The Euclidean vacuum state for linearized gravity on de Sitter spacetime

Christian Gérard
Université Paris-Saclay

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CCR$*$-algebras

Let $(\mathcal{Y}, q)$ be a Hermitian space.

- One can introduce the abstract CCR $*$-algebra $\mathrm{CCR}(\mathcal{Y}, q)$ generated by the symbols $\psi(y), \psi^*(y)$ for $y \in \mathcal{Y}$ with relations:

$$
\begin{array}{ll}
1) & \mathcal{Y} \ni y \mapsto \psi^*(y) \text{ resp. } \psi(y) \text{ linear resp. anti-linear,} \\
2) & [\psi(y_1), \psi^*(y_2)] = \overline{y}_1 \cdot q y_2 \mathbb{1}, \ \ y_1, y_2 \in \mathcal{Y}, \\
3) & [\psi(y_1), \psi(y_2)] = [\psi^*(y_1), \psi^*(y_2)] = 0, \ \ y_1, y_2 \in \mathcal{Y}, \\
4) & \psi(y)^* = \psi^*(y), \ \ y \in \mathcal{Y}.
\end{array}
$$

Quasi-free states

- A quasi-free state ω on $\mathrm{CCR}(\mathcal{Y}, q)$ is determined by a pair of Hermitian forms $\lambda^{\pm}$ on $\mathcal{Y}$ (called the covariances) by

$$\omega(\psi(y_1)\psi^*(y_2)) = \overline{y}_1 \cdot \lambda^+ y_2,$$
$$\omega(\psi^*(y_2)\psi(y_1)) = \overline{y}_1 \cdot \lambda^- y_2,$$
$$\omega(\psi(y_1)\psi(y_2)) = \omega(\psi^*(y_1)\psi^*(y_2)) = 0.$$

- Necessary and sufficient conditions for $\lambda^{\pm}$ to be covariances are

$$1) \quad \lambda^+ - \lambda^- = q \text{ (CCR)},$$
$$2) \quad \lambda^{\pm} \geq 0 \text{ (positivity)}.$$

- Useful to introduce $c^{\pm} =: \pm q^{-1} \circ \lambda^{\pm}$. Then $c^+ + c^- = \mathbb{1}$ and ω is pure iff $c^{\pm}$ are projections.

Quasi-free states for matter fields

- Let $(M, \mathbf{g})$ a globally hyperbolic spacetime, $V \xrightarrow{\pi} M$ a finite rank Hermitian bundle.
- Let D a second order differential operator acting on $C^\infty(M; V)$ such that $D = D^*$ with principal symbol $\xi \cdot \mathbf{g}^{-1} \xi \mathbb{1}_V$.
- standard example is the Klein-Gordon operator $D = -\Box$, acting on scalar functions.
- D has unique advanced/retarded inverses $G_{\rm ret/adv}$, $G := G_{\rm ret} - G_{\rm adv}$ is the commutator function.

The various Hermitian spaces

- 'off shell' Hermitian space is $\dfrac{C_0^\infty(M;V)}{DC_0^\infty(M;V)}$ with

$$\overline{[u]}\cdot Q[u] = \mathrm{i}(u|Gu)_V.$$

- 'on shell' Hermitian space is $\mathrm{Ker}_{\mathrm{sc}}\, D$ (space of solutions) with

$$\overline{u}\cdot qu = \mathrm{i}\int_\Sigma n^a J_a(u,u) dVol_{\mathrm{h}},$$

  $\Sigma \subset M$ (any) Cauchy surface, n future directed unit normal, $J_a(u,u)$ conserved current.
- 'Cauchy surface' Hermitian space is $C_0^\infty(\Sigma; V\otimes \mathbb{C}^2)$ with

$$\overline{f}\cdot q_\Sigma f = \int_\Sigma (f_1|f_0)_V + (f_0|f_1)_V dVol_{\mathrm{h}},\ f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

The various covariances

- All three Hermitian spaces are isomorphic. One can use any of the three to construct $\mathrm{CCR}(\mathcal{Y}, q)$.
- 'off shell' covariances: a pair $\Lambda^{\pm} : C_0^{\infty}(M; V) \to \mathcal{D}'(M; V)$ such that

$$
\begin{aligned}
&(1)\quad D \circ \Lambda^{\pm} = \Lambda^{\pm} \circ D = 0 \text{(field equation)}\\
&(2)\quad \Lambda^{+} - \Lambda^{-} = \mathrm{i}G,\ \text{(CCR)},\\
&(3)\quad (u|\Lambda^{\pm}u)_V \geq 0,\ u \in C_0^{\infty}(M; V),\ \text{(positivity)}.
\end{aligned}
$$

- 'Cauchy surface' covariances: a pair $\lambda_{\Sigma}^{\pm} : C_0^{\infty}(\Sigma; V \otimes \mathbb{C}^2) \to \mathcal{D}'(\Sigma; V \otimes \mathbb{C}^2)$ such that:

$$
\begin{aligned}
&(1)\quad \lambda_{\Sigma}^{+} - \lambda_{\Sigma}^{-} = q_{\Sigma},\ \text{(CCR)},\\
&(2)\quad (f|\lambda_{\Sigma}^{\pm}f)_{V\otimes\mathbb{C}^2} \geq 0, f \in C_0^{\infty}(\Sigma; V \otimes \mathbb{C}^2),\ \text{(positivity)}.
\end{aligned}
$$

The various covariances

- The two types of covariances are related by

$$\begin{aligned}\lambda^{\pm}_{\Sigma} &= (\rho^* q_{\Sigma})^* \Lambda^{\pm} (\rho^* q_{\Sigma}),\\ \Lambda^{\pm} &= (\rho G)^* \lambda^{\pm}_{\Sigma} (\rho G),\end{aligned}$$

where $\rho u = \begin{pmatrix} u\upharpoonright_{\Sigma} \\ \mathrm{i}^{-1}\nabla_n u\upharpoonright_{\Sigma} \end{pmatrix}$ is the trace of u on Σ.

The Hadamard condition

- The Hadamard condition on $\Lambda^{\pm}$ singles out the physically meaningful states:
- $$\mathrm{WF}(\Lambda^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm},$$
  where:
- $$\mathcal{N} = \{(x, \xi) \in T^*M \setminus o : \chi \cdot \mathbf{g}^{-1}(x)\xi = 0\},$$
  characteristic manifold aka lightcone,
  $$\mathcal{N}^{\pm} = \text{ positive/negative energy components of } \mathcal{N},$$
- $\mathrm{WF}(\Lambda^{\pm})' \subset T^*(M \times M) \setminus o$ is the wavefront set of $\Lambda^{\pm} \in \mathcal{D}'(M \times M; V \boxtimes V)$ (distributional kernel of $\Lambda^{\pm}$).

The Hadamard condition

- The Hadamard condition can also be formulated on the Cauchy surface covariances $\lambda^{\pm}_{\Sigma}$ (recall $\lambda^{\pm} = \pm q_{\Sigma} \circ c^{\pm}$):

$$\mathrm{WF}(U_{\Sigma} \circ c^{\pm})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^{*}\Sigma,$$

over $V \times \Sigma$, where:

- U_{Σ} solves the Cauchy problem for D, ie

$$\begin{cases} D \circ U_{\Sigma} = 0, \\ \rho_{\Sigma} \circ U_{\Sigma} = \mathbb{1}. \end{cases}$$

- $\mathcal{F} \subset T^{*}M \setminus o$ any conic set with $\mathcal{F} \cap \mathcal{N} = \emptyset$.

Wick rotation

- Assume that $M = I_t \times \Sigma$, $\mathbf{g} = -dt^2 + \mathrm{h}_t(x)dx^2$ and h_t real analytic in t near $t = 0$.
- Wick rotation amounts to set $t =: \mathrm{i}s$ ($dt = \mathrm{i}ds$ etc). We obtain $\tilde{M} = \tilde{I}_s \times \Sigma$ with a metric $\tilde{\mathbf{g}} = ds^2 + \mathrm{h}_{\mathrm{i}s}(x)dx^2$.
- Note that $\tilde{\mathbf{g}}$ is in general not Riemannian.
- The operator D becomes $\tilde{D}$, which is elliptic, at least near $s = 0$.

Calderón projectors

- Let $\Omega^{\pm} = \tilde{M} \cap \{\pm s > 0\}$. For $u \in \overline{C^{\infty}}(\Omega^{\pm})$ we set

$$\tilde{\rho} u = \begin{pmatrix} u\restriction_{\Sigma} \\ -\partial_s u\restriction_{\Sigma} \end{pmatrix}.$$

- Key fact: the spaces

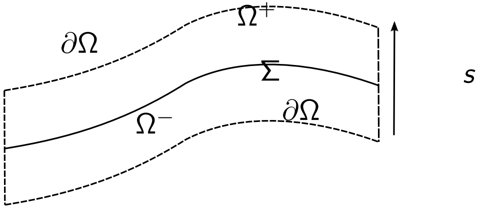
$$E^{\pm} = \{\tilde{\rho} u : u \in \overline{C^{\infty}}(\Omega^{\pm}),\ \tilde{D} u = 0 \text{ in } \Omega^{\pm}\}$$

are not equal to $C^{\infty}(\Sigma; \mathbb{C}^2)$: one cannot solve the Cauchy problem for an elliptic equation !

Calderón projectors

- The Calderón projectors $\tilde{c}^{\pm}$ are the projections on $E^{\pm}$ along $E^{\mp}$.
- This requires that $E^{+} \cap E^{-} = \{0\}$ or equivalently $\tilde{D}$ injective
- $E^{+} + E^{-} = C^{\infty}(\Sigma; \mathbb{C}^2)$ or equivalently $\tilde{D}$ surjective.
- To do this $\tilde{D}$ has to be defined as a linear operator, not only as a formal expression:
  put boundary conditions on $\partial\Omega$!

Calderón projectors


Ω^+
$\partial\Omega$
Σ
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s

Hadamard states from Calderón projectors

- For scalar fields one can put Dirichlet boundary conditions on $\partial\Omega$ to make $\tilde{D}$ invertible.

Theorem (GW)

Let

$$\lambda_{\Sigma}^{\pm} = \pm q_{\Sigma} \circ \tilde{c}^{\pm}.$$

Then $\lambda_{\Sigma}^{\pm}$ are the Cauchy surface covariances of a Hadamard state.

- For ultrastatic spacetimes $\mathbf{g} = -dt^2 + \mathrm{h}$, $\tilde{\mathbf{g}} = ds^2 + \mathrm{h}$, the state obtained with Calderón projectors with no boundary conditions (ie $\tilde{I} = \mathbb{R}$) is the vacuum state.

Einstein's equations

- $\mathbf{Ric}_{ab}(\mathbf{g}) = R_{acb}{}^{c}$, Ricci curvature.
- Einstein's equations:

$$\mathbf{Ric}(\mathbf{g}) = \Lambda \mathbf{g}, \ \Lambda \text{ cosmological constant}.$$

- non-linear system of PDE for $\mathbf{g}$.
- not hyperbolic.
- Cauchy problem ill posed: Einstein equations imply constraints on Cauchy data.
- gauge equivalence: two metrics $\mathbf{g}$ and $\chi^{*}\mathbf{g}$ where $\chi : M \to M$ diffeomorphism are physically equivalent.

Linearized gravity

- fix a background metric $\mathbf{g}$ solution of Einstein equations and linearize around $\mathbf{g}$, ie write

$$(\mathbf{Ric}-\Lambda)(\mathbf{g}+\epsilon u)=(\mathbf{Ric}-\Lambda)(\mathbf{g})+\epsilon Pu+O(\epsilon^2),$$

for $u\in C^\infty(M;\otimes_{\rm s}^2 T^*M)$.

- The equation

$$Pu=0$$

is called the linearized Einstein equations.

- Similarly linearize a diffeomorphism χ around $\mathbb{1}$: we obtain

$$\chi^*=\mathbb{1}+\epsilon\mathcal{L}_v+O(\epsilon^2),$$

$\mathcal{L}_v$ is the Lie derivative associated to the vector field v.

Some background

- Set $V_k = \mathbb{C} \otimes_{\mathrm{s}}^k T^*M$, $k = 0, 1, 2$.
  V_k equipped with the Hermitian form

$$(u|u)_{V_k} = k!\overline{u}\cdot(\mathbf{g}^{\otimes k}(x))^{-1}u, \ \ u \in V_k(x),$$

$$(u|v)_{V_k} = \int_M (u(x)|v(x))_{V_k} d\mathrm{Vol}_{\mathbf{g}}$$

For example

$$(\mathbf{g}|\mathbf{g})_{V_2} = 8, \ \ (\mathbf{g}|u)_{V_2} = 2\mathrm{tr}_{\mathbf{g}} u = 2\mathbf{g}^{ab}u_{ab}.$$

physical Hermitian form: $(u|v)_{I,V_2} := (u|Iv)_{V_2}$, I *trace reversal* (see below).)

Symmetric differential and co-differential

- symmetric differential: we set

$$d: \begin{array}{l} C^\infty(M; V_k) \to C^\infty(M; V_{k+1}) \\ (du)_{a_1\dots,a_{k+1}} = \nabla_{(a_1} u_{a_2\dots,a_{k+1})}, \end{array}$$

  $u_{(a_1\dots a_k)}$ is the symmetrization of $u_{a_1\dots a_k}$,
- symmetric co-differential

$$\delta: \begin{array}{l} C^\infty(M; V_k) \to C^\infty(M; V_{k-1}) \\ (\delta u)_{a_1,\dots,a_{k-1}} = -k\nabla^a u_{a a_1\dots a_{k-1}}. \end{array}$$

  $d^* = \delta$ w.r.t. the Hermitian form $(\cdot|\cdot)_{V_k}$.

Trace reversal

- Trace reversal: I is orthogonal symmetry w.r.t. $\mathbb{C}\mathbf{g}$:

$$Iu_2 = u_2 - \frac{1}{4}\mathbf{g}(\mathbf{g}|u_2)_{V_2},$$

- one has

$$I^2 = \mathbb{1}, I\mathbf{g} = -\mathbf{g}, \ I = I^* \text{ for } (\cdot|\cdot)_{V_2}.$$

Linearized gravity as a gauge theory

- Tedious computation shows that:

$$P = -I \circ \Box - d \circ \delta \circ I + 2I \circ \mathbf{Riem},$$

for

- $$\mathbf{Riem}\, u_{ab} = \mathbf{R}^{c}{}_{ab}{}^{d}\, u_{cd},$$

  (preserves symmetric 2-tensors because of symmetries of the Riemann tensor), and

$$P \circ d = 0.$$

- The map $u_2 \to u_2 + du_1$ corresponds to linearized gauge transformations (preserves solutions of $Pu = 0$).

Linearized gravity as a gauge theory

- replace u_2 by Iu_2.
- P becomes
$$P = -\Box - I \circ d \circ \delta + 2\mathbf{Riem},$$
- set
$$K := I \circ d.$$
- The gauge invariance of P is expressed by
$$P \circ K = 0,$$
- u_2 and $u_2 + Ku_1$ are equivalent solutions of $Pu_2 = 0$.
- the Hermitian space is
$$\frac{\mathrm{Ker}_{\mathrm{sc}}\, P}{\mathrm{Ran}_{\mathrm{sc}} K},$$
ie solutions of linearized Einstein modulo gauge equivalence.

Quantization of linearized gravity

- To quantize linearized gravity we need to equip $\dfrac{\mathrm{Ker}_{\mathrm{sc}} P}{\mathrm{Ran}_{\mathrm{sc}} K}$ with a Hermitian form.
- To do this one uses gauge fixing. We follow here the nice exposition in [Hack-Schenkel]:
- one adds the gauge condition $K^\star u = 0$ ie $\delta u = 0$ (harmonic gauge condition).
- here $A^\star$ is the adjoint w.r.t. the physical Hermitian form

$$(u|u)_{I,V_2} = (u|Iu)_{V_2}.$$

- for any u_2 with $Pu_2 = 0$ there exists u_1 such that $K^\star(u_2 + Ku_1) = 0$.
- u_1 is unique modulo a solution of $K^\star K v_1 = 0$ (residual gauge freedom).

- It follows that

$$\frac{\mathrm{Ker}_{\mathrm{sc}}\, P}{\mathrm{Ran}_{\mathrm{sc}} K} \sim \frac{\mathrm{Ker}_{\mathrm{sc}}\, D_2 \cap \mathrm{Ker}_{\mathrm{sc}}\, K^\star}{K\, \mathrm{Ker}_{\mathrm{sc}}\, D_1},$$

where

-

$$D_2 := P + K \circ K^\star = -\Box + 2\mathbf{Riem},$$

$$D_1 := K^\star \circ K = -\Box + \Lambda,$$

- $D_i = D_{i,L} - 2\Lambda$ where $D_{i,L}$ are the Lichnerowicz d' Alembertians, D_i are hyperbolic operators.
- They admit advanced/retarded inverses.

Further gauge fixing

- It is possible to impose further gauge fixing conditions, for example the traceless gauge

$$K_0^\star u_2 = 0$$

for $K_0^\star u_2 = -\mathrm{tr}_{\mathbf{g}} u_2$, $K_0 u_0 = u_0 \mathbf{g}$.

- One obtains then the equivalent Hermitian space

$$\frac{\mathrm{Ker}_{\mathrm{sc}}\, D_2 \cap \mathrm{Ker}_{\mathrm{sc}}\, K^\star \cap \mathrm{Ker}_{\mathrm{sc}}\, K_0^\star}{K\, \mathrm{Ker}_{\mathrm{sc}}\, D_1 \cap \mathrm{Ker}_{\mathrm{sc}}\, K_0^\star},$$

- It is also possible to change the gauge fixing condition. For example the condition:

$$\delta u_2 + \epsilon d \mathrm{tr}_{\mathbf{g}} u_2 = 0,$$

for $\epsilon \in \mathbb{R}$ has been used in the Euclidean framework.

- Leads to different operators D_i (leading term no more scalar).

- $\dfrac{\mathrm{Ker_{sc}}\,P}{\mathrm{Ran_{sc}}K}$ represents the 'on shell' Hermitian space.
- the corresponding ' off shell' Hermitian space is

$$\mathcal{V} = \frac{\mathrm{Ker_c}\,K^\star}{\mathrm{Ran_c}P}.$$

- One equips it with the Hermitian form

$$\overline{[u]}\cdot Q[u] = \mathrm{i}(u|G_2 u)_{I,V_2},$$

for $G_2 = G_{2\mathrm{ret}} - G_{2\mathrm{adv}}$.

Cauchy surface Hermitian space

- We have $D_2 \circ K = K \circ D_1$ and (taking adjoints) $K^\star \circ D_2 = D_1 \circ K^\star$.
- therefore
$$K : \mathrm{Ker}_{\mathrm{sc}}\, D_1 \to \mathrm{Ker}_{\mathrm{sc}}\, D_2,$$
$$K^\star : \mathrm{Ker}_{\mathrm{sc}}\, D_2 \to \mathrm{Ker}_{\mathrm{sc}}\, D_1$$
- We denote by K_Σ, $K_\Sigma^\dagger$ the 'Cauchy data' versions of K, K^*.
- For example if $D_1 u_1 = 0$, $f_1 = \rho_{1\Sigma} u_1$, then
$$K_\Sigma f_1 = \rho_{2\Sigma} K u_1.$$
- Since $D_1 = K^\star \circ K$ we have
$$K_\Sigma^\dagger \circ K_\Sigma = 0.$$

Cauchy surface Hermitian space

- We have $I \circ D_2 = D_2 \circ I$, so $I : \mathrm{Ker}_{\mathrm{sc}}\, D_2 \to \mathrm{Ker}_{\mathrm{sc}}\, D_2$. ($I$ = trace reversal).
- We denote by I_Σ the Cauchy data version of I.
- The Cauchy surface Hermitian space is

$$\frac{\mathrm{Ker}_{\mathrm{c}}\, K_\Sigma^\dagger}{\mathrm{Ran}_{\mathrm{c}} K_\Sigma},$$

equipped with the Hermitian form

$$\overline{[f]} \cdot q_{2,I}[f] = (f|q_{2\Sigma} \circ I_\Sigma f)_{V_2 \otimes \mathbb{C}^2}, \ \ f \in \mathrm{Ker}_{\mathrm{c}}\, K_\Sigma^\dagger.$$

off shell covariances

- Let $\Lambda_2^\pm : C_0^\infty(M; V_2) \to \mathcal{D}'(M; V_2)$ be such that

$$\begin{aligned}
&(1) \quad D_2 \circ \Lambda_2^\pm = \Lambda_2^\pm \circ D_2 = 0 \text{ (field equation)},\\
&(2) \quad \Lambda_2^+ - \Lambda_2^- = \mathrm{i} G_2 \text{ on } \operatorname{Ker}_{\mathrm{c}} K^\star \text{ (CCR)},\\
&(3) \quad \Lambda_2^\pm : \mathrm{Ran}_{\mathrm{c}} K \to \mathrm{Ran} K \text{ (gauge invariance)},\\
&(4) \quad (u|I\Lambda_2^\pm u)_{V_2} \geq 0, \ \forall u \in \operatorname{Ker}_{\mathrm{c}} K^\star \text{ (positivity)}.
\end{aligned}$$

Then

- $$\overline{[u]} \cdot \Lambda^\pm [u] = (u|I \circ \Lambda_2^\pm u)_{V_2}, \ [u] \in \frac{\operatorname{Ker}_{\mathrm{c}} K^\star}{\mathrm{Ran}_{\mathrm{c}} P}$$

are the (off-shell) covariances of a quasi-free state on $\mathrm{CCR}(\mathcal{V}, Q)$.

Cauchy surface covariances

If we use the Cauchy surface Hermitian space, we obtain analogous conditions.

- Let $\lambda^{\pm}_{2\Sigma} : C^{\infty}_{0}(\Sigma; V_2 \otimes \mathbb{C}^2) \to \mathcal{D}'(\Sigma; V_2 \otimes \mathbb{C}^2)$. We set as before
$$\lambda^{\pm}_{2\Sigma} =: \pm q_{2\Sigma} \circ c^{\pm}_{2}.$$
- The analogous conditions on $\lambda^{\pm}_{2\Sigma}$ are:

$$\begin{aligned}
&(1) \quad c^{+}_{2} + c^{-}_{2} = \mathbb{1} \text{ on } \operatorname{Ker}_{\mathrm{c}} K^{\dagger}_{\Sigma} \text{ (CCR)},\\
&(2) \quad c^{\pm}_{2} : \mathrm{Ran}_{\mathrm{c}} K_{\Sigma} \to \mathrm{Ran} K_{\Sigma} \text{ (gauge invariance)},\\
&(3) \quad \pm(f | I_{\Sigma} q_{2\Sigma} c^{\pm}_{2} f)_{V_2 \otimes \mathbb{C}^2} \geq 0, \ \forall f \in \operatorname{Ker}_{\mathrm{c}} K^{\star}_{\Sigma} \text{ (positivity)}.
\end{aligned}$$

Hadamard condition

- In addition to the above conditions, we require the Hadamard condition ie
- $$\mathrm{WF}(\Lambda_2^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm},$$

  or equivalently

  $$\mathrm{WF}(U_{2\Sigma} \circ c_2^{\pm})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma,$$

Existence of Hadamard states

- Theorem (G)

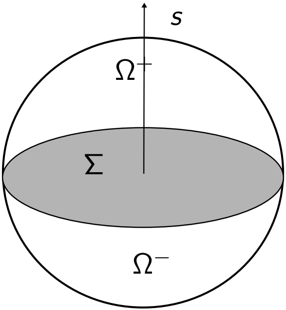
  Let $(M, \mathbf{g})$ be any Einstein manifold with compact Cauchy surfaces. Then there exist gauge invariant Hadamard states for linearized gravity on $(M, \mathbf{g})$.

  - The proof relies on pdo calculus and uses full gauge fixing:
  - this amounts to find a convenient supplementary space to $\mathrm{Ran} K_{\Sigma}$ inside Ker $K_{\Sigma}^{\dagger}$. The delicate gauge invariance property can now be forgotten.

de Sitter spacetime

- the de Sitter spacetime dS^4 is $\mathbb{R}_t \times \mathbb{S}^3$, equipped with the metric
$$\mathbf{g} = -dt^2 + \cosh^2(t)\mathrm{h},$$
h canonical metric on $\mathbb{S}^3 = \Sigma$.
- By Wick rotation $t \mapsto is$ we obtain the metric $\tilde{\mathbf{g}} = ds^2 + \cos^2(s)\mathrm{h}$, $s \in]-\pi/2, \pi/2[$,
- ie the sphere $\mathbb{S}^4$ by setting
$$x_0 = \sin s, \ (x_1, \dots, x_4) = \cos s\ \omega, \ \omega \in \mathbb{S}^3.$$
- The Wick rotations of D_i are denoted by $\tilde{D}_i$.
- They are selfadjoint for the natural scalar products on $\mathbb{S}^4$.

Wick rotated de Sitter spacetime



Euclidean dS^4

Wick rotated de Sitter spacetime

- The Wick rotation of dS^4 is compact: no need for boundary conditions to define $\tilde{D}_1$, $\tilde{D}_2$!
- $\tilde{D}_2$ is invertible,
- $\tilde{D}_2 \geq 2$ on $\mathrm{Ker}(\tilde{\mathbf{g}}|$ (traceless symmetric 2-tensors on $\mathbb{S}^4$).
- $\tilde{D}_1$ is not invertible,
- $\mathrm{Ker}\, \tilde{D}_1 = \mathrm{Ker}\, d = \mathrm{Ker}\, d \cap \mathrm{Ker}\, \delta =$ space of Killing 1-forms on $\mathbb{S}^4$.

Calderón projectors

- By Wick rotating the identity $D_2 \circ K = K \circ D_1$ we obtain

$$\tilde{D}_2 \circ \tilde{K} = \tilde{K} \circ \tilde{D}_1, \ (\tilde{K} = \tilde{I} \circ \tilde{d}).$$

- If Calderón projectors $\tilde{c}_i^{\pm}$ exist for $\tilde{D}_i$ then one would have

$$\tilde{c}_2^{\pm} \circ \tilde{K}_\Sigma = \tilde{K}_\Sigma \circ \tilde{c}_1^{\pm}$$

hence $\tilde{c}_2^{\pm}$ preserves $\mathrm{Ran}\tilde{K}_\Sigma$: one would get gauge invariance !

Calderón projectors

- $\tilde{c}_2^{\pm}$ exist since $\tilde{D}_2$ is invertible.
- $\tilde{c}_1^{\pm}$ do not exist on the whole space $C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$, since $\tilde{D}_1$ is not invertible.
- this problem is due to the existence of Killing one-forms !
- it is still present with any of the alternative gauge fixing conditions explained above.

Calderón projectors

- One can show that Calderón projectors $\tilde{c}_1^{\pm}$ exist on the subspace:

$$E_1 := (\tilde{\rho}_1 \,\mathrm{Ker}\, \tilde{D}_1)^{q_1},$$

  the q_1-orthogonal of $\tilde{\rho}_1 \,\mathrm{Ker}\, \tilde{D}_1$, of codimension 10 in $\tilde{V}_1 \times \mathbb{C}^2$.
- There is a corresponding natural subspace E_2 of codimension 10 in $\tilde{V}_2 \times \mathbb{C}^2$ such that

$$\tilde{K}_\Sigma : E_1 \to E_2$$

The Euclidean vacuum state

- Let us define Cauchy surface covariances

$$\overline{f}_2 \cdot \lambda^{\pm}_{2\Sigma} f_2 = \pm (f_2 | q_{2\Sigma} I_\Sigma \tilde{c}_2^{\pm} f_2)_{V_2 \otimes \mathbb{C}^2}.$$

- We call the associated (pseudo) state ω the Euclidean vacuum.

Theorem (GW)

The Euclidean vacuum satisfies

(1) *the Hadamard condition,*
(2) *the CCR on* $\mathrm{Ker}\, K^\dagger_\Sigma \cap \mathrm{Ker}\, K^\dagger_{0\Sigma}$, *(TT gauge subspace)*
(3) *the gauge invariance under* E_1,
(4) *the positivity on* $\mathrm{Ker}\, K^\dagger_\Sigma \cap \mathrm{Ker}\, K^\dagger_{0\Sigma} \cap E_2$,
(5) *the invariance under all de Sitter isometries.*

The Euclidean vacuum state

- one can show that $\tilde{c}_2^{\pm}$ do not preserve the full $\mathrm{Ran}\tilde{K}_\Sigma$: in fact

$$\tilde{c}_2^{\pm}\tilde{K}_\Sigma f \in \mathrm{Ran}\tilde{K}_\Sigma \text{ iff } f \in E_1!$$

- We recall that Ker $K_\Sigma^\dagger \cap$ Ker $K_{0\Sigma}^\dagger$ correspond to Cauchy data of transverse, traceless tensors.
- This is one of the many equivalent Hermitian spaces appearing in linearized gravity.

Modified Euclidean vacuum

- one can improve the gauge invariance and positivity of ω by additional gauge fixing.
- This amounts to replace $\lambda^{\pm}_{2\Sigma}$ by

$$\overline{f}_2 \cdot \lambda^{\pm}_{2\Sigma} f_2 = \pm (Tf_2 | q_{2\Sigma} I_\Sigma \tilde{c}^{\pm}_2 Tf_2)_{V_2 \otimes \mathbb{C}^2},$$

for T a natural projection on E_2.

- Theorem (GW)

The modified Euclidean vacuum satisfies

(1) *the Hadamard condition,*
(2) *the CCR on* $\mathrm{Ker}\, K^{\dagger}_{\Sigma} \cap \mathrm{Ker}\, K^{\dagger}_{0\Sigma}$,
(3) *the full gauge invariance*
(4) *the positivity on* $\mathrm{Ker}\, K^{\dagger}_{\Sigma} \cap \mathrm{Ker}\, K^{\dagger}_{0\Sigma}$,
(5) *the invariance under de Sitter isometries preserving* Σ.

- Thank you for your attention !