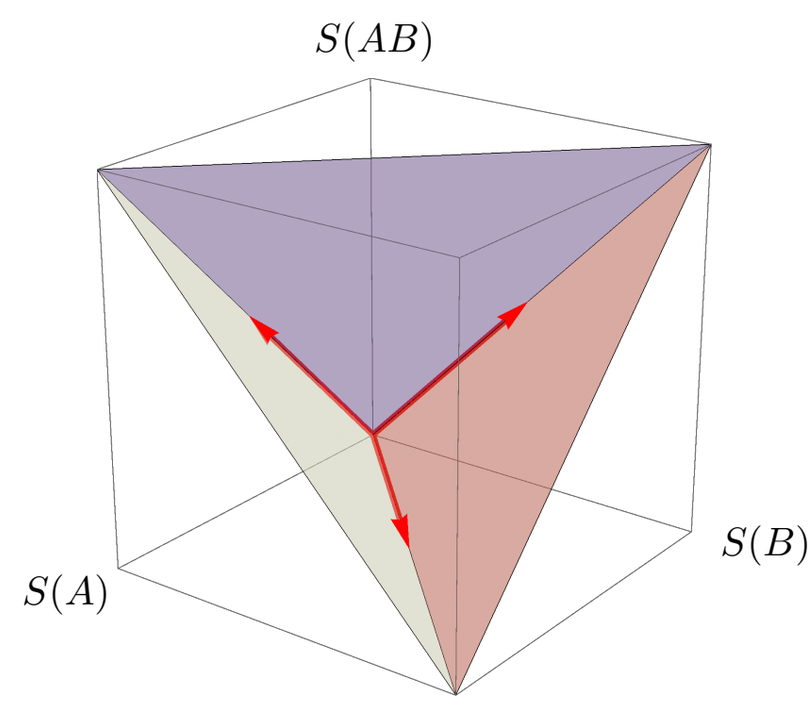
2 useful characterizations of a polyhedral cone:

intersection of half-spaces
delineated by entropy inequalities (facets)

convex hull of extreme rays

eg. at N=2:





Entropy cone

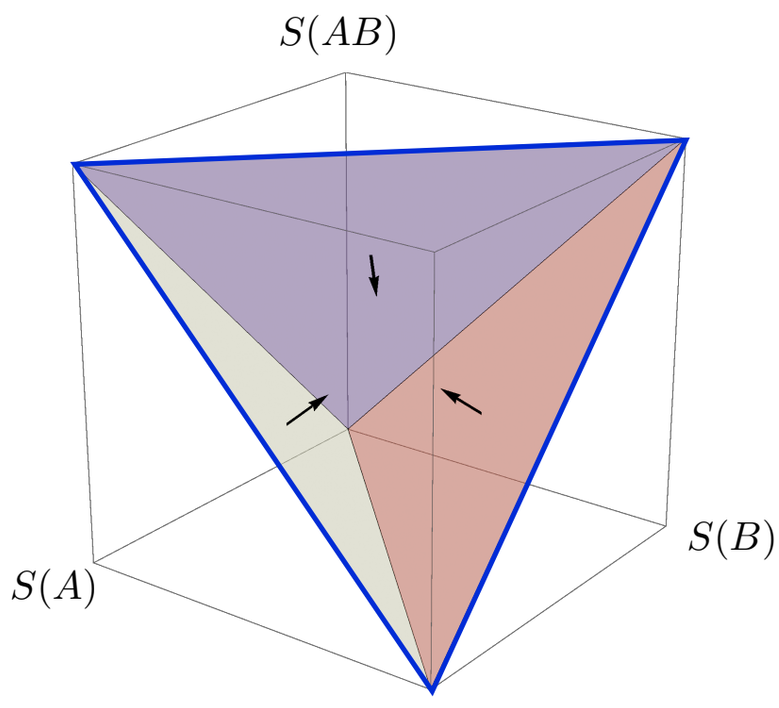
{All physically allowed entropy vectors} = convex cone in entropy space

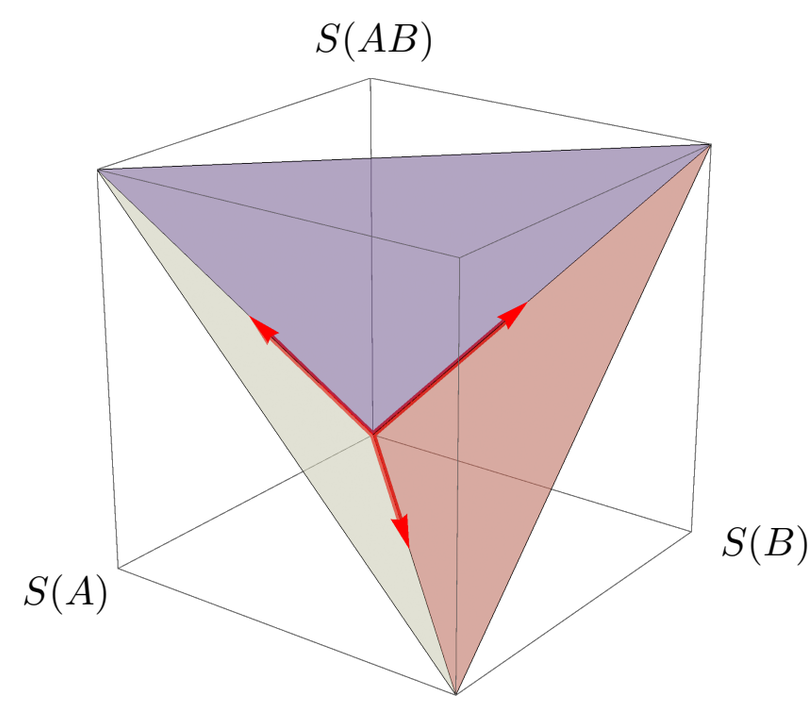
2 useful characterizations of a polyhedral cone:

intersection of half-spaces
delineated by entropy inequalities (facets)

convex hull of extreme rays

eg. at N=2:





when restricted to geometric states in holography ⇝ *holographic entropy cone* (HEC)

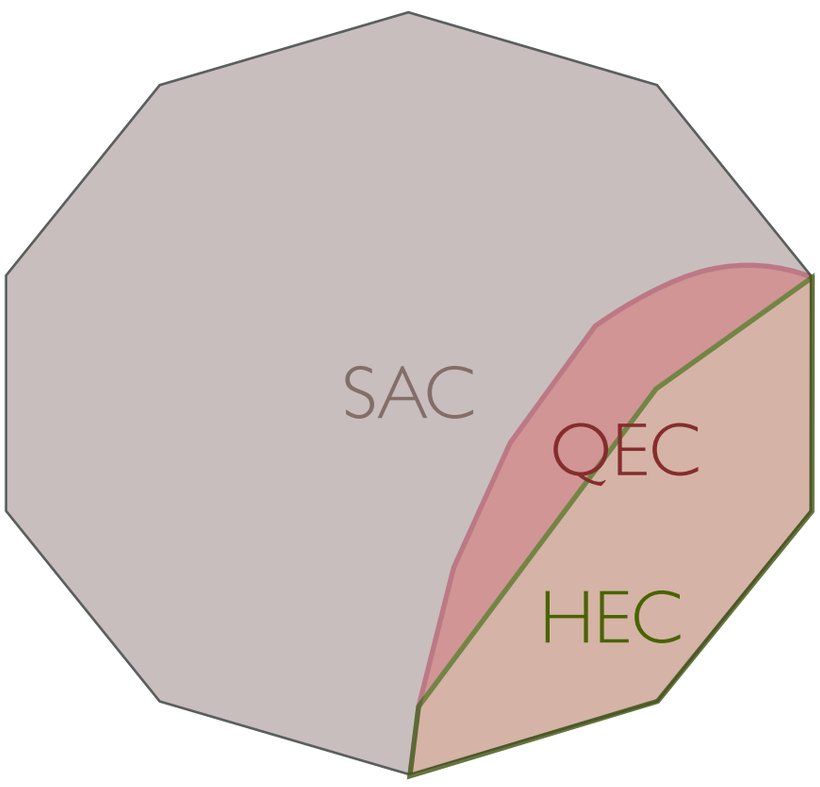
[Bao, Nezami, Ooguri, Stoica, Sully, Walter '15]

Hierarchy of cones in entropy space

- Consider entropy space for fixed N
  - holographic entropy cone: HEC = { holographically realizable $\vec{S}$ }
  - quantum entropy cone: QEC = { physically realizable $\vec{S}$ }
  - subadditivity cone: SAC = { $\vec{S}$ compatible with all instances of SA }

- These are nested convex sets:

SAC ⊃ QEC ⊃ HEC



Hierarchy of cones in entropy space

- Consider entropy space for fixed N
  - holographic entropy cone: HEC = { holographically realizable $\vec{S}$ }
  - quantum entropy cone: QEC = { physically realizable $\vec{S}$ }
  - subadditivity cone: SAC = { $\vec{S}$ compatible with all instances of SA }

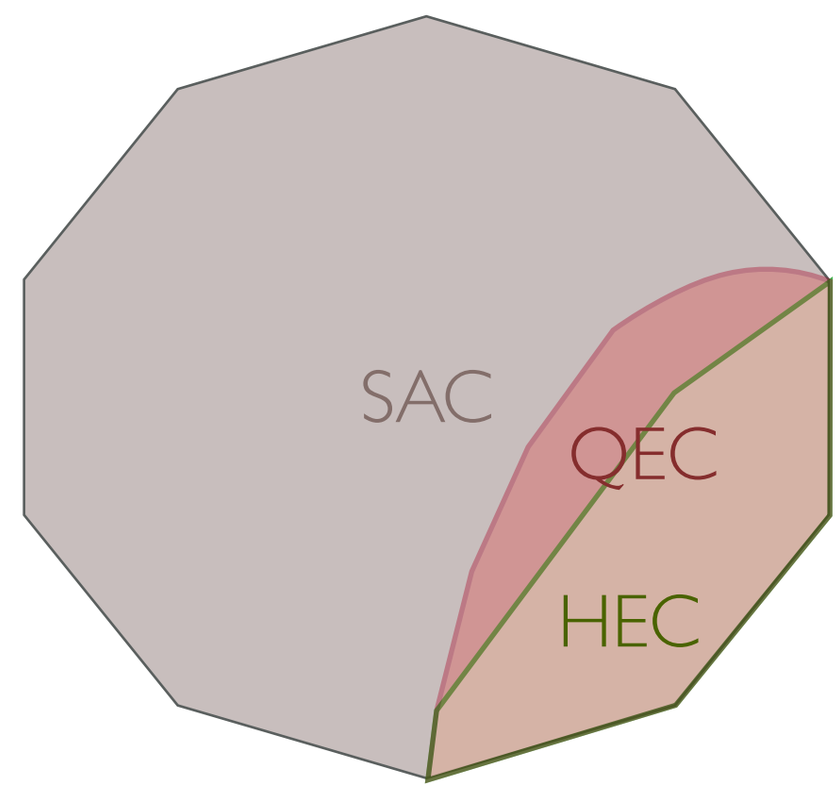
- These are nested convex sets:

SAC ⊃ QEC ⊃ HEC

- Many more...

{ $\vec{S}$ compatible w/ SA & SSA }

convex hull of SAC ERs in HEC



Hierarchy of cones in entropy space

- Consider entropy space for fixed N
  - holographic entropy cone: HEC = { holographically realizable $\vec{S}$ }
  - quantum entropy cone: QEC = { physically realizable $\vec{S}$ }
  - subadditivity cone: SAC = { $\vec{S}$ compatible with all instances of SA }

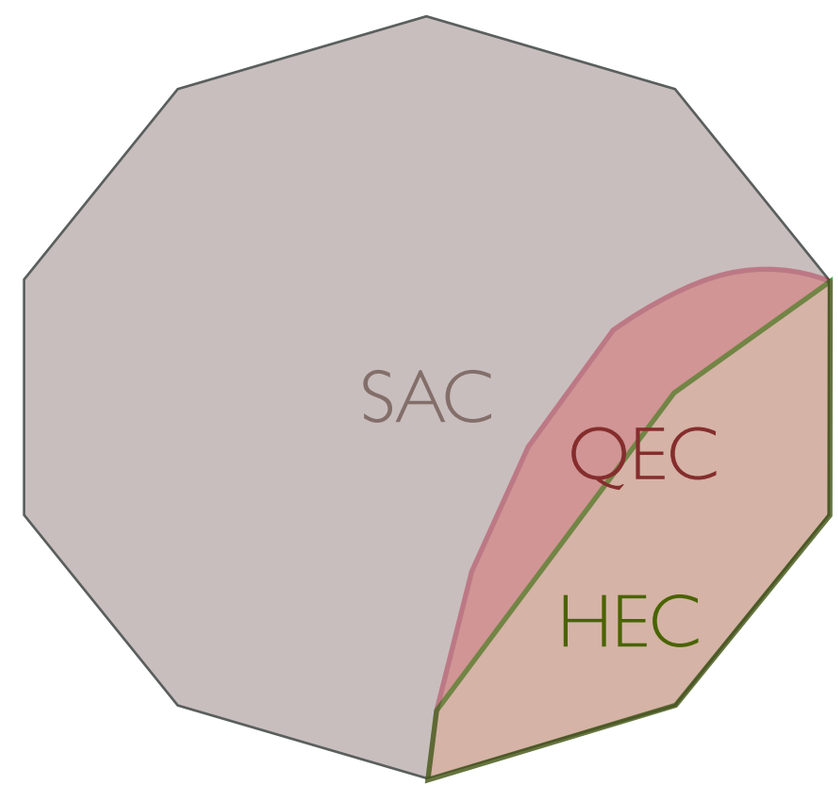
- These are nested convex sets:

  SAC ⊃ QEC ⊃ HEC

- Many more...

  { $\vec{S}$ compatible w/ SA & SSA }

  convex hull of SAC ERs in HEC



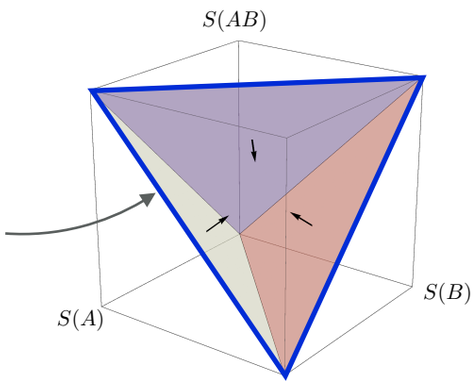
- At N=2 all these cones in $\mathbb{R}^3$ coincide, at N>2 they are strictly nested

N=3 example

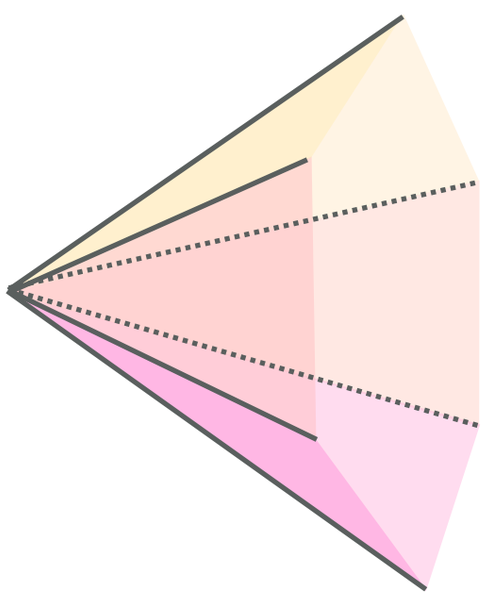
cartoon of 3-d cross-section of $\mathbb{R}^7$ (not including the origin)

analogous to

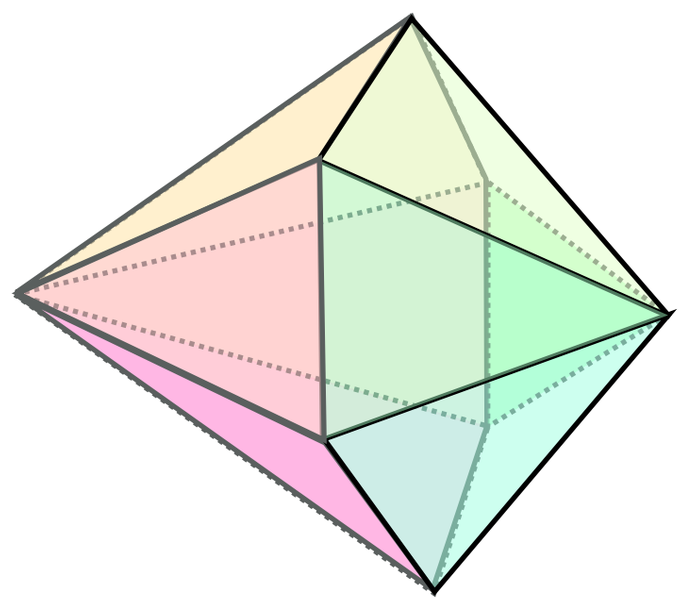
for N=2



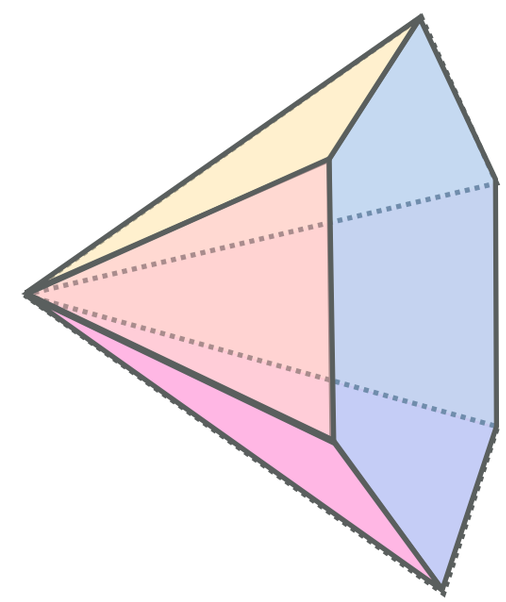
SAC



QEC



HEC



delimited by:

{SA}

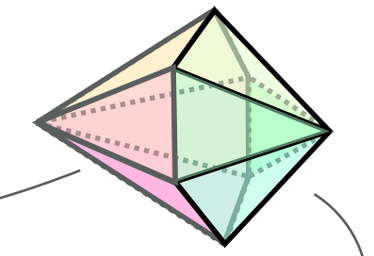
{SA, SSA}

{SA, MMI}

Entropy relations for N=3

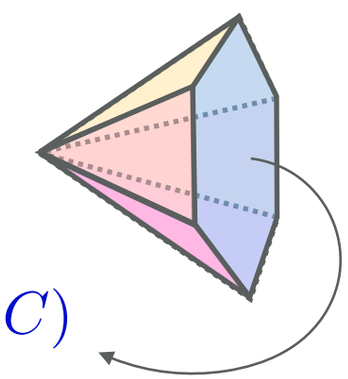
Recall:

- Universal:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$
    - ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AC) + S(BC) \geq S(C) + S(ABC)$
    - ⇨ Mutual information monotonicity $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$



- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

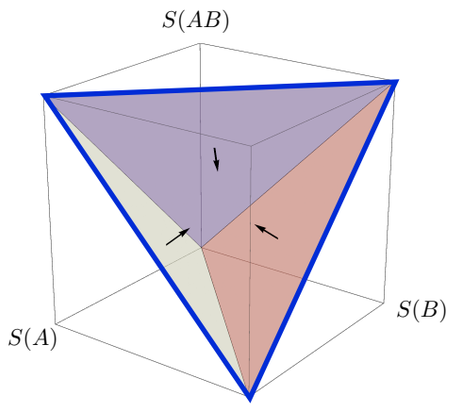


⇨ Tripartite information $I_3(A:B:C) \equiv I(A:B) + I(A:C) - I(A:BC) \leq 0$

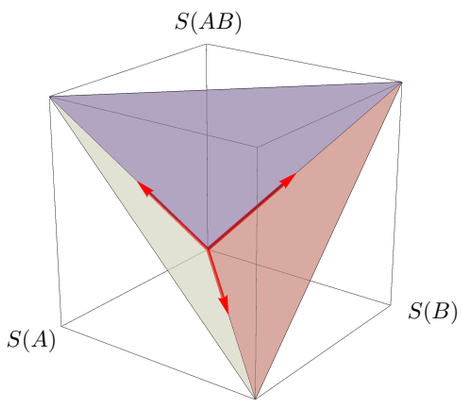
- Note: SSA becomes redundant since SSA = SA + MMI

OUTLINE

- HEC in terms of facets
  - previously-known results: N≤5
  - rewriting in tripartite form
  - new holographic entropy inequalities for N=6
  - no-go for correlation measures



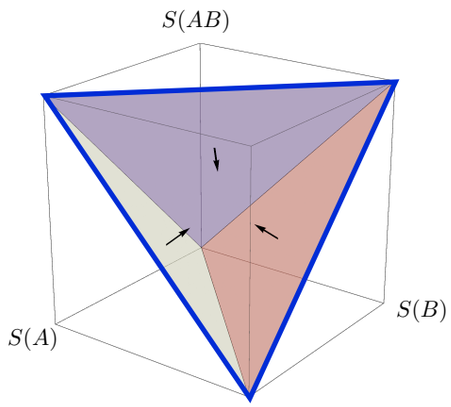
- HEC in terms of extreme rays
  - previously-known results: N≤5
  - HEC from SAC and marginal independence
  - new extreme rays for N=6
  - gap between holographic and quantum SAC ERs



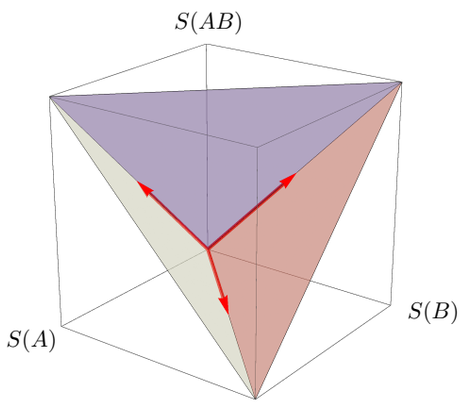
- Summary & future directions

OUTLINE

- HEC in terms of facets
  - previously-known results: N≤5
  - rewriting in tripartite form
  - new holographic entropy inequalities for N=6
  - no-go for correlation measures



- HEC in terms of extreme rays
  - previously-known results: N≤5
  - HEC from SAC and marginal independence
  - new extreme rays for N=6
  - gap between holographic and quantum SAC ERs



- Summary & future directions

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

- N=5 HEC has 5 further inequalities
  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]
  & proved to be the complete set in [Hernández-Cuenca, '19]

⇝ N=5 HEC has 8 orbits of facets (total 372)

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

- N=5 HEC has 5 further inequalities
  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]
  & proved to be the complete set in [Hernández-Cuenca, '19]

⇝ N=5 HEC has 8 orbits of facets (total 372)

e.g.: $$0 \leq -\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{EA}} - \mathtt{S}_{\mathrm{ABCDE}} + \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{CDE}} + \mathtt{S}_{\mathrm{DEA}} + \mathtt{S}_{\mathrm{EAB}}$$

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

- N=5 HEC has 5 further inequalities
  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]
  & proved to be the complete set in [Hernández-Cuenca, '19]

⇝ N=5 HEC has 8 orbits of facets (total 372)

e.g.: $$0 \leq -\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{EA}} - \mathtt{S}_{\mathrm{ABCDE}} + \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{CDE}} + \mathtt{S}_{\mathrm{DEA}} + \mathtt{S}_{\mathrm{EAB}}$$

$$0 \leq \frac{-2\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ABCE}} - 2\mathtt{S}_{\mathrm{ABDE}} - \mathtt{S}_{\mathrm{ACDE}} - 2\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - 2\mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{AE}} - 2\mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{CE}} - 2\mathtt{S}_{\mathrm{DE}}}{3\mathtt{S}_{\mathrm{ABC}} + 3\mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + 3\mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}}$$

HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⇝ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

- N=5 HEC has 5 further inequalities
  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]
  & proved to be the complete set in [Hernández-Cuenca, '19]

⇝ N=5 HEC has 8 orbits of facets (total 372)

e.g.: $$0 \leq -\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{EA}} - \mathtt{S}_{\mathrm{ABCDE}} + \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{CDE}} + \mathtt{S}_{\mathrm{DEA}} + \mathtt{S}_{\mathrm{EAB}}$$

$$0 \leq \frac{-2\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ABCE}} - 2\mathtt{S}_{\mathrm{ABDE}} - \mathtt{S}_{\mathrm{ACDE}} - 2\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - 2\mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{AE}} - 2\mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{CE}} - 2\mathtt{S}_{\mathrm{DE}}}{3\mathtt{S}_{\mathrm{ABC}} + 3\mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + 3\mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}}$$

?: How do we find HEC systematically & understand its meaning / implications?

HEI in terms of I_n

- Re-write in terms of more compact expressions
  - Multipartite informations with 'singleton' arguments form a basis

$$\{I_1(A), \ldots, I_2(A:B), \ldots, I_3(A:B:C), \ldots, I_{\mathsf{N}}(A:B:\ldots)\}$$

w/ tripartite information:

$$I_3(\mathrm{A{:}B{:}C}) = \mathtt{S}_{\mathrm{A}} + \mathtt{S}_{\mathrm{B}} + \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - \mathtt{S}_{\mathrm{BC}} + \mathtt{S}_{\mathrm{ABC}}$$

& more generally, multipartite information:

$$\mathtt{I}_{\mathcal{J}} = \sum_{\mathcal{K}\subseteq\mathcal{J}} (-1)^{|\mathcal{K}|+1}\, \mathtt{S}_{\mathcal{K}}$$

HEI in terms of I_n

- Re-write in terms of more compact expressions
  - Multipartite informations with 'singleton' arguments form a basis

$$\{I_1(A), \ldots, I_2(A:B), \ldots, I_3(A:B:C), \ldots, I_{\mathsf{N}}(A:B:\ldots)\}$$

w/ tripartite information:

$$I_3(\mathrm{A{:}B{:}C}) = \mathtt{S}_{\mathrm{A}} + \mathtt{S}_{\mathrm{B}} + \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - \mathtt{S}_{\mathrm{BC}} + \mathtt{S}_{\mathrm{ABC}}$$

& more generally, multipartite information:

$$\mathtt{I}_{\mathcal{I}} = \sum_{\mathcal{K}\subseteq\mathcal{I}} (-1)^{|\mathcal{K}|+1}\, \mathtt{S}_{\mathcal{K}}$$

- Multipartite informations with composite arguments:

e.g. $$I_3(\mathrm{A{:}B{:}CD}) = \mathtt{S}_{\mathrm{A}} + \mathtt{S}_{\mathrm{B}} + \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{ACD}} - \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{ABCD}}$$

Tripartite form of HEIs

- In the I-basis, HEIs are simpler [He, Headrick, VH], but not simple enough...

e.g.: $0 \leq Q = -\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$+ \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$

vs. $Q = \mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$

Tripartite form of HEIs

- In the I-basis, HEIs are simpler [He, Headrick, VH], but not simple enough...

e.g.: $0 \leq Q = -\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$+ \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$

vs. $Q = \mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$

- However, written in terms of I_3 and its conditional form, w/ composite arguments, and all signs negative $\equiv$ "tripartite form":

$$Q = \sum_i -I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \mathrm{W}_i)$$

w/ $I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \varnothing) := I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i)$

HEIs become more compact [WIP: Hernández-Cuenca, VH, Jia]:

Tripartite form of HEIs

- In the I-basis, HEIs are simpler [He, Headrick, VH], but not simple enough...

e.g.: $0 \leq Q = -\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$+ \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$

vs. $Q = \mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$

- However, written in terms of I_3 and its conditional form, w/ composite arguments, and all signs negative ≡ "tripartite form":

$$Q = \sum_i -I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \mathrm{W}_i)$$

w/ $I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \varnothing) := I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i)$

HEIs become more compact [WIP: Hernández-Cuenca, VH, Jia]:

vs. $Q = -I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{B{:}D{:}E|C}) - I_3(\mathrm{A{:}C{:}E|D})$

Tripartite form of HEIs

- In the I-basis, HEIs are simpler [He, Headrick, VH], but not simple enough...

e.g.: $0 \leq Q = -\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$+ \mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$

vs. $Q = \mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$

- However, written in terms of I_3 and its conditional form, w/ composite arguments, and all signs negative ≡ "tripartite form":

$$Q = \sum_i -I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \mathrm{W}_i)$$

w/ $I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i \,|\, \varnothing) := I_3(\mathrm{X}_i : \mathrm{Y}_i : \mathrm{Z}_i)$

HEIs become more compact [WIP: Hernández-Cuenca, VH, Jia]:

vs. $Q = -I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{B{:}D{:}E|C}) - I_3(\mathrm{A{:}C{:}E|D})$

- in other examples, 2 terms instead of 11 in I-basis or 13 in S-basis

HEIs for N=5

$\mathsf{N} = 5$ HEI information quantities
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{C}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ACDE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{A{:}B{:}E}) - I_3(\mathrm{C{:}D{:}E}\vert\mathrm{B})$
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{CE}} - \mathtt{S}_{\mathrm{DE}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + 2\mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ACDE}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{BDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{B{:}D{:}E}\vert\mathrm{C}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-\mathtt{S}_{\mathrm{ABCE}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABD}} - \mathtt{S}_{\mathrm{AC}} - \mathtt{S}_{\mathrm{AE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BE}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{DE}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCE}} + \mathtt{I}_{\mathrm{ACDE}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCE}} - \mathtt{I}_{\mathrm{CDE}}$
$-I_3(\mathrm{A{:}B{:}D}) - I_3(\mathrm{B{:}C{:}E}\vert\mathrm{A}) - I_3(\mathrm{C{:}D{:}E}\vert\mathrm{B}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-2\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ABCE}} - 2\mathtt{S}_{\mathrm{ABDE}} - \mathtt{S}_{\mathrm{ACDE}} - 2\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - 2\mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{AE}} - 2\mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{CE}} - 2\mathtt{S}_{\mathrm{DE}}$
$3\mathtt{S}_{\mathrm{ABC}} + 3\mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + 3\mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}$
$2\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ABCE}} + 2\mathtt{I}_{\mathrm{ABDE}} + \mathtt{I}_{\mathrm{ACDE}} - \mathtt{I}_{\mathrm{ABD}} - 2\mathtt{I}_{\mathrm{ABE}} - 2\mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{BDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{AE{:}B{:}D}) - I_3(\mathrm{A{:}B{:}E}\vert\mathrm{C}) - I_3(\mathrm{A{:}B{:}E}\vert\mathrm{D}) - I_3(\mathrm{A{:}C{:}D}\vert\mathrm{B}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$

New HEIs for N=6

- This provides a useful HEI generating technique: [WIP: Hernández-Cuenca, VH, Jia]
  - Posit form,
  - check if true ineq. (via contraction map) vs. not sign definite (via explicit evaluation),
  - if true, check if non-redundant (spanned by D-1 l.i. hol. entropy vectors) or sum of facets

New HEIs for N=6

- This provides a useful HEI generating technique: [WIP: Hernández-Cuenca, VH, Jia]
  - Posit form,
  - check if true ineq. (via contraction map) vs. not sign definite (via explicit evaluation),
  - if true, check if non-redundant (spanned by D-1 l.i. hol. entropy vectors) or sum of facets
- For N=6 we obtained >300 new *orbits* of HEIs
  - So far at least 384 orbits for N=6
  - e.g.:

$-\mathtt{S}_{\mathrm{ABCF}} - \mathtt{S}_{\mathrm{ABDE}} - \mathtt{S}_{\mathrm{ABEF}} - \mathtt{S}_{\mathrm{CDEF}} - \mathtt{S}_{\mathrm{AE}} - \mathtt{S}_{\mathrm{AF}} - \mathtt{S}_{\mathrm{BE}} - \mathtt{S}_{\mathrm{BF}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{CF}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{B}}$
$\mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ABF}} + \mathtt{S}_{\mathrm{ACF}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{AEF}} + \mathtt{S}_{\mathrm{BCF}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{BEF}} + \mathtt{S}_{\mathrm{CDE}} + \mathtt{S}_{\mathrm{CDF}} + \mathtt{S}_{\mathrm{AB}}$
$\mathtt{I}_{\mathrm{ABCF}} + \mathtt{I}_{\mathrm{ABDE}} + \mathtt{I}_{\mathrm{ABEF}} + \mathtt{I}_{\mathrm{CDEF}} - \mathtt{I}_{\mathrm{ABC}} - \mathtt{I}_{\mathrm{ABD}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ABF}} - \mathtt{I}_{\mathrm{CEF}} - \mathtt{I}_{\mathrm{DEF}}$
$-I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}D}\vert\mathrm{E}) - I_3(\mathrm{A{:}B{:}C}\vert\mathrm{F})$
$-\mathtt{S}_{\mathrm{ABCDF}} - \mathtt{S}_{\mathrm{ACEF}} - \mathtt{S}_{\mathrm{BDEF}} - \mathtt{S}_{\mathrm{CDF}} - 2\mathtt{S}_{\mathrm{AF}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{EF}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{B}} - 2\mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}}$
$\mathtt{S}_{\mathrm{BCDF}} + \mathtt{S}_{\mathrm{ABF}} + \mathtt{S}_{\mathrm{ACF}} + \mathtt{S}_{\mathrm{ADF}} + \mathtt{S}_{\mathrm{AEF}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CEF}} + \mathtt{S}_{\mathrm{DEF}} + \mathtt{S}_{\mathrm{AC}} + \mathtt{S}_{\mathrm{CD}}$
$-\mathtt{I}_{\mathrm{ABCDF}} + \mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ABCF}} + \mathtt{I}_{\mathrm{ABDF}} + \mathtt{I}_{\mathrm{ACDF}} + \mathtt{I}_{\mathrm{ACEF}} + \mathtt{I}_{\mathrm{BDEF}}$
$-\mathtt{I}_{\mathrm{ABC}} - \mathtt{I}_{\mathrm{ABD}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{ACF}} - \mathtt{I}_{\mathrm{BDF}} - \mathtt{I}_{\mathrm{BEF}} - \mathtt{I}_{\mathrm{CDF}}$
$-I_3(\mathrm{AF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD}\vert\mathrm{F})$
$-2\mathtt{S}_{\mathrm{ABCDF}} - \mathtt{S}_{\mathrm{ACEF}} - \mathtt{S}_{\mathrm{BDEF}} - \mathtt{S}_{\mathrm{CDF}} - \mathtt{S}_{\mathrm{AF}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{EF}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{B}} - 2\mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}}$
$\mathtt{S}_{\mathrm{ABCF}} + \mathtt{S}_{\mathrm{ABDF}} + \mathtt{S}_{\mathrm{ACDF}} + \mathtt{S}_{\mathrm{BCDF}} + \mathtt{S}_{\mathrm{AEF}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CEF}} + \mathtt{S}_{\mathrm{DEF}} + \mathtt{S}_{\mathrm{AC}} + \mathtt{S}_{\mathrm{CD}}$
$-2\mathtt{I}_{\mathrm{ABCDF}} + 2\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ABCF}} + \mathtt{I}_{\mathrm{ABDF}} + \mathtt{I}_{\mathrm{ACDF}} + \mathtt{I}_{\mathrm{ACEF}} + \mathtt{I}_{\mathrm{BCDF}} + \mathtt{I}_{\mathrm{BDEF}}$
$-\mathtt{I}_{\mathrm{ABC}} - \mathtt{I}_{\mathrm{ABD}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{ACF}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{BDF}} - \mathtt{I}_{\mathrm{BEF}} - \mathtt{I}_{\mathrm{CDF}}$
$-I_3(\mathrm{ABF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD}\vert\mathrm{F})$

Structural relations

- Advantage of tripartite form
  - much more compact
  - easier to see structural relations
  - may suggest how to "bootstrap" to higher N

e.g.:

Structural relations

- Advantage of tripartite form
  - much more compact
  - easier to see structural relations
  - may suggest how to "bootstrap" to higher N

e.g.:

N

$$
\begin{aligned}
Q^{[5]}_{\{4,2,0\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{5,3,0\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{6,5,1\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{7,6,1\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F})
\end{aligned}
$$

$\{\#I_3, \#I_4, \#I_5\}$

Structural relations

- Advantage of tripartite form
  - much more compact
  - easier to see structural relations
  - may suggest how to "bootstrap" to higher N

e.g.:

N

$$
\begin{aligned}
Q^{[5]}_{\{4,2,0\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{5,3,0\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{6,5,1\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[6]}_{\{7,6,1\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F})
\end{aligned}
$$

$\{\#I_3, \#I_4, \#I_5\}$

or:

$$
\begin{aligned}
Q^{[6]}_{\{8,6,1\}} &= -I_3(\mathrm{AF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F}) \\
Q^{[6]}_{\{8,6,1\}} &= -I_3(\mathrm{BF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F}) \\
Q^{[6]}_{\{9,8,2\}} &= -I_3(\mathrm{ABF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F})
\end{aligned}
$$

Higher HEIs & correlation measures

- Correlation measure:
  - Non-negative
  - Monotonically increasing under inclusion

Higher HEIs & correlation measures

- Correlation measure:
  - Non-negative
  - Monotonically increasing under inclusion
- cf. mutual information:
  - SA: $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - SSA: $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$

Higher HEIs & correlation measures

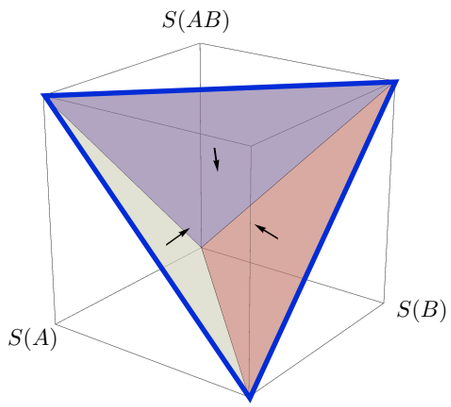
- Correlation measure:
  - Non-negative
  - Monotonically increasing under inclusion
- cf. mutual information:
  - SA: $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - SSA: $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$
- Higher HEIs are *superbalanced* [He, VH, Rangamani]
  - finite on any configuration
  - in I-basis, contain only I_n for $n \geq 3$
  - $\Rightarrow$ not monotonic under inclusion [WIP: Hernández-Cuenca, VH, Jia]

Higher HEIs & correlation measures

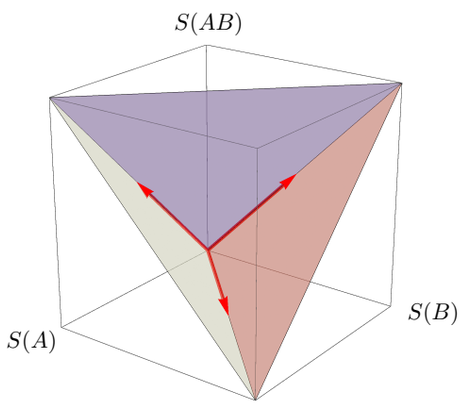
- Correlation measure:
  - Non-negative
  - Monotonically increasing under inclusion
- cf. mutual information:
  - SA: $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - SSA: $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$
- Higher HEIs are *superbalanced* [He, VH, Rangamani]
  - finite on any configuration
  - in I-basis, contain only I_n for $n \geq 3$
  - $\Rightarrow$ not monotonic under inclusion [WIP: Hernández-Cuenca, VH, Jia]
- Higher HEI quantities are <u>not</u> a correlation measures

OUTLINE

- HEC in terms of facets
  - previously-known results: N≤5
  - rewriting in tripartite form
  - new holographic entropy inequalities for N=6
  - no-go for correlation measures



- HEC in terms of extreme rays
  - previously-known results: N≤5
  - HEC from SAC and marginal independence
  - new extreme rays for N=6
  - gap between holographic and quantum SAC ERs



- Summary & future directions

HEC in terms of ERs

- ERs correspond to special/extreme states
  - HEC = convex hull of ERs
  - Simultaneously saturate (D-1) independent HEIs
  - Holographically correspond to multi-boundary wormholes

- Useful toolkit:
  - Holographic graph model
  - Pattern of marginal independence (PMI)

HEC in terms of ERs

- ERs correspond to special/extreme states
  - HEC = convex hull of ERs
  - Simultaneously saturate (D-1) independent HEIs
  - Holographically correspond to multi-boundary wormholes

- Useful toolkit:
  - Holographic graph model ← $\vec{S}$ from spacetime
  - Pattern of marginal independence (PMI) ← entanglement structure from $\vec{S}$

HEC in terms of ERs

- ERs correspond to special/extreme states
  - HEC = convex hull of ERs
  - Simultaneously saturate (D-1) independent HEIs
  - Holographically correspond to multi-boundary wormholes

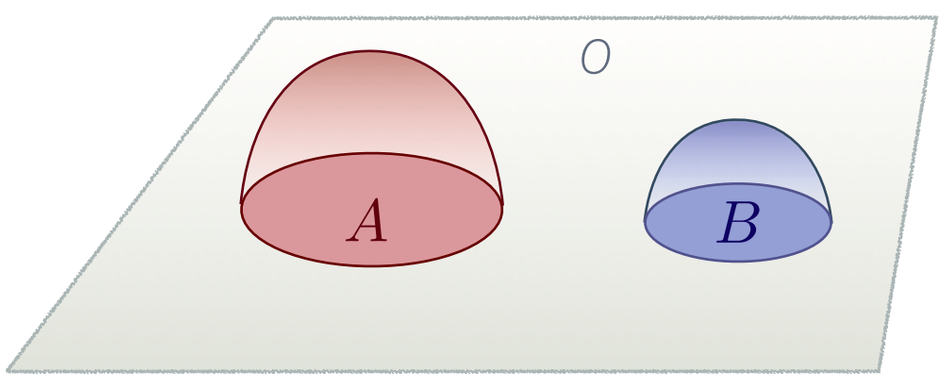
- Useful toolkit:
  - Holographic graph model ← $\vec{S}$ from spacetime
  - Pattern of marginal independence (PMI) ← entanglement structure from $\vec{S}$

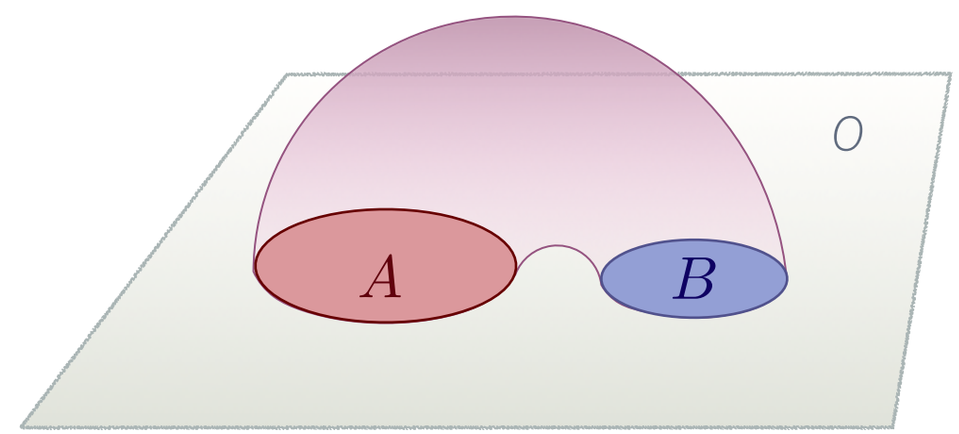
These will lead us to study the ERs of the SAC.

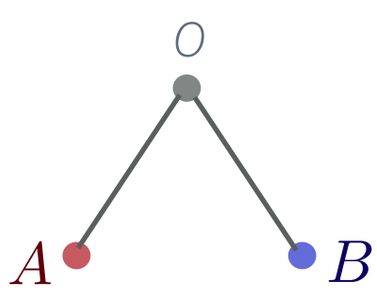
Holographic graph models

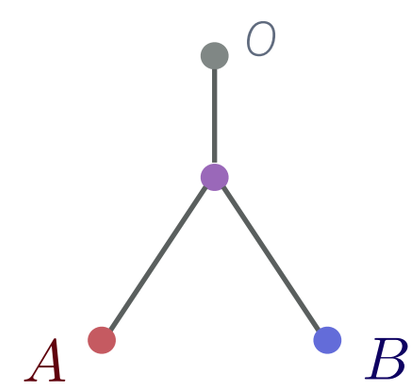
cf. [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

- Distill the discrete elements from a holographic configuration
  - vertices = cells in RT surface network
  - edges = pieces of RT surfaces separating neighboring regions, with weight = corresponding area
  - EE = min-cut





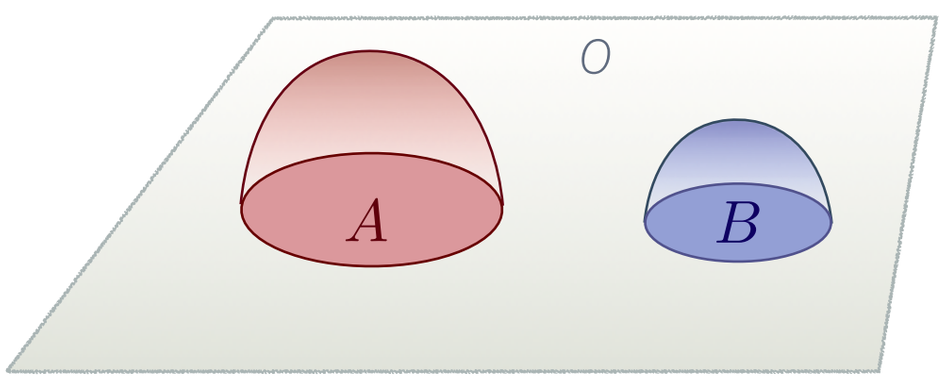


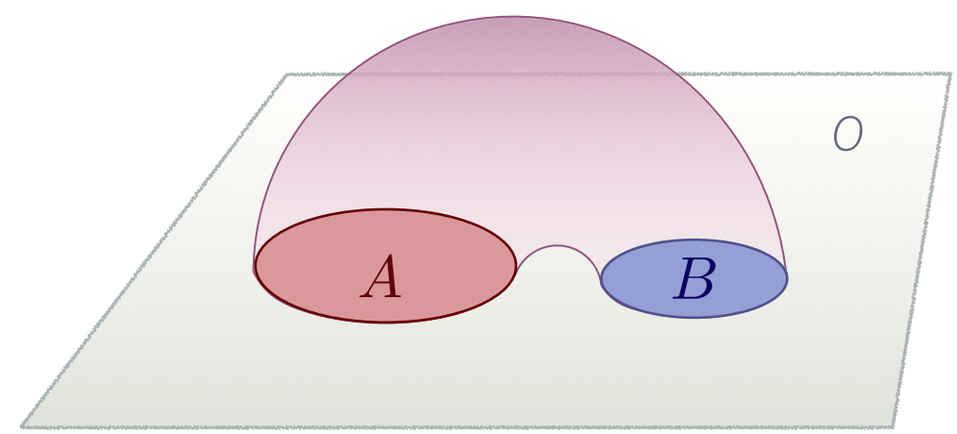


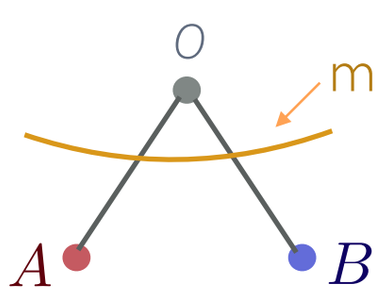
Holographic graph models

cf. [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

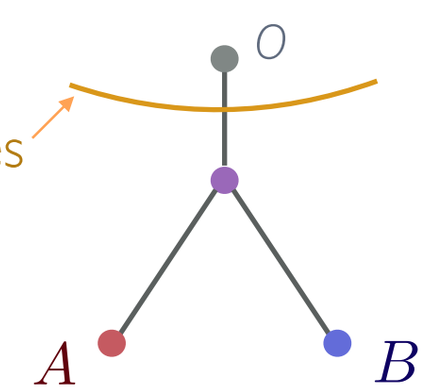
- Distill the discrete elements from a holographic configuration
  - vertices = cells in RT surface network
  - edges = pieces of RT surfaces separating neighboring regions, with weight = corresponding area
  - EE = min-cut







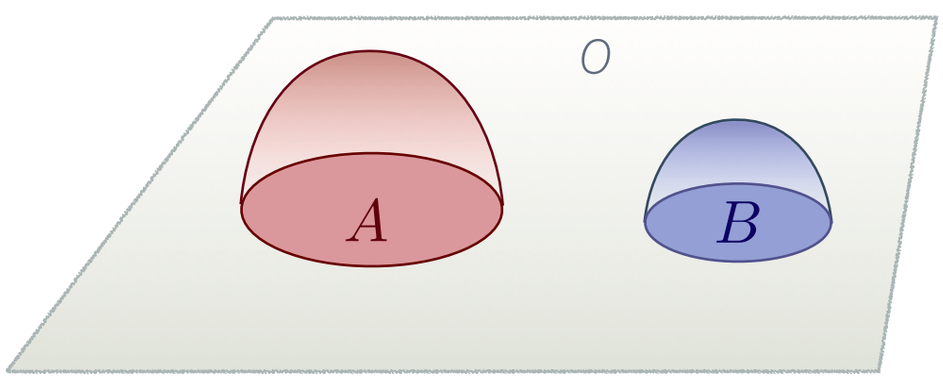
min-cut structure specified by cut edges

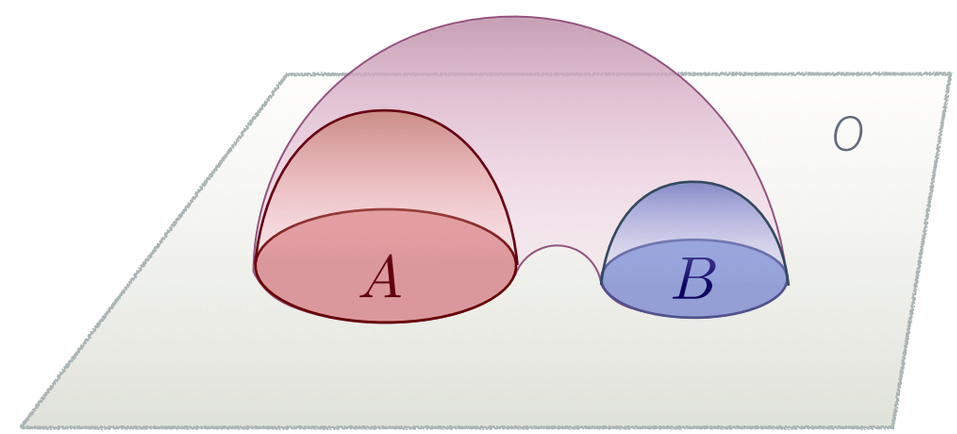


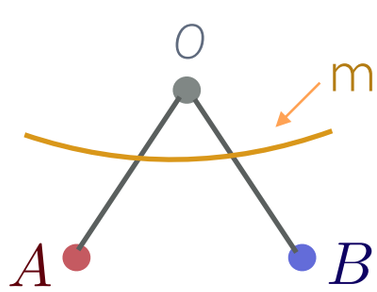
Holographic graph models

cf. [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

- Distill the discrete elements from a holographic configuration
  - vertices = cells in RT surface network
  - edges = pieces of RT surfaces separating neighboring regions, with weight = corresponding area
  - EE = min-cut

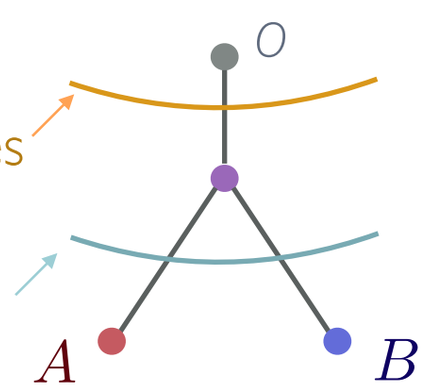






min-cut structure specified by cut edges

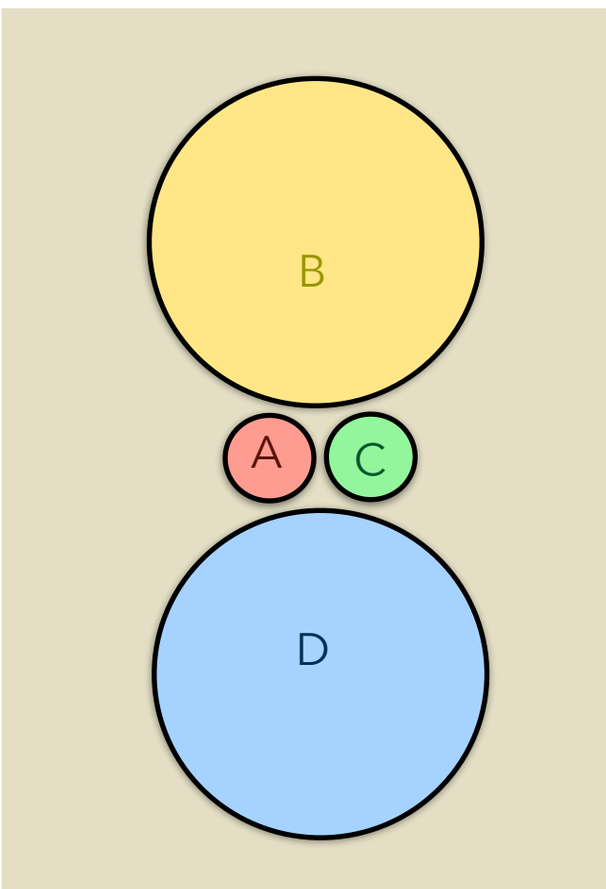
first phase incorporated by selecting alternate mincut

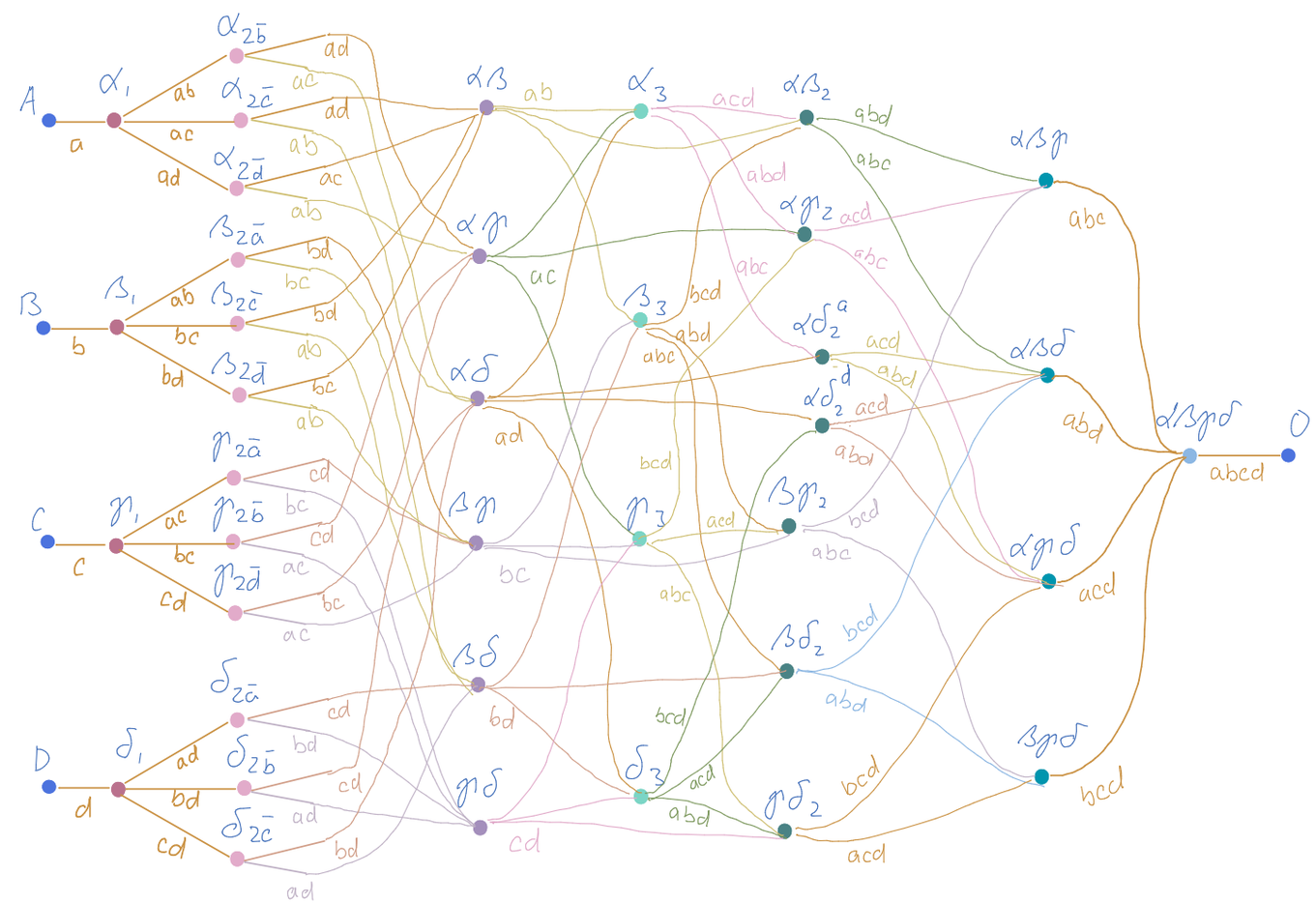


Holographic graph models

- Graphs can get complicated...

e.g. graph model for



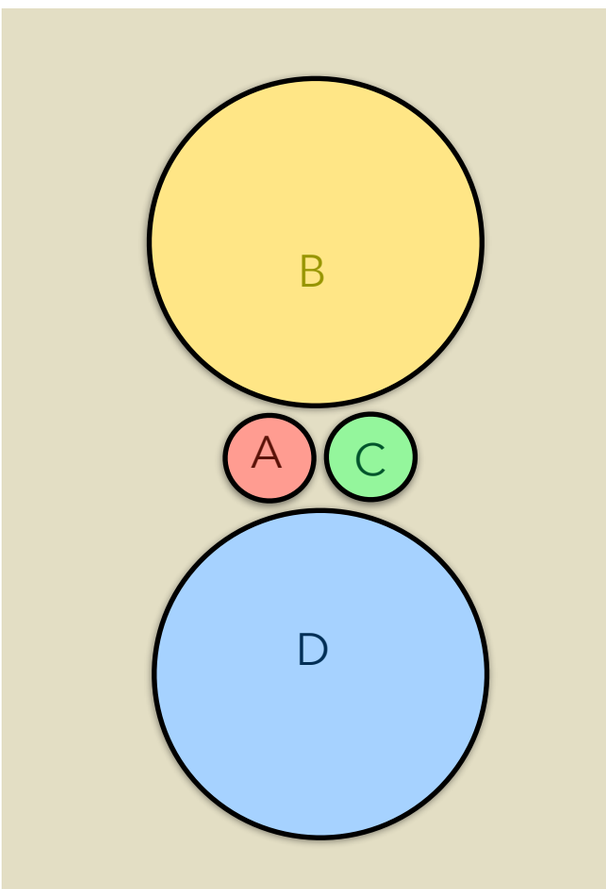


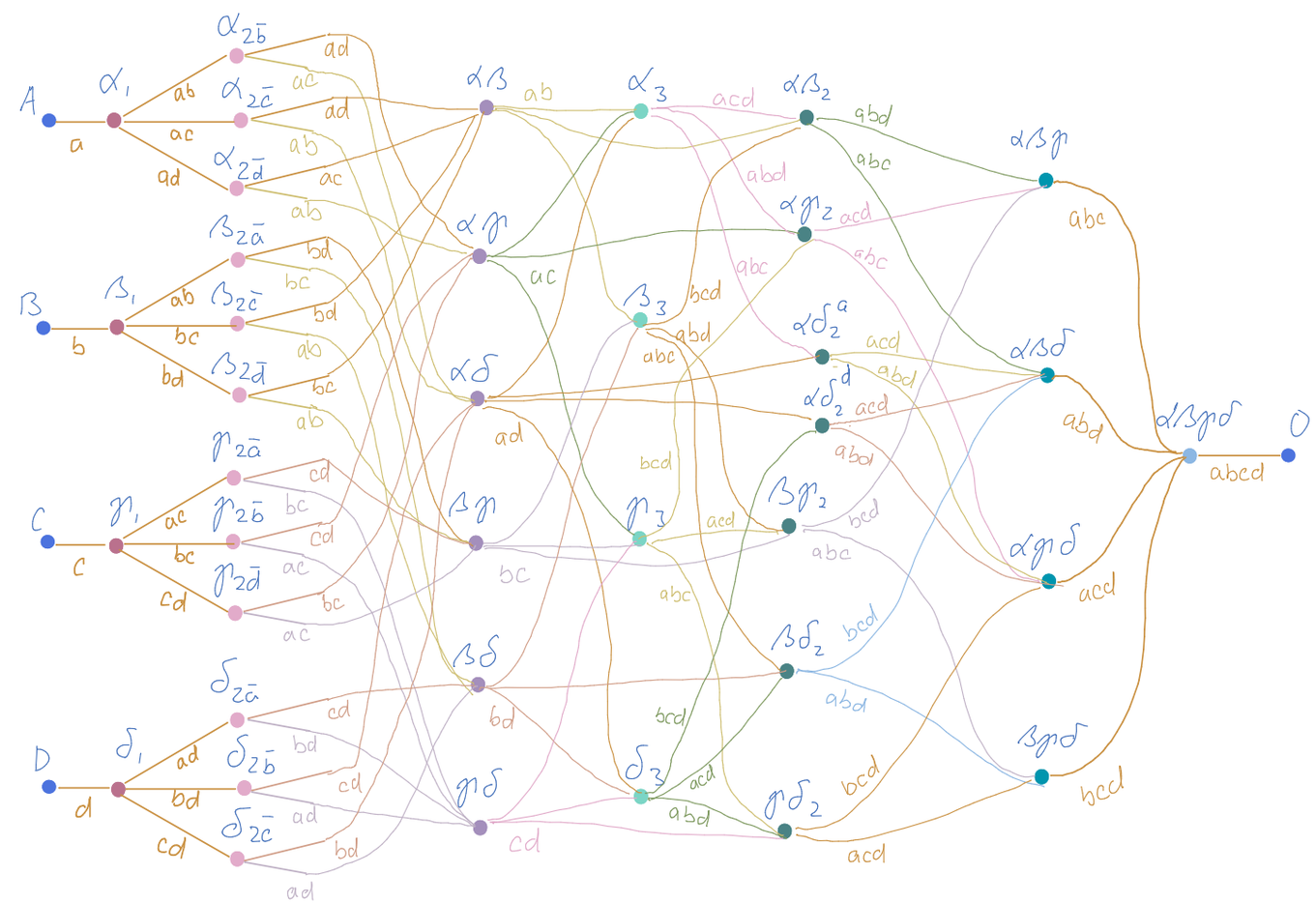
has 43 vertices & 88 edges

Holographic graph models

- Graphs can get complicated...

e.g. graph model for





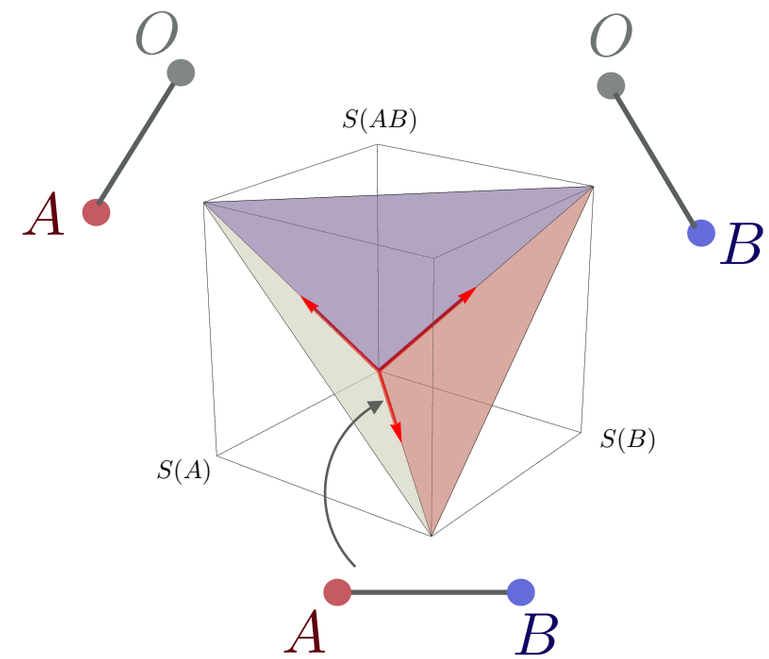
has 43 vertices & 88 edges

- But graph cone = HEC

[Bao, Nezami, Ooguri, Stoica, Sully, Walter]

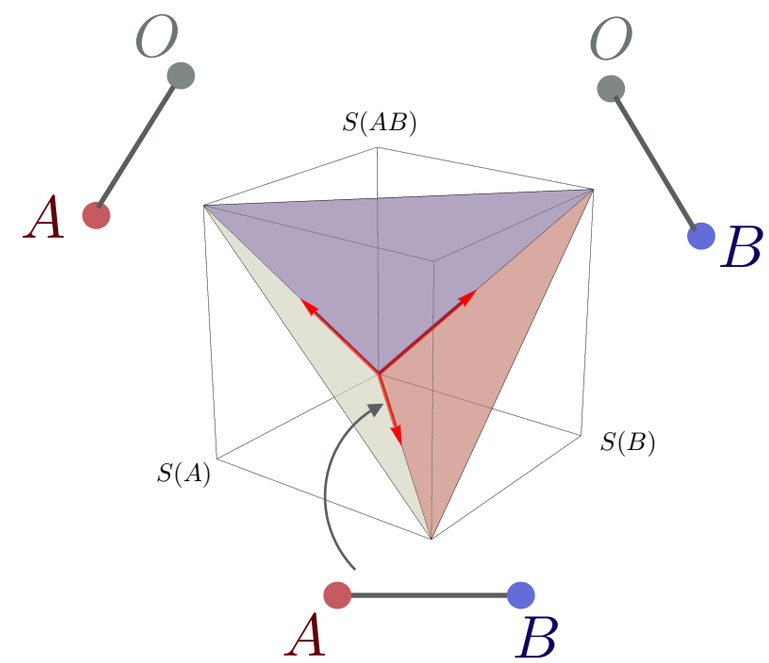
Graph representation of HEC ERs

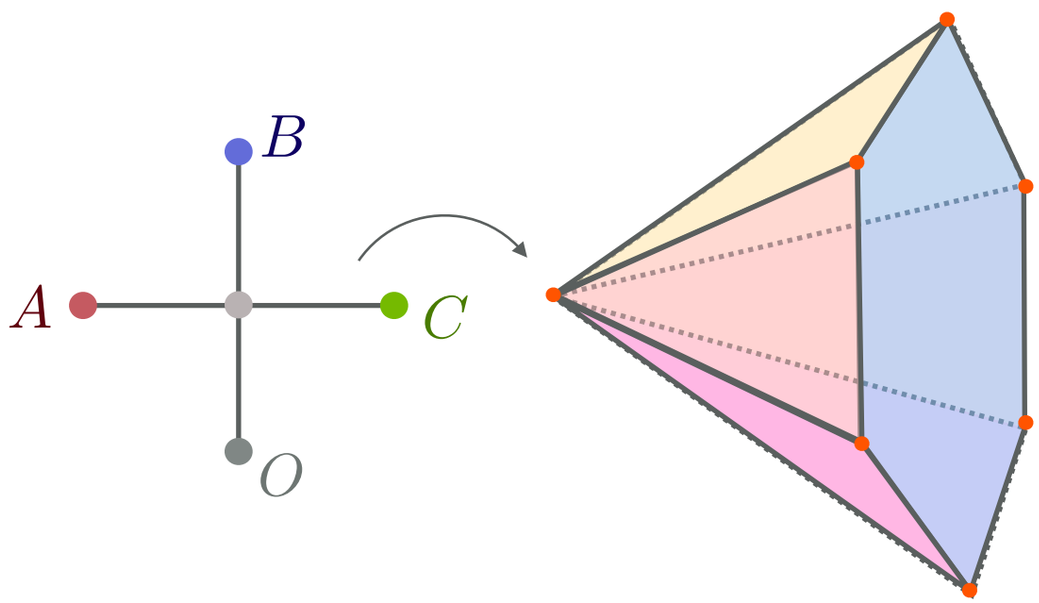
- N=2 HEC has 3 extreme rays (1 orbit)



Graph representation of HEC ERs

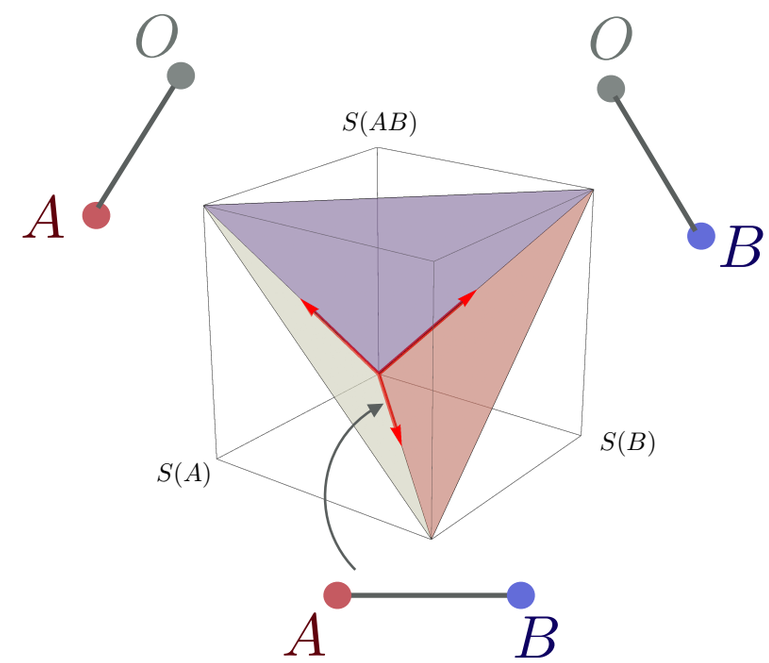
- N=2 HEC has 3 extreme rays (1 orbit)
- N=3 HEC has 7 extreme rays (2 orbits)

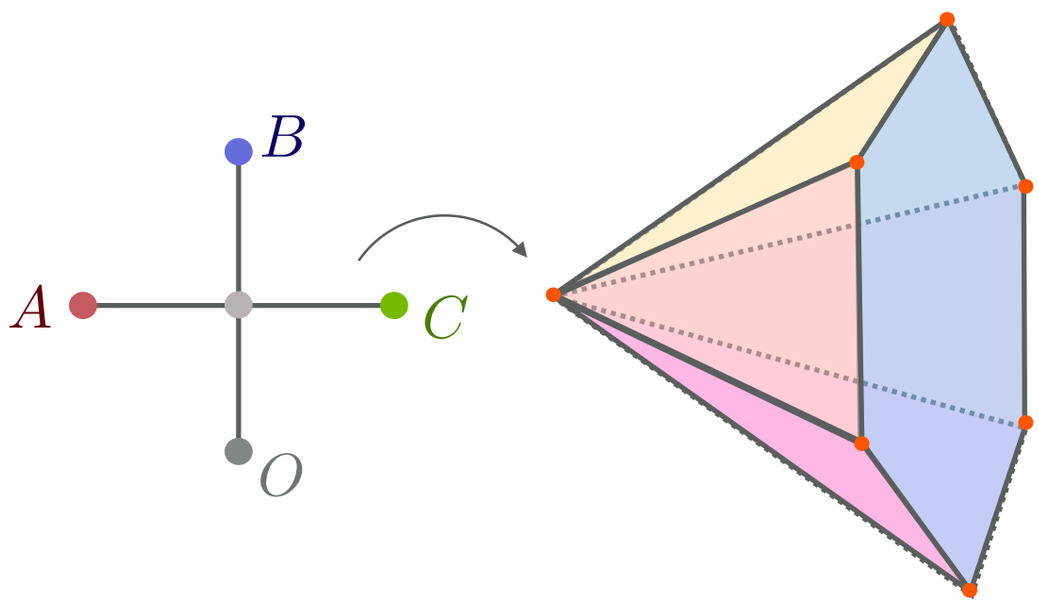




Graph representation of HEC ERs

- N=2 HEC has 3 extreme rays (1 orbit)
- N=3 HEC has 7 extreme rays (2 orbits)

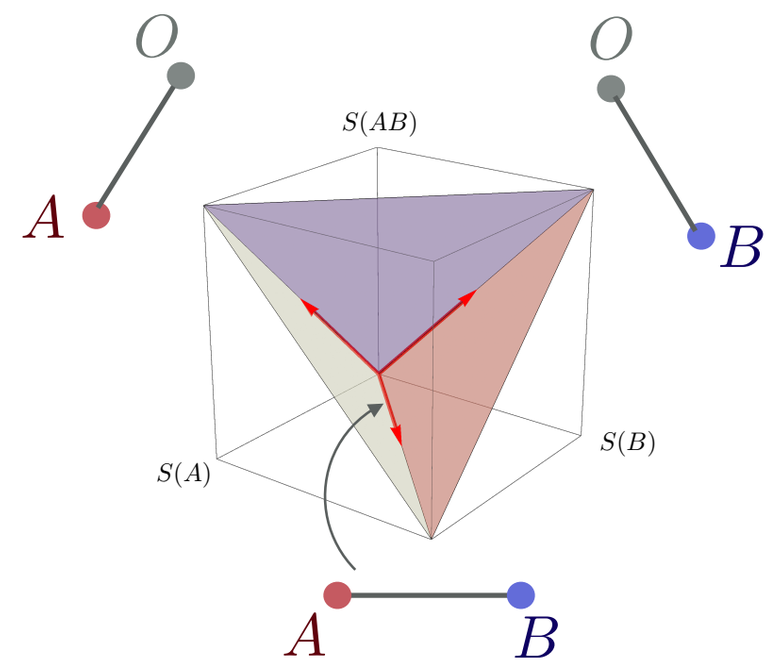




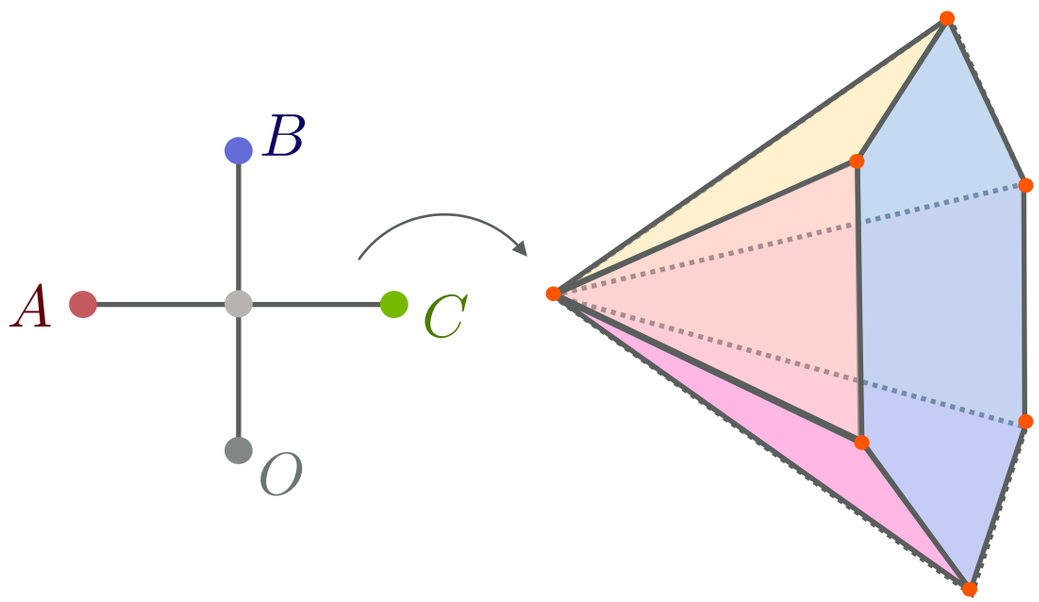
- N=4 HEC has 20 extreme rays (3 orbits) adds

Graph representation of HEC ERs

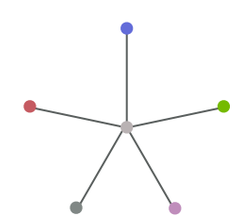
- N=2 HEC has 3 extreme rays (1 orbit)



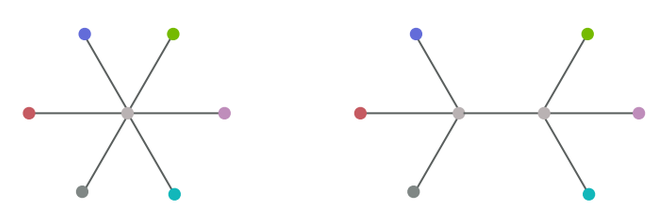
- N=3 HEC has 7 extreme rays (2 orbits)



- N=4 HEC has 20 extreme rays (3 orbits) adds



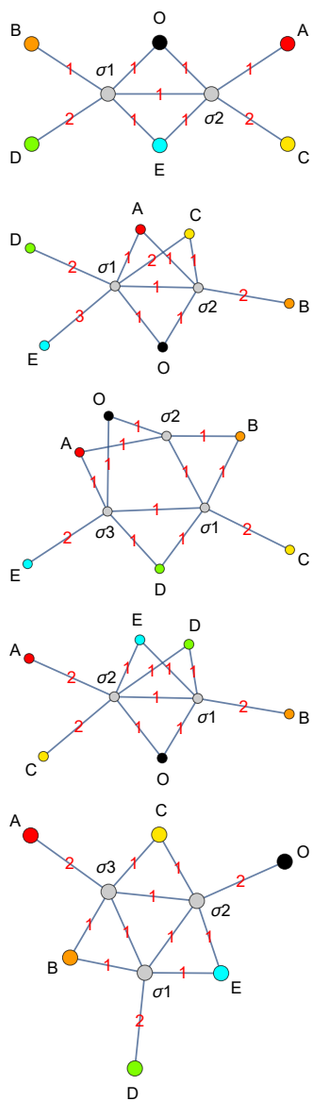
- N=5 HEC has 2267 extreme rays (19 orbits) adds

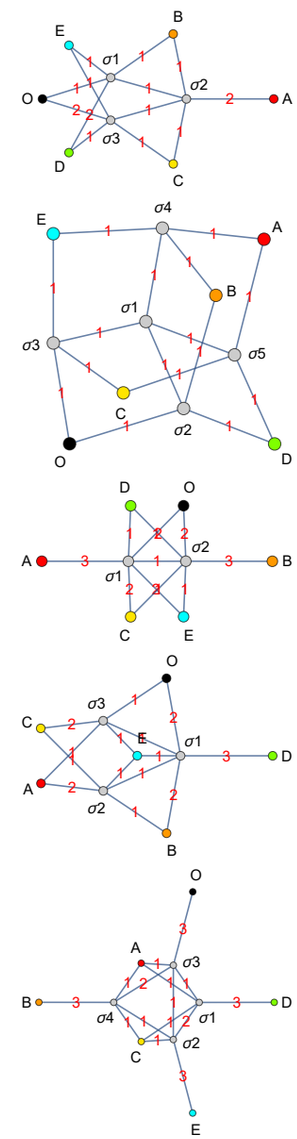


and:

Graph representation of HEC_5 ERs

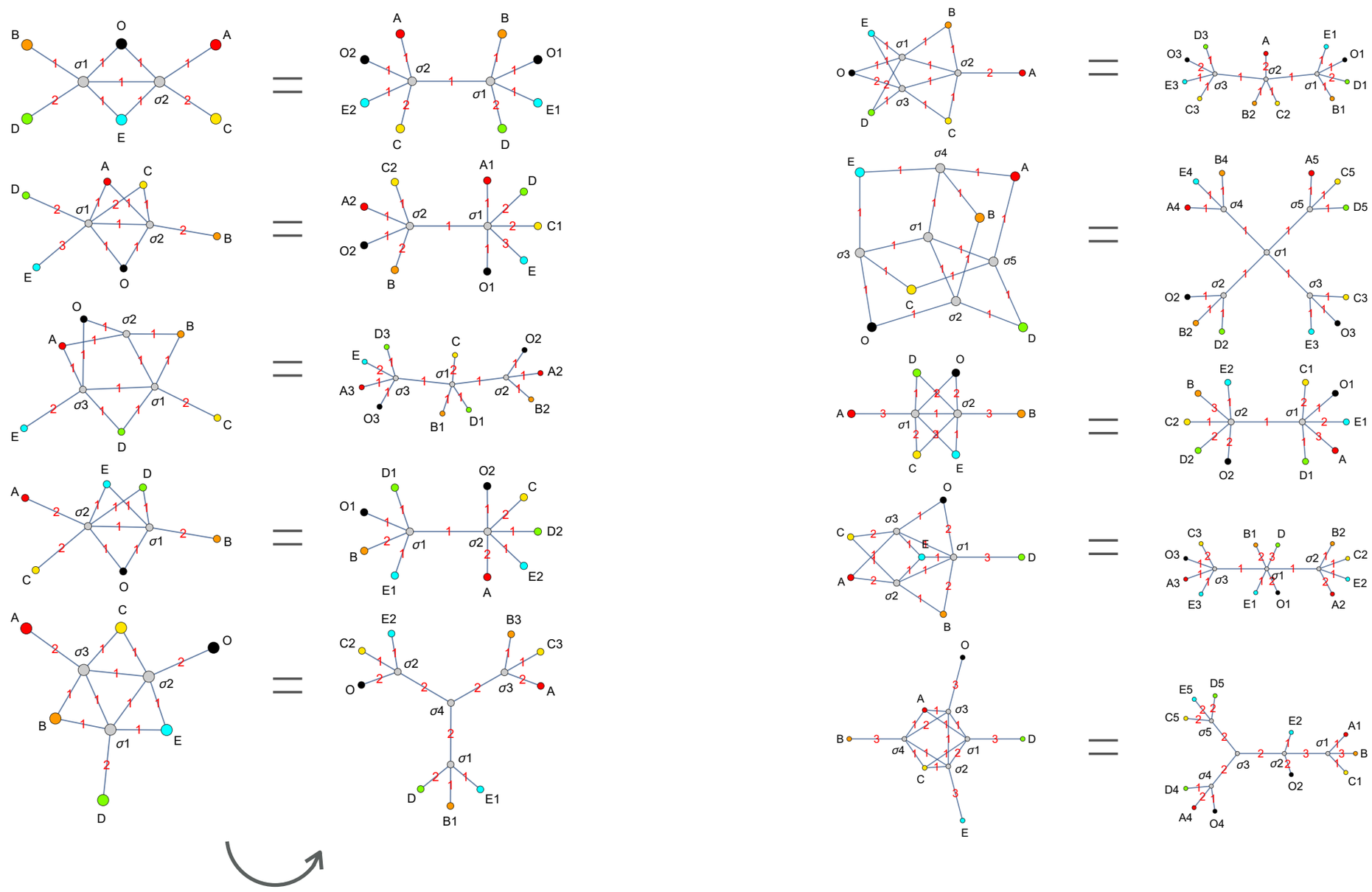
[Hernández-Cuenca, '19]





Graph representation of HEC_5 ERs

[Hernández-Cuenca, '19]



- All deformable to tree graphs (though with multiple vertices of same color)

[Hernández-Cuenca, VH, Rota '22]

PMI

- **Def**: *Pattern of Marginal Independence* (PMI) is a specification of full set of subsystems { X , Y } for which $I(X{:}Y) = 0$.
  - *marginals* = reduced density matrices
  - *independent* if factorized structure

PMI

- **Def**: *Pattern of Marginal Independence* (PMI) is a specification of full set of subsystems { X , Y } for which $I(X{:}Y) = 0$.
  - *marginals* = reduced density matrices
  - *independent* if factorized structure

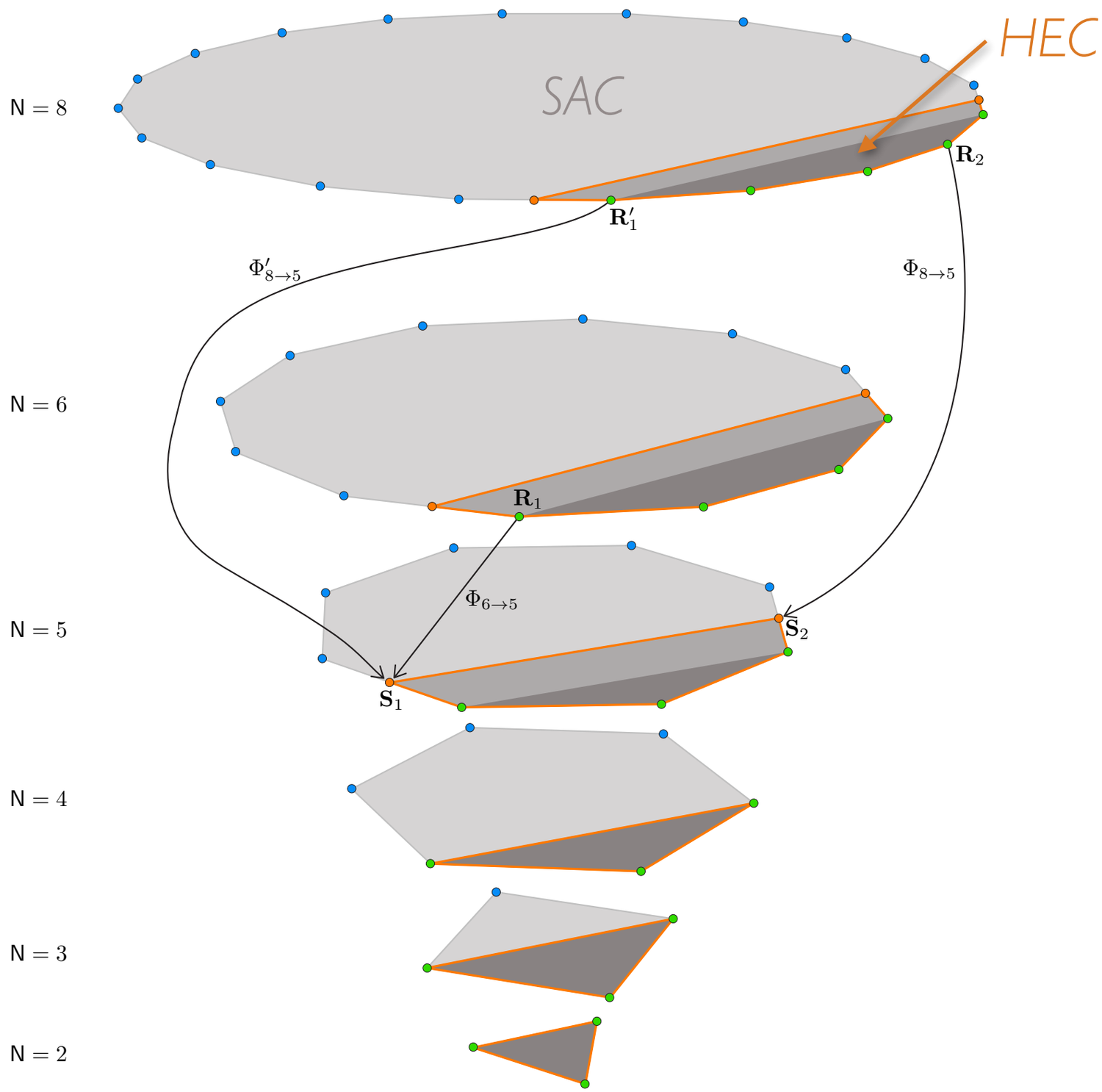
- Meaning & Utility
  - holographically: bdy region pairs, s.t. the joint entanglement wedge is disconnected (since connected XY ent. wedge $\Longrightarrow$ $I(X{:}Y) > 0$)
  - every entropy vector $\vec{S}$ has a unique PMI
  - linear subspace of entropy space
  - = intersection of all saturated SA or AL hyperplanes
  - $\Rightarrow$ discrete structure
  - for physical $\vec{S}$, PMI = span of a face of the SAC

HEC from SAC

- Utilize graph model
  - ER graph has maximal min-cut degeneracy (cf. phase transition of RT surfaces)
  - Conjecture: ERs can be rendered as tree graphs (has strong evidence)
- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

[Hernández-Cuenca, VH, Rota]

Cartoon of HEC vs. SAC



HEC from SAC

- Utilize graph model
  - ER graph has maximal min-cut degeneracy (cf. phase transition of RT surfaces)
  - Conjecture: ERs can be rendered as tree graphs (has large evidence)
- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

[Hernández-Cuenca, VH, Rota]

- This in principle allows us to construct the full HEC_N for any N
  - In practice complicated: requires correct set of ERs of $SAC_{N'}$ for all N', projecting, taking convex hull, extracting ERs, and constructing facets...
  - But conceptually demystifies the HEC (and entanglement structure of holographic states)

HEC from SAC

- Utilize graph model
  - ER graph has maximal min-cut degeneracy (cf. phase transition of RT surfaces)
  - Conjecture: ERs can be rendered as tree graphs (has large evidence)
- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

[Hernández-Cuenca, VH, Rota]

- This in principle allows us to construct the full HEC_N for any N
  - In practice complicated: requires correct set of ERs of $SAC_{N'}$ for all N', projecting, taking convex hull, extracting ERs, and constructing facets...
  - But conceptually demystifies the HEC (and entanglement structure of holographic states)
- Crux: how can we characterize the requisite set of SAC ERs?
  - Formulate in terms of PMIs

Marginal Independence Problem

- Not all marginal independence bipartitions are possible, due to:
  - mathematical inconsistency
  - physical inconsistency

Marginal Independence Problem

- Not all marginal independence bipartitions are possible, due to:
  - mathematical inconsistency

    e.g. violates the identity $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$

  - physical inconsistency

    violates entropy inequality, e.g. SSA $\Rightarrow$ $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$

Marginal Independence Problem

- Not all marginal independence bipartitions are possible, due to:
  - mathematical inconsistency

    e.g. violates the identity $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$

  - physical inconsistency

    violates entropy inequality, e.g. SSA $\Rightarrow$ $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$

    Klein's condition (KC) [He, VH, Rota]

Marginal Independence Problem

- Not all marginal independence bipartitions are possible, due to:
  - mathematical inconsistency

    e.g. violates the identity $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$

  - physical inconsistency

    violates entropy inequality, e.g. SSA $\Rightarrow$ $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$

    Klein's condition (KC) [He, VH, Rota]

- *Marginal Independence Problem* (MIP): what PMIs are realizable?
  - QMIP: what PMIs are realizable in QM?

    considered in [Hernández-Cuenca, VH, Rangamani, Rota]
  - HMIP: what PMIs are realizable by geometric states in holography?

    [He, VH, Rota] & WIP

Exploring the SAC

- No convenient characterization of all ERs directly
- Easy characterization of facets: saturation of any single SA
- Hence define ERs as 1-d PMIs
  - ⇝ collection of D-1 independent MIs { I(X:Y) }

Exploring the SAC

- No convenient characterization of all ERs directly
- Easy characterization of facets: saturation of any single SA
- Hence define ERs as 1-d PMIs
  - ⇝ collection of D-1 independent MIs { I(X:Y) }

- Convenient mathematical framework:
  - Matroid theory (abstractifies notion of linear dependence in combinatorial language):
    use to implement mathematical consistency
  - Oriented matroids:
    use to implement consistency w/ SA
  - Lattice theory: poset order by inclusion (in MI arguments):
    use to implement Klein's condition as approximation to SSA
  - Closure theory: unifying framework for the above (on MI powerset)

Power of restrictions

- Linear dependence: e.g. $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$
  - Implemented by matroid circuits

- KC: i.e. $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$
  - Implemented as a down-set in MI poset
  - Reduces the # of potentially viable PMIs drastically:

Power of restrictions

- Linear dependence: e.g. $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$
  - Implemented by matroid circuits

- KC: i.e. $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$
  - Implemented as a down-set in MI poset
  - Reduces the # of potentially viable PMIs drastically:

N=3: # of subspaces of a given dimension d ≤ D:

d	0	1	2	3	4	5	6	7
in PMI lattice	1	11	48	107	127	75	18	1
in KC lattice	1	7	21	35	32	15	6	1

Power of restrictions

- Linear dependence: e.g. $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$
  - Implemented by matroid circuits

- KC: i.e. $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$
  - Implemented as a down-set in MI poset
  - Reduces the # of potentially viable PMIs drastically:

N=3: # of subspaces of a given dimension d ≤ D:

d	0	1	2	3	4	5	6	7
in PMI lattice	1	11	48	107	127	75	18	1
in KC lattice	1	7	21	35	32	15	6	1

N=4:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3085	66005	532585	2254005	5719656	9301825	10032200	7275805	3541900	1138826	234470	29455	2100	75	1
1	20	175	840	2465	4843	6345	5875	4100	2300	1072	430	150	45	10	1

Classes of ERs of SAC

Let $\mathcal{R} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{PMI}} : \dim(\mathbb{P}) = 1\}$ = set of all extreme rays of the SAC

$\mathcal{R}_{\mathrm{KC}} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{KC}} : \dim(\mathbb{P}) = 1\}$ = set of KC-compatible ERs of the SAC

$$\mathcal{R}_{\mathrm{SSA}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is SSA-compatible}\}$$

$$\mathcal{R}_{\mathrm{Q}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a quantum state}\}$$

$$\mathcal{R}_{\mathrm{H}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a graph model}\},$$

$$\Rightarrow \quad \mathcal{R}_{\mathrm{H}} \subseteq \mathcal{R}_{\mathrm{Q}} \subseteq \mathcal{R}_{\mathrm{SSA}} \subseteq \mathcal{R}_{\mathrm{KC}} \subseteq \mathcal{R}$$

Classes of ERs of SAC

Let $\mathcal{R} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{PMI}} : \dim(\mathbb{P}) = 1\}$ = set of all extreme rays of the SAC

$\mathcal{R}_{\mathrm{KC}} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{KC}} : \dim(\mathbb{P}) = 1\}$ = set of KC-compatible ERs of the SAC

$$\mathcal{R}_{\mathrm{SSA}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is SSA-compatible}\}$$

$$\mathcal{R}_{\mathrm{Q}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a quantum state}\}$$

$$\mathcal{R}_{\mathrm{H}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a graph model}\},$$

$$\Rightarrow \quad \mathcal{R}_{\mathrm{H}} \subseteq \mathcal{R}_{\mathrm{Q}} \subseteq \mathcal{R}_{\mathrm{SSA}} \subseteq \mathcal{R}_{\mathrm{KC}} \subseteq \mathcal{R}$$

- Observations:
  - For N=2, $\mathcal{R}_{\mathrm{SSA}} = \mathcal{R}$; but otherwise $|\mathcal{R}_{\mathrm{SSA}}| \ll |\mathcal{R}|$

Classes of ERs of SAC

Let $\mathcal{R} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{PMI}} : \dim(\mathbb{P}) = 1\}$ = set of all extreme rays of the SAC

$\mathcal{R}_{\mathrm{KC}} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{KC}} : \dim(\mathbb{P}) = 1\}$ = set of KC-compatible ERs of the SAC

$$\mathcal{R}_{\mathrm{SSA}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is SSA-compatible}\}$$

$$\mathcal{R}_{\mathrm{Q}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a quantum state}\}$$

$$\mathcal{R}_{\mathrm{H}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a graph model}\},$$

$$\Rightarrow \quad \mathcal{R}_{\mathrm{H}} \subseteq \mathcal{R}_{\mathrm{Q}} \subseteq \mathcal{R}_{\mathrm{SSA}} \subseteq \mathcal{R}_{\mathrm{KC}} \subseteq \mathcal{R}$$

- Observations:
  - For N=2, $\mathcal{R}_{\mathrm{SSA}} = \mathcal{R}$; but otherwise $|\mathcal{R}_{\mathrm{SSA}}| \ll |\mathcal{R}|$
  - For N ≤ 5, $\mathcal{R}_{\mathrm{H}} = \mathcal{R}_{\mathrm{SSA}}$
  - Original hope: this prevails ∀ N, which would give $\mathcal{R}_{\mathrm{H}} = \mathcal{R}_{\mathrm{Q}}$
  - Establishing this would be useful since little known about QEC

Holographic - quantum gap

- However, ∃ a counter-example at N=6: [He, VH, Rota, '23]

The following ER of SAC_6 :

$$\mathsf{R}_6 = (2,1,1,1,2,2;\ 3,3,3,4,4,2,2,3,3,2,3,3,3,3,4;\ 2,4,5,5,4,5,5,3,5,4,3,4,4,4,4,5,4,4,5,3;\ 3,4,4,4,4,3,4,4,3,3,3,5,4,4,4;\ 3,3,2,2,2,3;\ 1)$$

w/ components ordered as $(A,\ldots,F;AB,AC,\ldots,EF;ABC,\ldots;ABCDEF)$

violates MMI, since $-\mathsf{I}_3\left(A:BC:DE\right) = -2 \ngeq 0$

$$\Longrightarrow \mathsf{R}_6 \notin \mathcal{R}_{\mathrm{H}}$$

Holographic - quantum gap

- However, ∃ a counter-example at N=6: [He, VH, Rota, '23]

The following ER of SAC_6 :

$$\mathsf{R}_6 = (2,1,1,1,2,2;\ 3,3,3,4,4,2,2,3,3,2,3,3,3,3,4;\ 2,4,5,5,4,5,5,3,5,4,3,4,4,4,4,5,4,4,5,3;\ 3,4,4,4,4,3,4,4,3,3,3,5,4,4,4;\ 3,3,2,2,2,3;\ 1)$$

w/ components ordered as $(A,\ldots,F; AB, AC,\ldots,EF; ABC,\ldots; ABCDEF)$

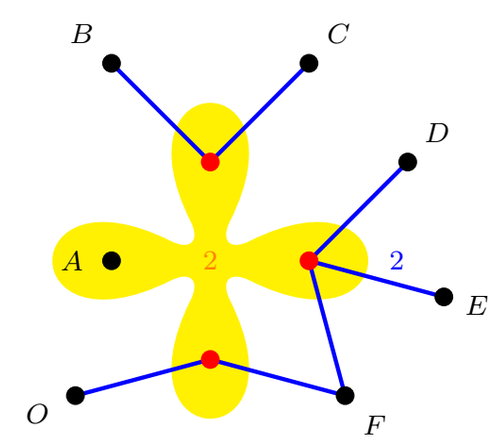
violates MMI, since $-\mathsf{I}_3\,(A:BC:DE) = -2 \ngeq 0$

$$\Longrightarrow \mathsf{R}_6 \notin \mathcal{R}_{\mathrm{H}}$$

But, ∃ a hypergraph which realizes R_6 , so it describes a stabilizer state.

[Walter & Witteveen]

$$\Longrightarrow \mathsf{R}_6 \in \mathcal{R}_{\mathrm{Q}}$$



Holographic - quantum gap

- However, ∃ a counter-example at N=6: [He, VH, Rota, '23]

The following ER of SAC_6 :

$$\mathsf{R}_6 = (2,1,1,1,2,2;\ 3,3,3,4,4,2,2,3,3,2,3,3,3,3,4;\ 2,4,5,5,4,5,5,3,5,4,3,4,4,4,4,5,4,4,5,3;\ 3,4,4,4,4,3,4,4,3,3,3,5,4,4,4;\ 3,3,2,2,2,3;\ 1)$$

w/ components ordered as $(A,\ldots,F; AB, AC,\ldots,EF; ABC,\ldots; ABCDEF)$

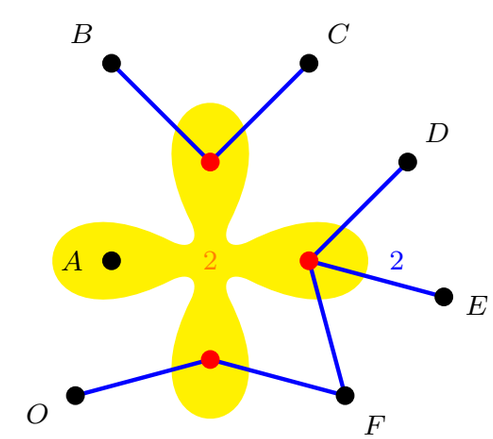
violates MMI, since $-\mathsf{I}_3\,(A:BC:DE) = -2 \ngeq 0$

$$\Longrightarrow \mathsf{R}_6 \notin \mathcal{R}_{\mathrm{H}}$$

But, ∃ a hypergraph which realizes R_6 , so it describes a stabilizer state.

[Walter & Witteveen]

$$\Longrightarrow \mathsf{R}_6 \in \mathcal{R}_{\mathrm{Q}}$$



$\Longrightarrow \mathcal{R}_{\mathrm{H}} \neq \mathcal{R}_{\mathrm{Q}}$ i.e. ∃ a gap between holographic and quantum ERs of SAC.

Summary & future directions

- Holographic entropy inequalities
  - Can be packaged efficiently using the tripartite form
  - Constructed 384 orbits of holographic entropy inequalities for N=6
  - These manifest rich structural relations
  - How can we bootstrap these to generate new HEIs for higher N?
  - Are all HEIs guaranteed to admit the tripartite form?
  - Is there an even better packaging?
- Interpretation?
  - Not correlation measures (since not monotonic under inclusion)
  - More generalized multipartite correlation?
  - Operational meaning?

Summary & future directions

- Main point:
  - HEC_N can be fully reconstructed from far simpler structure, at finer N':

    Holographically realizable ERs of $SAC_{N'}$ (← solution to HMIP)

    ↓

    describe using PMIs, specified by { I(X:Y) }

Summary & future directions

- Main point:
  - HEC_N can be fully reconstructed from far simpler structure, at finer N':

    Holographically realizable ERs of $SAC_{N'}$ (← solution to HMIP)

    ↓

    describe using PMIs, specified by { I(X:Y) }
- Implications:
  - { HEC_N ∀ N } ⟸ { $HMIP_N$ ∀ N } ("Holographic entropy cone from marginal independence")
  - Any fixed N contaminated by structural (combinatorial) artifacts
  - Seemingly minimal dependence on holography (SAC universal)

Summary & future directions

- Main point:
  - HEC_N can be fully reconstructed from far simpler structure, at finer N':
    Holographically realizable ERs of $SAC_{N'}$ (← solution to HMIP)

    ↓

    describe using PMIs, specified by { I(X:Y) }

- Implications:
  - { HEC_N ∀ N } ⟸ { $HMIP_N$ ∀ N } ("Holographic entropy cone from marginal independence")
  - Any fixed N contaminated by structural (combinatorial) artifacts
  - Seemingly minimal dependence on holography (SAC universal)

    ↓

    - arbitrarily refined partition (N)
    - classicality: admits phase transitions

Future directions

- Within the present context:
  - Complete soln. to HMIP
  - Explain HEIs
  - Internal structure of HEC
  - Bootstrapping ERs and HEIs to higher N

- Beyond the present context:
  - Beyond classical bulk (quantum and stringy corrections)
  - Other QI quantities
  - …

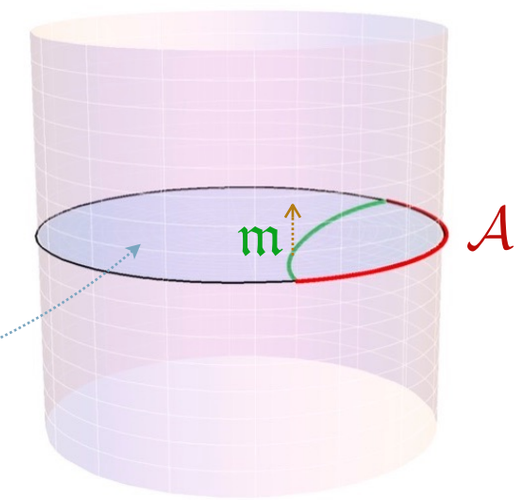
Thank you

Appendices

Covariant Holographic EE

The RT prescription for holographic EE is not well-defined outside the context of static configurations:

- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…

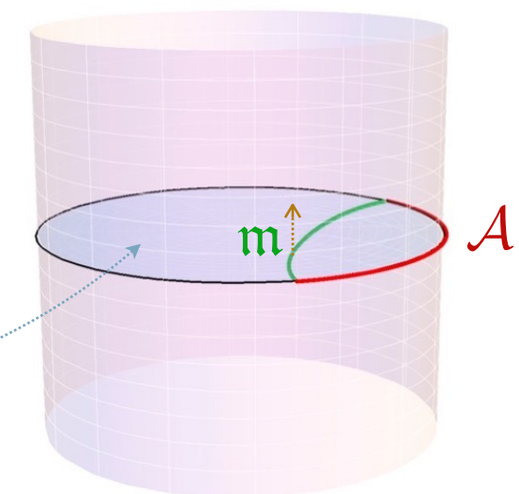


In *time-dependent* situations, RT prescription must be covariantized:

Covariant Holographic EE

The RT prescription for holographic EE is not well-defined outside the context of static configurations:

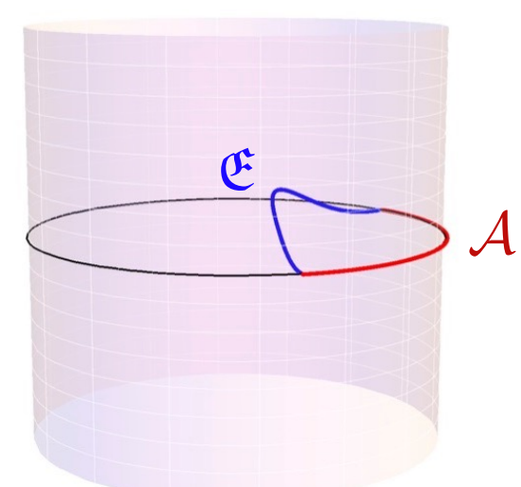
- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…



In *time-dependent* situations, RT prescription must be covariantized:

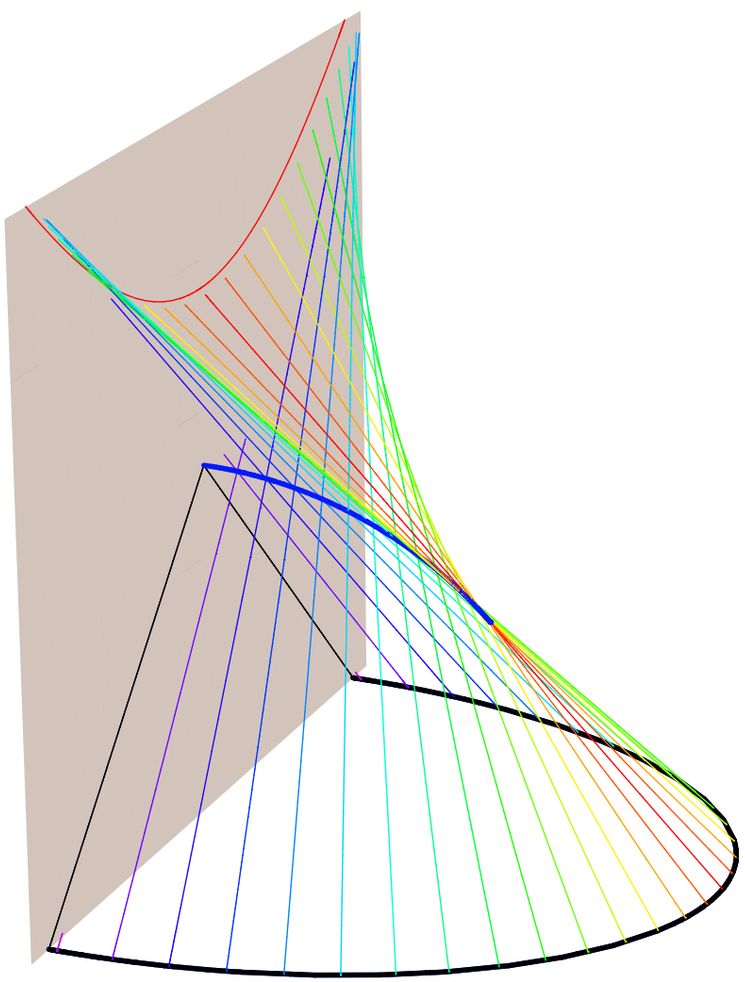
[HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk

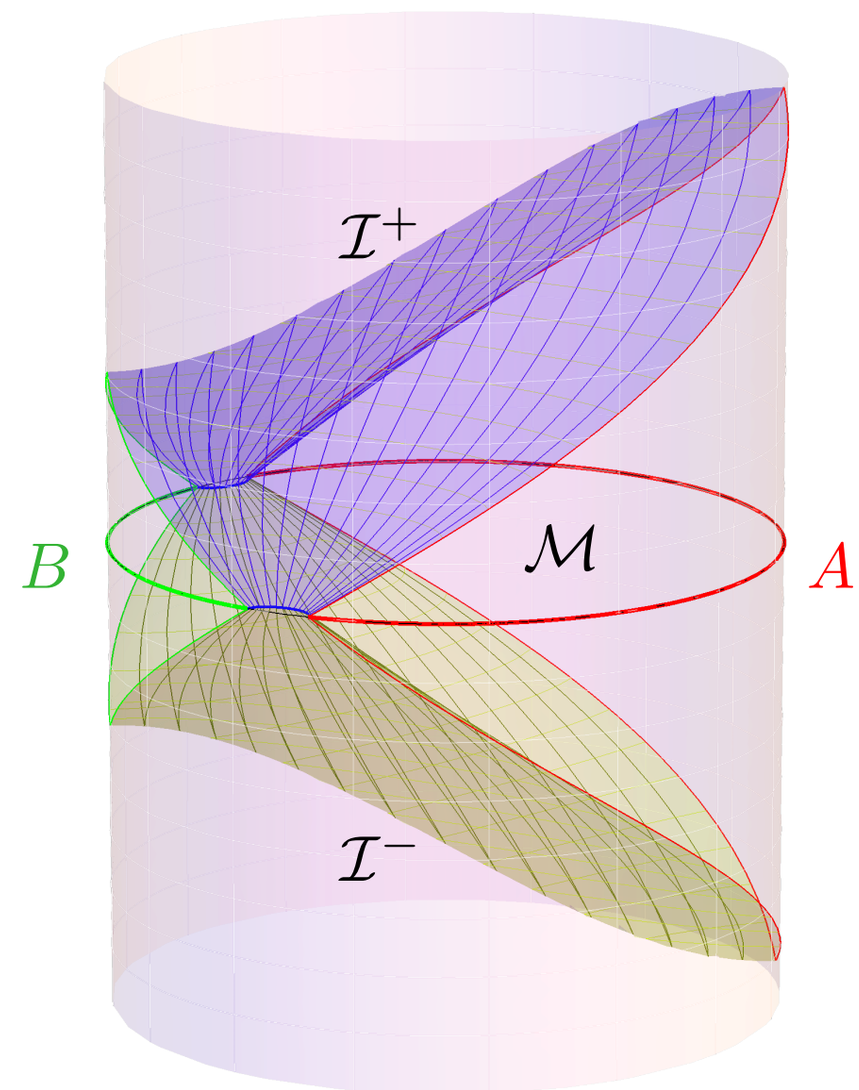


Entanglement wedge

Structure of generators of EW horizon



EW for composite region AB



RT cone = HRT cone

previously shown only for 2-d CFT [Czech, Dong]

- Hitherto graph representation only for static situations (RT)
  - Useful toolkit (e.g. contraction map to prove HEIs, constructing ERs, ...)

RT cone = HRT cone

previously shown only for 2-d CFT [Czech, Dong]

- Hitherto graph representation only for static situations (RT)
  - Useful toolkit (e.g. contraction map to prove HEIs, constructing ERs, ...)
- Covariant reformulation of holographic entanglement entropy
  - HRT: extremal surface
  - maximin: maximize over Cauchy slices the minimal surface area on slice [Wall]
  - minimax: minimize over 'timesheets' the maximal surface area on sheet [Headrick, VH]

RT cone = HRT cone

previously shown only for 2-d CFT [Czech, Dong]

- Hitherto graph representation only for static situations (RT)
  - Useful toolkit (e.g. contraction map to prove HEIs, constructing ERs, ...)
- Covariant reformulation of holographic entanglement entropy
  - HRT: extremal surface
  - maximin: maximize over Cauchy slices the minimal surface area on slice [Wall]
  - minimax: minimize over 'timesheets' the maximal surface area on sheet [Headrick, VH]
- Minimax (=partitioning bulk by timesheets) allows for graph model even in general time-dependent situations

  [WIP: Grado-White, Grimaldi, Headrick, VH]
  - Hence the toolkit developed for static case applies in general.
  - All HEIs proved for static case are valid in time-dependent case as well.

Klein's condition (KC)

- Klein inequality:

$$\mathrm{Tr}(f(\rho) - f(\sigma) - (\rho - \sigma) f'(\sigma)) \geq 0$$

convex fn. Hermitian matrices

$$= \quad \Longleftrightarrow \quad \rho = \sigma$$

- Apply to relative entropy

$$\mathsf{R}\,(\rho \,||\, \sigma) \;=\; 0 \qquad \Longleftrightarrow \qquad \rho = \sigma$$

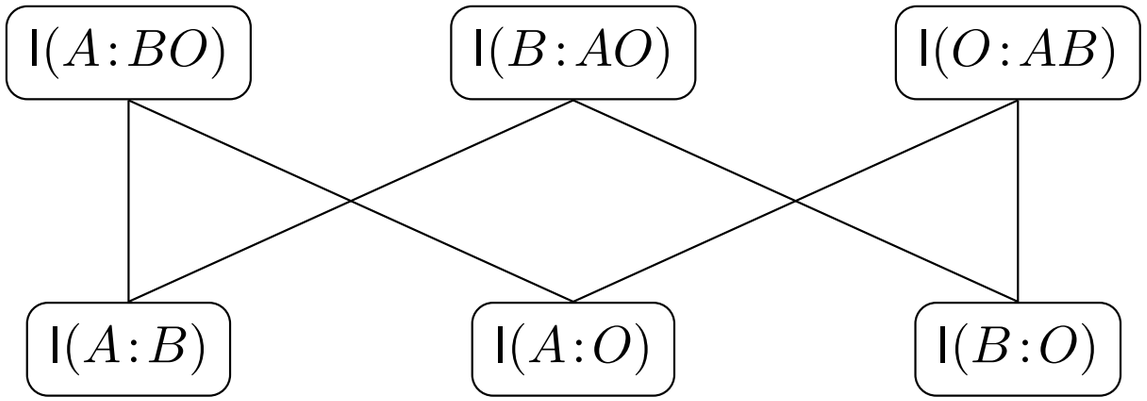
$\rho_{\underline{\mathcal{J}\mathcal{K}}}$ $\rho_{\underline{\mathcal{J}}} \otimes \rho_{\underline{\mathcal{K}}}$

- ⇝ KC for MI:

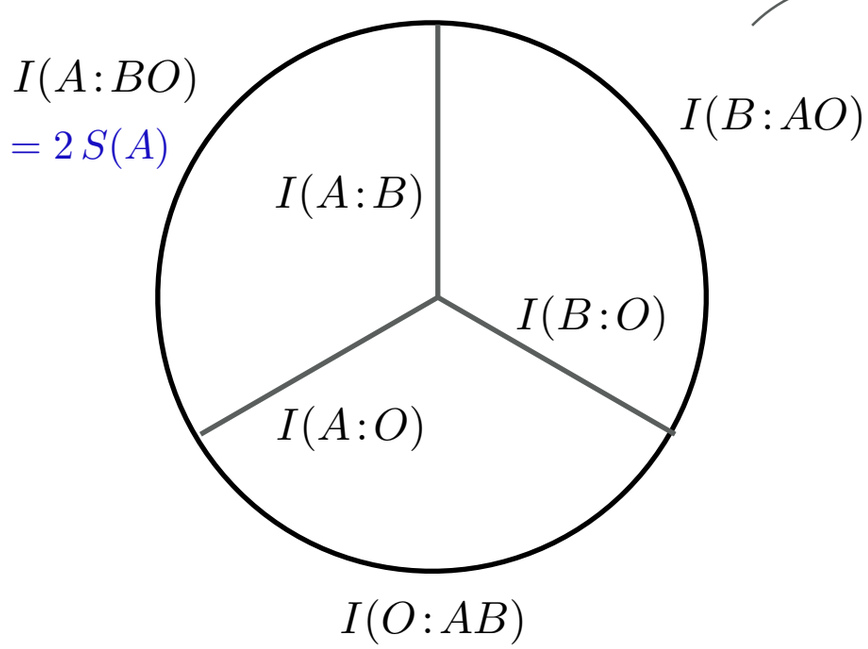
$$\begin{aligned} & \mathsf{I}(\underline{\mathcal{J}} : \underline{\mathcal{J}}\underline{\mathcal{K}}) = 0 \\ \Longrightarrow \quad & \rho_{\underline{\mathcal{J}}\underline{\mathcal{J}}\underline{\mathcal{K}}} = \rho_{\underline{\mathcal{J}}} \otimes \rho_{\underline{\mathcal{J}}\underline{\mathcal{K}}} \\ \Longrightarrow \quad & \rho_{\underline{\mathcal{J}}\underline{\mathcal{J}}} = \rho_{\underline{\mathcal{J}}} \otimes \rho_{\underline{\mathcal{J}}} \quad \text{and} \quad \rho_{\underline{\mathcal{J}}\underline{\mathcal{K}}} = \rho_{\underline{\mathcal{J}}} \otimes \rho_{\underline{\mathcal{K}}} \\ \Longrightarrow \quad & \mathsf{I}(\underline{\mathcal{J}} : \underline{\mathcal{J}}) = 0 \quad \text{and} \quad \mathsf{I}(\underline{\mathcal{J}} : \underline{\mathcal{K}}) = 0\,. \end{aligned}$$

N=2 SAC in terms of MIs

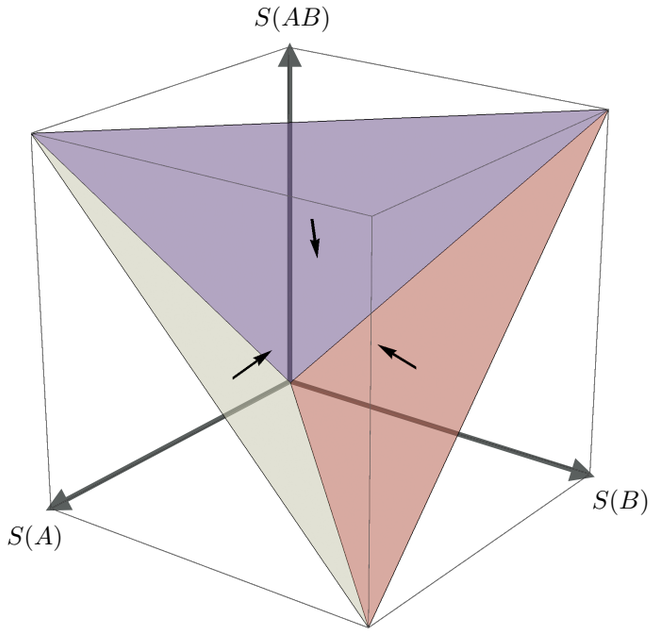
- MI poset
  - order by inclusion:



- Matroid structure
  - represented by a "whirl"

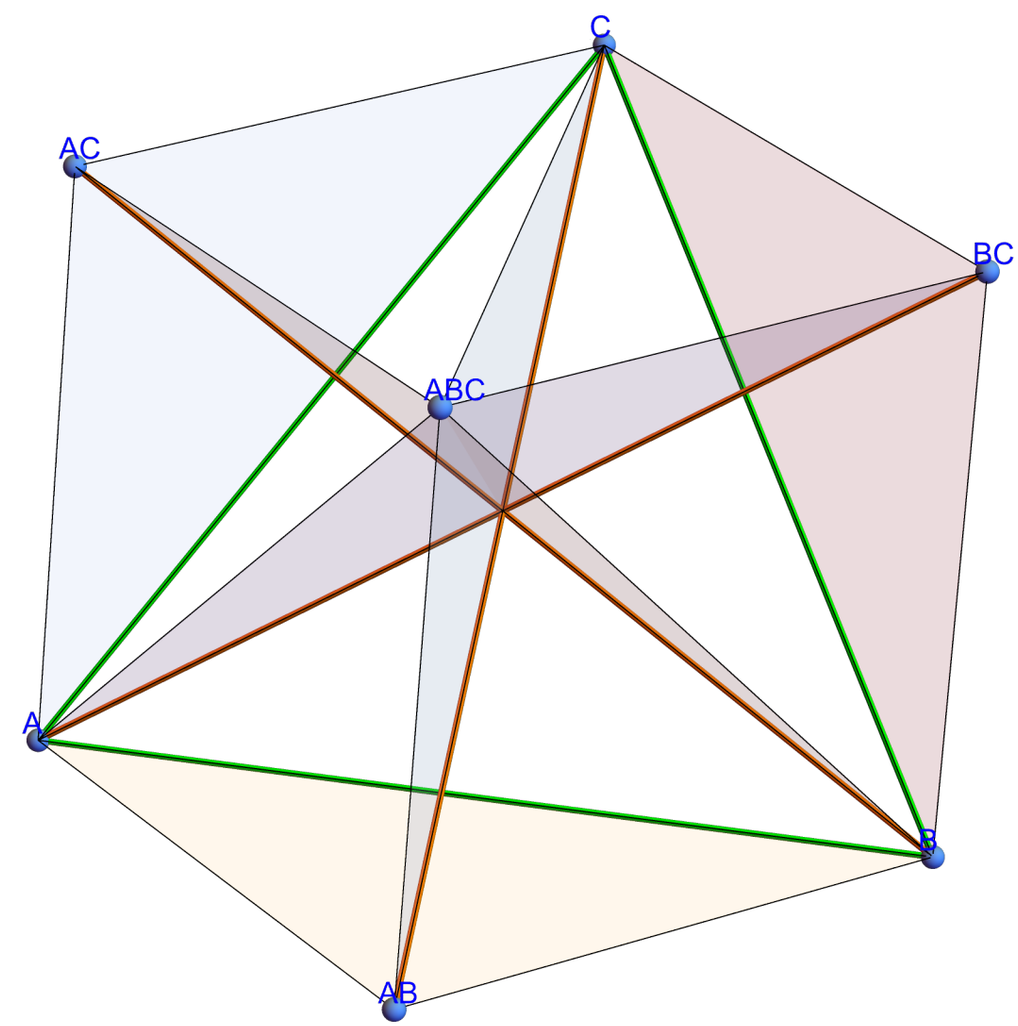


each segment = plane in



Graphical representation of {I(X:Y)}

- Organize into whirls ⇝ W-complex:
  - each triangle = whirl
  - each edge = MI
  - each vertex = entropy / subsystem

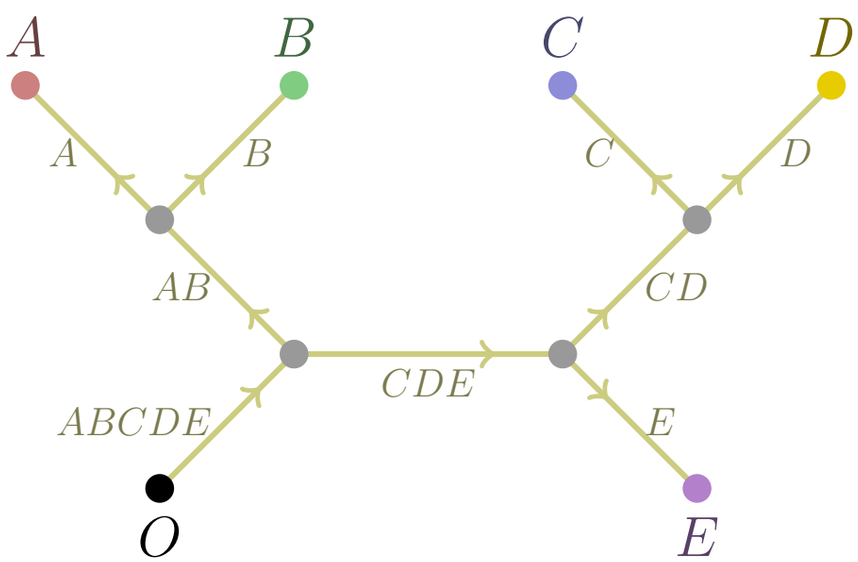


- e.g. for N=3
  - 6 whirls
  - 18 (subsystem) MIs
  - 7 entropies

- Properties
  - whirls join at vertices
  - specified by 1 generating edge per whirl
  - automatically implements permutation & purification symmetry

Simple tree graphs

- But tree graphs have much simpler structure

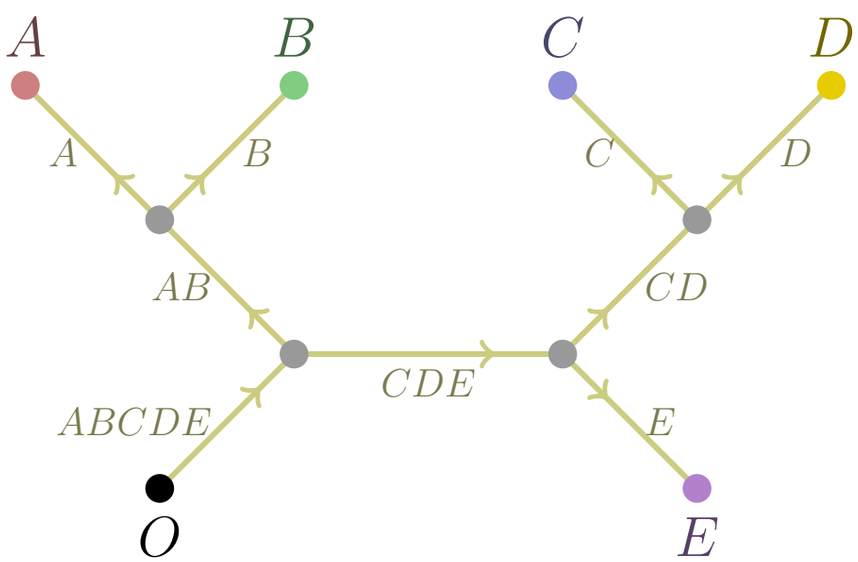


- each edge specified by a unique collection of boundary vertices
- PMI follows directly from specification of min-cut structure

- **Def**: graph is *simple* if every edge defines some subsystem cut; or equivalently, if each boundary vertex has different "color"

Simple tree graphs

- But tree graphs have much simpler structure



- each edge specified by a unique collection of boundary vertices
- PMI follows directly from specification of min-cut structure

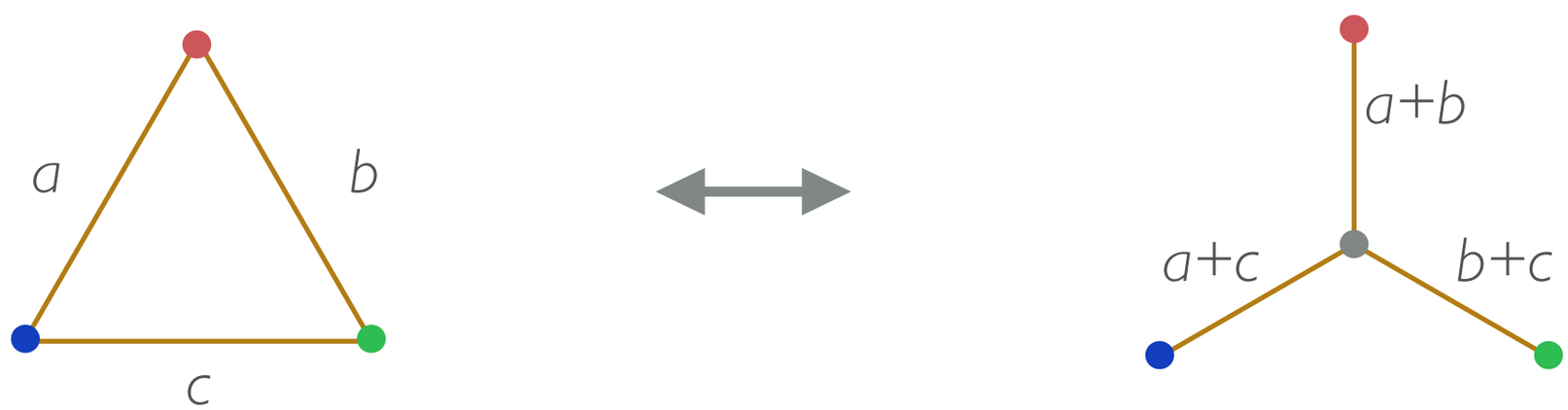
- **Def**: graph is *simple* if every edge defines some subsystem cut; or equivalently, if each boundary vertex has different "color"

- **Thm**: For *simple* tree graphs, min-cut subspace = PMI

Conjecture

- **Conjecture:** We can convert any holographic graph model into a tree while preserving the min-cut subspace.

  - e.g.



  - similarly, can collapse any isolated k-cycle
  - Holds true for all ERs for N=5 HEC and all (hitherto-known) N=6 HEC ERs.
  - generically gives a non-simple tree, but can be trivially made simple by `fine-graining':

Coarse-graining & fine-graining

- Change N → N'
  - Changes dimensionality D → D' of entropy space
  - Aspects of entanglement structure preserved (inherited)

- Coarse-graining = declare multiple colors indistinguishable
  - projection of entropy vectors
  - corresponding projection of linear subspaces (V-space & PMI)

- Fine-graining = reverse of coarse-graining
  - Hence can obtain simple graph from non-simple one by fine-graining

Main theorem

- Consider ERs of N-party holographic entropy cone (HEC_N)
  - The HEC_N is a convex hull of these; so ERs (in principle) determine all holographic entropy inequalities

- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

- Idea of Pf:
  - Start w/ HEC_N ER & obtain graph representation G (has 1-d V-space)
  - Use Conjecture & fine-grain to transform into a simple tree G'
  - Resulting V-space = PMI in D'-dimensional entropy space
  - Reduce to 1-d PMI (if uplifting increased V-space dimensionality)
  - = ER for $SAC_{N'}$