Renormalised stress energy tensor (RSET) in four-dimensional anti-de Sitter space-time (adS4) with Robin boundary conditions

(Morley, Namasivayam, Winstanley; arXiv: 2308.05623)

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Aims

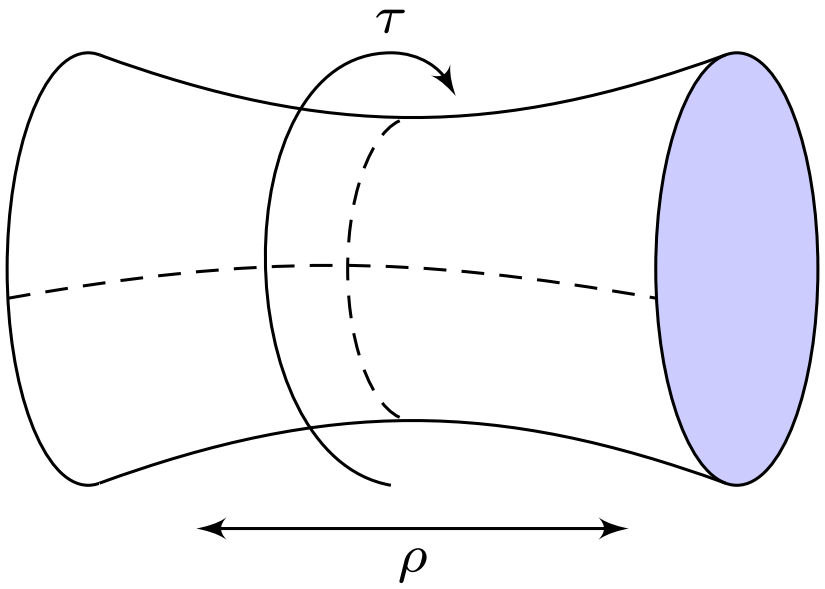
Semi-classical Approach to Quantum Gravity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G \left\langle \hat{T}_{\mu\nu} \right\rangle$$

- vacuum (v.e.v) and thermal (t.e.v.) expectation values of different components of $\langle \hat{T}_{\mu\nu} \rangle$
  - thermal: inverse temperature $\beta = 1/T$
- massless, conformally coupled scalar field, Φ
- adS4
- effects of boundary conditions on $\langle \hat{T}_{\mu\nu} \rangle$
- pressure deviator , Π, measure of $\langle \hat{T}^{\rho}_{\rho} \rangle - \langle \hat{T}^{\theta}_{\theta} \rangle$
  - difference between quantum state and classical

$\hbar = c = 1$

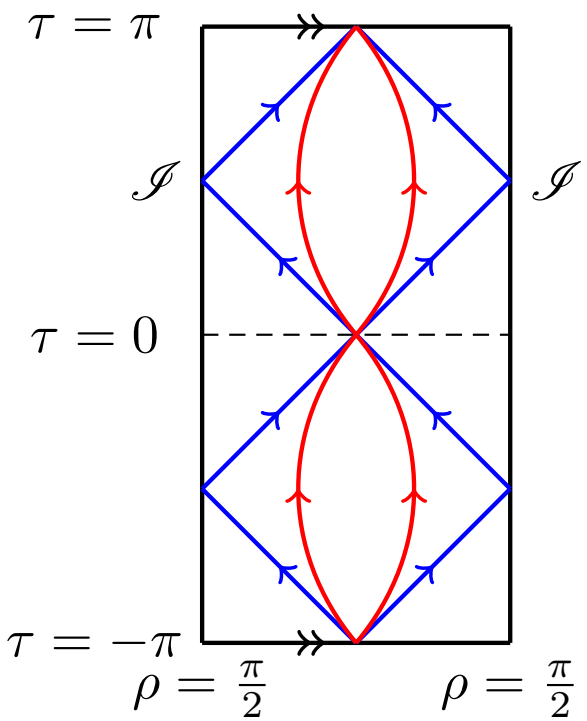
Anti-de Sitter space: adS4



- 4D hyperboloid embedded in a 5D Minkowski spacetime
- $ds^2 = L^2 \sec^2\rho\,[-dt^2 + d\rho^2 + \sin^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)]$
- $0 \leq \rho \leq \pi/2, \quad 0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad -\pi \leq t \leq \pi$
- closed time-like curves...covering space (CadS)

Properties of anti-de Sitter space-time

- maximally symmetric solution to Einstein's field equation
- constant negative curvature $\Lambda = -3/L^2$
- time-like boundary at $\rho = \pi/2$
- adS not a globally hyperbolic space-time
- need to impose boundary conditions [1]



[1] Avis, Isham, Storey, *PRD*; 1978; Dappiaggi, Ferreira, Marta, *PRD*; 2018

Boundary conditions

- Impose boundary conditions on radial solution of $\Phi(t, \mathbf{x})$ at the time-like boundary, $\rho = \pi/2$. $(\mathbf{x} = \rho, \theta, \phi)$
- Dirichlet b.c.

$$\Phi(t, \mathbf{x}) = 0$$

- Neumann b.c

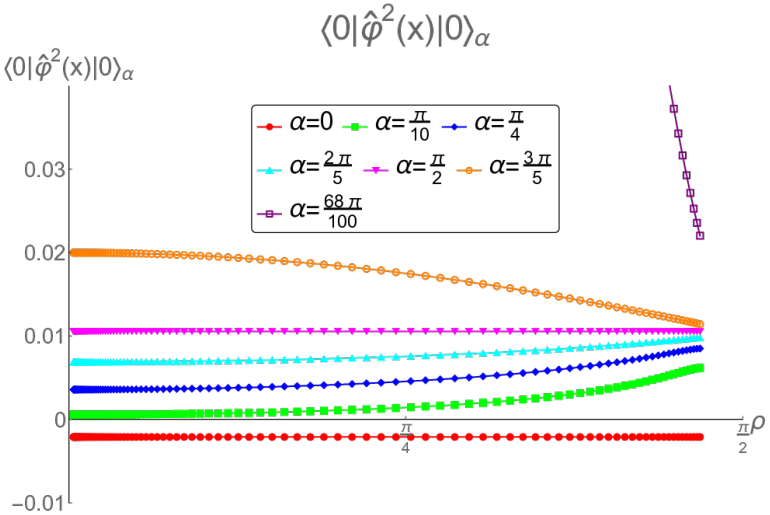
$$\frac{\partial \Phi(t, \mathbf{x})}{\partial \rho} = 0$$

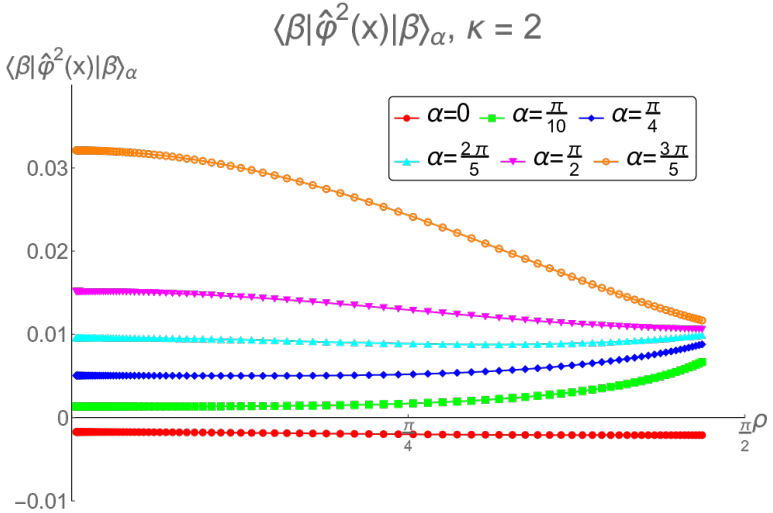
- Robin b.c

$$\Phi(t, \mathbf{x}) \cos\zeta + \frac{\partial \Phi(t, \mathbf{x})}{\partial \rho} \sin\zeta = 0 \quad \text{for} \quad \zeta \in [0, \pi]$$

- $\zeta = 0 \rightarrow$ Dirichlet $\quad \zeta = \pi/2 \rightarrow$ Neumann

Vacuum polarisation (VP) in adS4 with Robin b.c. [3]





- Neumann b.c. gives the generic behaviour of VP at the space-time boundary
- Does the same occur with $\langle \hat{T}_{\mu\nu} \rangle$?

[3] Morley, Taylor and Winstanley, *CQG*; 2021

Euclidean space

- Wick rotation $t \to i\tau$
- $ds^2 = L^2 \sec^2 \rho \, [d\tau^2 + d\rho^2 + \sin^2 \rho \, (d\theta^2 + \sin^2 \theta \, d\phi^2)]$
- Green's function unique and well defined
- no need for '$i\epsilon$' prescription
- simplifies numerical calculations

Calculating $\langle\psi|\,\hat{T}_{\mu\nu}|\psi\rangle$

Euclidean Green's function, $G^E(x,x')$

$$\langle\psi|\,\hat{T}_{\mu\nu}|\psi\rangle = \lim_{x'\to x}\left\{\mathcal{T}_{\mu\nu}(x,x')G^{\mathrm{E}}(x,x')\right\},$$

where

$$\mathcal{T}_{\mu\nu} = \frac{2}{3}g_{\nu}{}^{\nu'}\nabla_{\mu}\nabla_{\nu'} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma'}\nabla_{\rho}\nabla_{\sigma'} - \frac{1}{3}g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}\nabla_{\mu'}\nabla_{\nu'} + \frac{1}{3}g_{\mu\nu}\nabla_{\rho}\nabla^{\rho} + \frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

$g_{\mu\nu'}$ is the bivector of parallel transport

Decanini and Folacci, *PRD*; 2008

Euclidean Green's functions

massless, conformally coupled scalar field on adS4 [5]

$$G^{\mathrm{E}}_{\zeta}(x,x') = G^{\mathrm{E}}_{\mathrm{D}}(x,x')\cos^2\zeta + G^{\mathrm{E}}_{\mathrm{N}}(x,x')\sin^2\zeta + G^{\mathrm{E}}_{\mathrm{R}}(x,x')\sin 2\zeta,$$

$$\begin{aligned} G^{\mathrm{E}}_{\mathrm{D},0}(x,x') = {} & \frac{1}{16\pi^2 L^2}\frac{\cos\rho\cos\rho'}{\sqrt{\sin\rho\sin\rho'}}\int_{\omega=-\infty}^{\infty} d\omega\, e^{i\omega\Delta\tau} \\ & \times \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)|\Gamma(\ell+1+i\omega)|^2 P^{-\ell-1/2}_{i\omega-1/2}(\cos\rho_<) \\ & \times \left[P^{-\ell-1/2}_{i\omega-1/2}(-\cos\rho_>) - P^{-\ell-1/2}_{i\omega-1/2}(\cos\rho_>)\right] \end{aligned}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\phi$

[5] Morley, Taylor, Winstanley, *CQG*; 2021

Regular/Robin contribution

$$G^{\mathrm{E}}_{\mathrm{R},0}(x,x') = \frac{1}{16\pi^2 L^2}\frac{\cos\rho\,\cos\rho'}{\sqrt{\sin\rho\,\sin\rho'}}\int_{\omega=-\infty}^{\infty} d\omega\, e^{i\omega\Delta\tau}$$
$$\times \sum_{\ell=0}^{\infty} D^{\zeta}_{\omega\ell} P_\ell(\cos\gamma) P^{-\ell-1/2}_{i\omega-1/2}(\cos\rho)\, P^{-\ell-1/2}_{i\omega-1/2}(\cos\rho')$$

where

$$D^{\zeta}_{\omega\ell} = (2\ell+1)|\Gamma(1+\ell+i\omega)|^2 \left[\frac{2|\Gamma(\frac{i\omega+\ell+2}{2})|^2\cos\zeta - |\Gamma(\frac{i\omega+\ell+1}{2})|^2\sin\zeta}{2|\Gamma(\frac{i\omega+\ell+2}{2})|^2\sin\zeta + |\Gamma(\frac{i\omega+\ell+1}{2})|^2\cos\zeta}\right]$$

$0 \le \zeta < \zeta_{\mathrm{crit}}$ where $\zeta_{\mathrm{crit}} \sim 0.68\pi$

calculating $\langle T_{\mu\nu} \rangle^{\zeta}$

$$G^{\mathrm{E}}_{\zeta}(x,x') = G^{\mathrm{E}}_{\mathrm{D}}(x,x')\cos^2\zeta + G^{\mathrm{E}}_{\mathrm{N}}(x,x')\sin^2\zeta + G^{\mathrm{E}}_{\mathrm{R}}(x,x')\sin 2\zeta,$$

$$\langle \hat{T}_{\mu\nu} \rangle^{\zeta} = \lim_{x'\to x} \mathcal{T}_{\mu\nu}(x,x')\Big\{ G^{\mathrm{E}}_{\mathrm{D}}(x,x')\cos^2\zeta + G^{\mathrm{E}}_{\mathrm{N}}(x,x')\sin^2\zeta + G^{\mathrm{E}}_{\mathrm{R}}(x,x')\sin 2\zeta \Big\}.$$

$$\langle \hat{T}_{\mu\nu} \rangle^{\zeta}_{\mathrm{ren}} = \langle \hat{T}_{\mu\nu} \rangle^{\mathrm{D}}_{\mathrm{ren}}\cos^2\zeta + \langle \hat{T}_{\mu\nu} \rangle^{\mathrm{N}}_{\mathrm{ren}}\sin^2\zeta + \lim_{x'\to x}\Big\{ \mathcal{T}_{\mu\nu}(x,x')G^{\mathrm{E}}_{\mathrm{R}}(x,x') \Big\}\sin 2\zeta$$

$\langle \hat{T}_{\mu\nu} \rangle^{\mathrm{D}}_{\mathrm{ren}}$ and $\langle \hat{T}_{\mu\nu} \rangle^{\mathrm{N}}_{\mathrm{ren}}$

$$
\begin{aligned}
\langle \hat{T}_{\mu\nu} \rangle^{\mathrm{D/N}}_{\beta} = \\
&\frac{1}{8\pi^2 L^4} \left\{ \left[-\frac{1}{120} + \frac{4}{3} \cos^4 \rho \; f_3 \left(\frac{\beta}{L} \right) \right] g_{\mu\nu} + \left[\frac{16}{3} \cos^4 \rho \, f_3 \left(\frac{\beta}{L} \right) \right] \tau_\mu \tau_\nu \right\} \\
&\pm \frac{\cot \rho}{8\pi^2 L^4} \left\{ \left[-\frac{1}{6} \csc^2 \rho \; \cos 2\rho \; S_0 \left(\frac{\beta}{L}, \rho \right) + \frac{1}{3} \cot \rho \; C_1 \left(\frac{\beta}{L}, \rho \right) + \frac{2}{3} \cos^2 \rho \; S_2 \left(\frac{\beta}{L}, \rho \right) \right] g_{\mu\nu} \right. \\
&+ \left[\frac{1}{6} (3 - \cot^2 \rho) S_0 \left(\frac{\beta}{L}, \rho \right) + \cot \rho \left(1 - \frac{2}{3} \cos^2 \rho \right) C_1 \left(\frac{\beta}{L}, \rho \right) + 2 \cos^2 \rho \; S_2 \left(\frac{\beta}{L}, \rho \right) \right] \tau_\mu \tau_\nu \\
&\left. + \left[\frac{1}{6} (3 \csc^2 \rho - 4) S_0 \left(\frac{\beta}{L}, \rho \right) + \cot \rho \left(\frac{2}{3} \sin^2 \rho - 1 \right) C_1 \left(\frac{\beta}{L}, \rho \right) - \frac{2}{3} \cos^2 \rho \; S_2 \left(\frac{\beta}{L}, \rho \right) \right] \rho_\mu \rho_\nu \right\}
\end{aligned}
$$

$$
f_m(x) = \sum_{n=1}^{\infty} n^m (e^{nx} - 1)^{-1}, \quad S_m(x, \rho) = \sum_{n=1}^{\infty} n^m (-1)^n (e^{nx} - 1)^{-1} \sin(2n\rho),
$$

$$
C_m(x, \rho) = \sum_{n=1}^{\infty} n^m (-1)^n (e^{nx} - 1)^{-1} \cos(2n\rho).
$$

Allen, Folacci, Gibbons, *Phys. Lett.*; 1987

v.e.v with Dirichlet and Neumann b.c.

- retains maximum symmetry of adS

$$\langle \hat{T}_{\mu\nu} \rangle_0^{\mathrm{D/N}} = \alpha g_{\mu\nu}, \qquad \alpha = \langle \hat{T}^{\mu}_{\mu} \rangle_0^{\mathrm{D/N}} / 4$$

- m.c.c scalar field ... trace anomaly in adS4

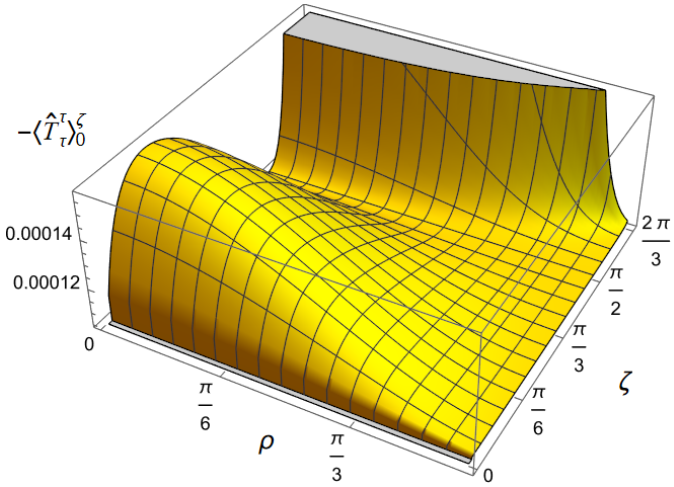
$$\langle \hat{T}^{\mu}_{\mu} \rangle_0^{\mathrm{D/N}} = -\frac{1}{240\pi^2 L^4} \tag{1}$$

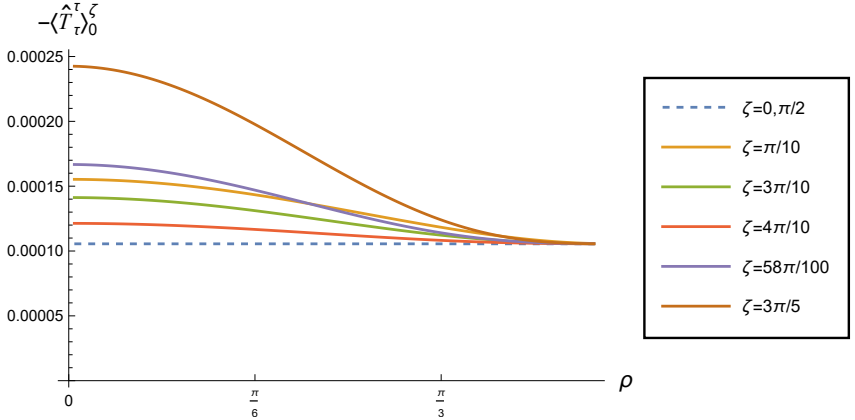
- independent of boundary conditions ($D = N$)

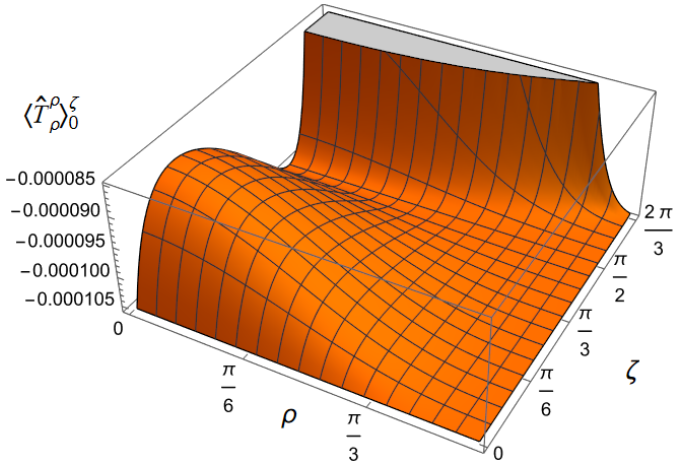
$$\langle \hat{T}^{t}_{t} \rangle_0^{D/N} = \langle \hat{T}^{\rho}_{\rho} \rangle_0^{D/N} = \langle \hat{T}^{\theta}_{\theta} \rangle_0^{D/N} = -\frac{1}{960\pi^2 L^4} \tag{2}$$

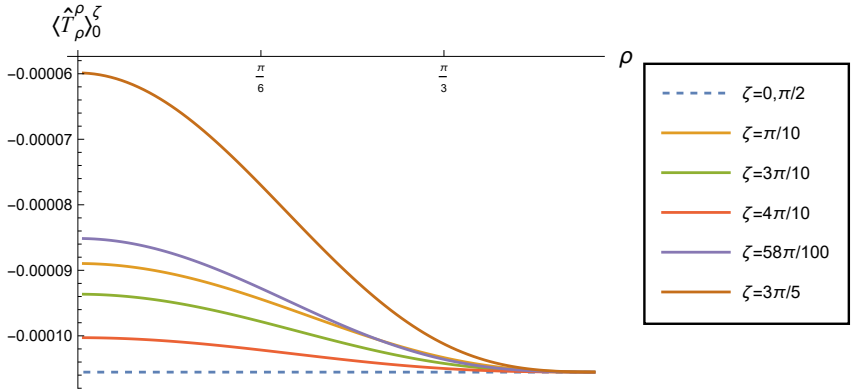
Allen, Folacci, Gibbons, *Phys. Lett.*; 1987: Kent, Winstanley, *PRD*; 2015

v.e.v with Robin b.c

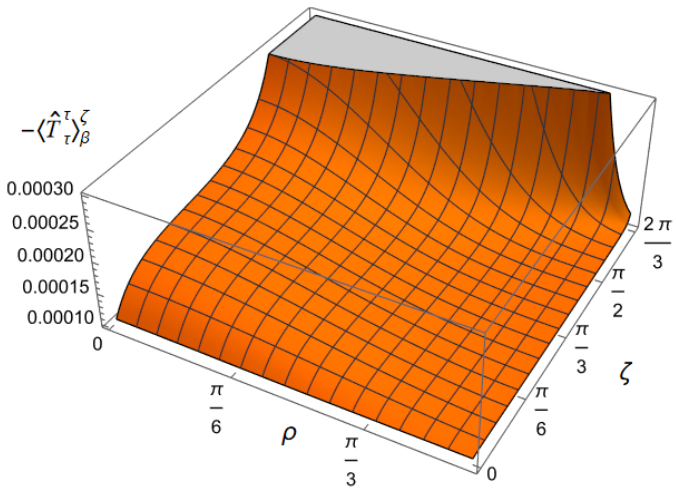


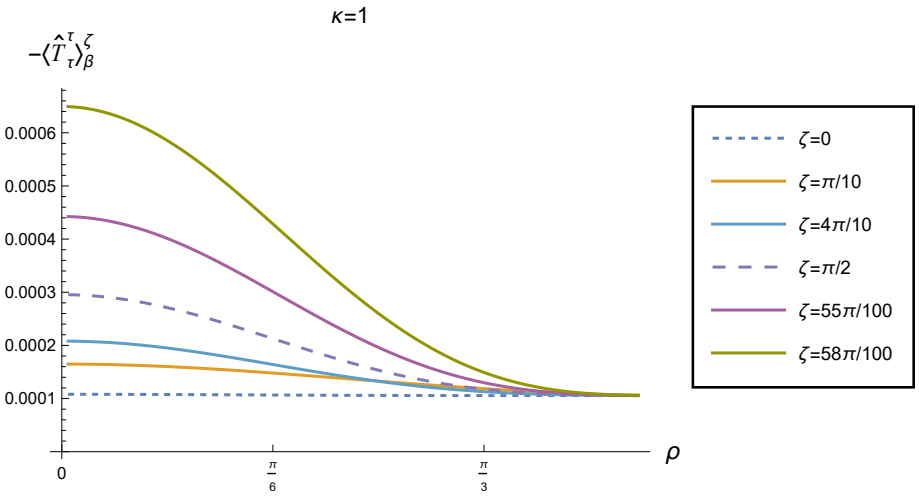


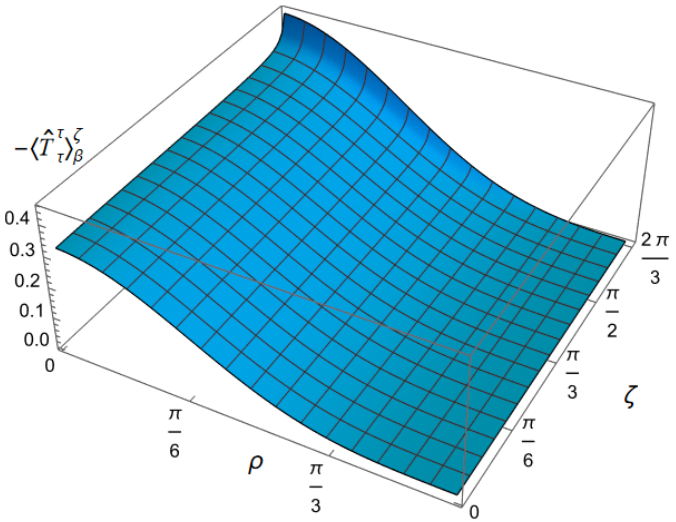


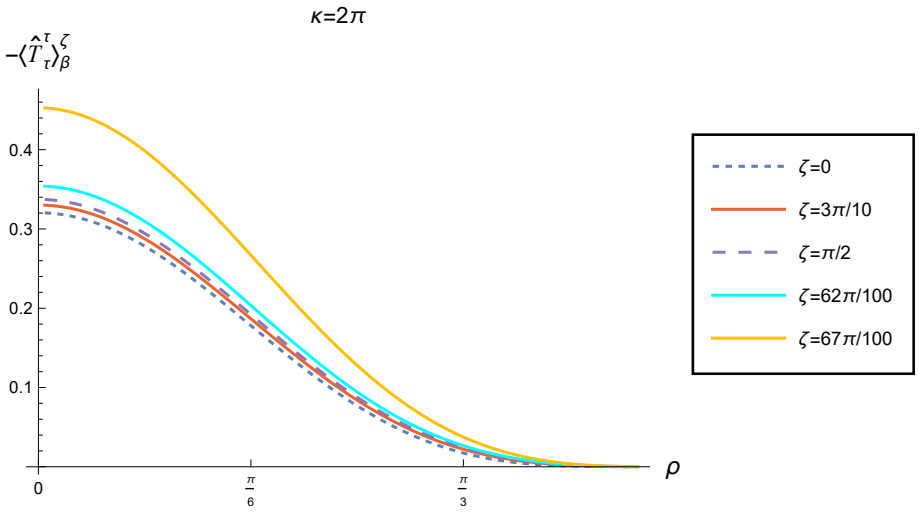


t.e.v. with Robin b.c. $(\kappa = 2\pi/\beta)$

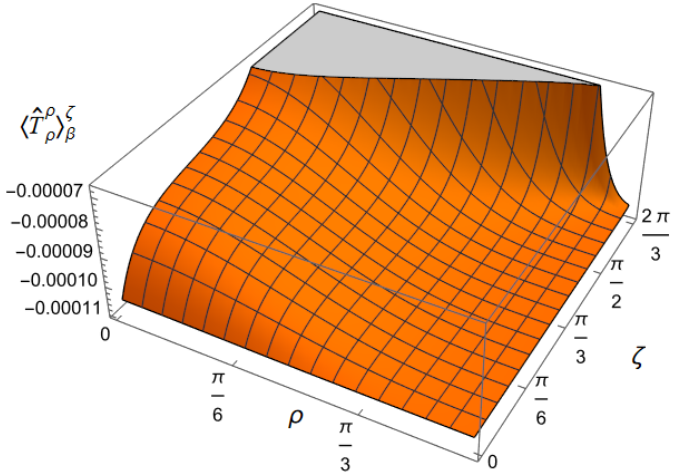


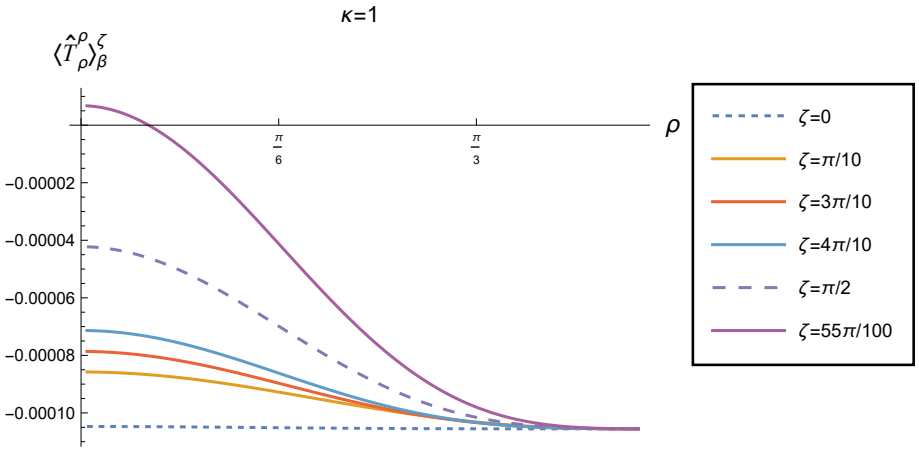


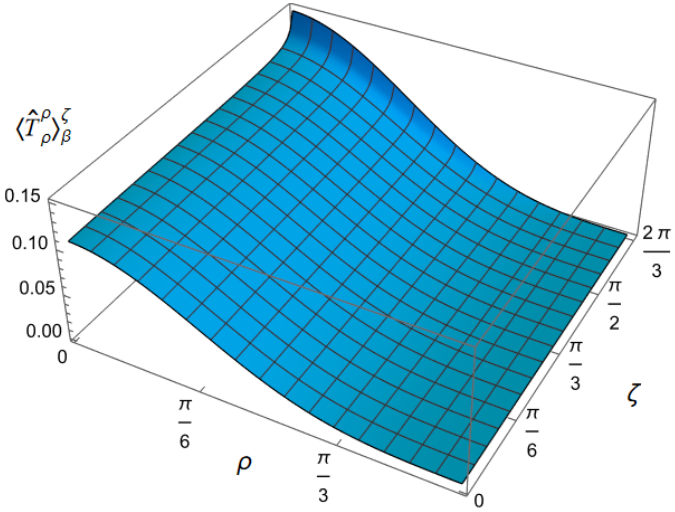


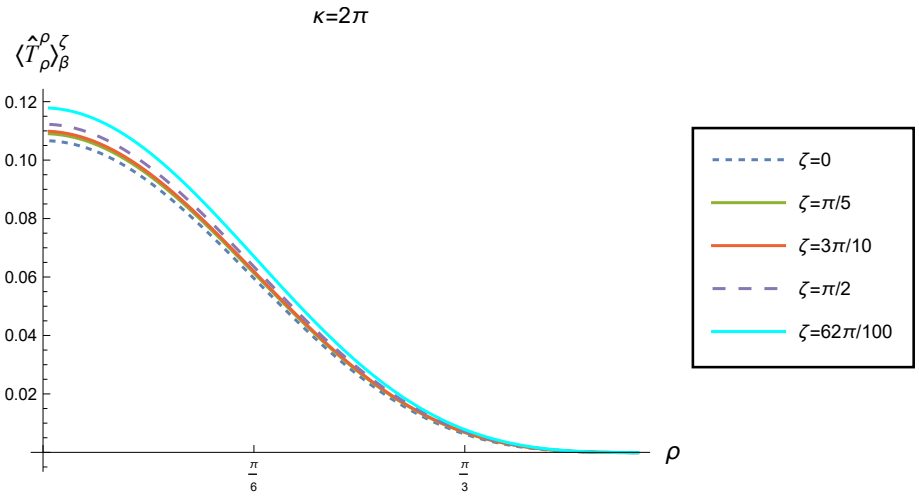


t.e.v with Robin b.c

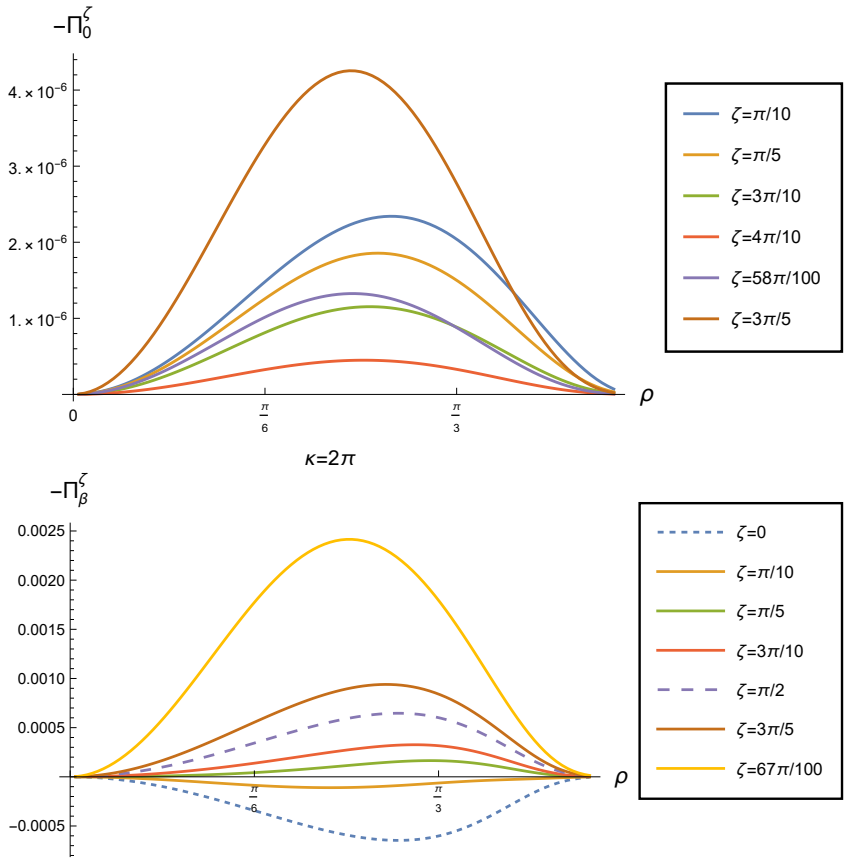








pressure deviators, Π^{ζ}, with Robin b.c.



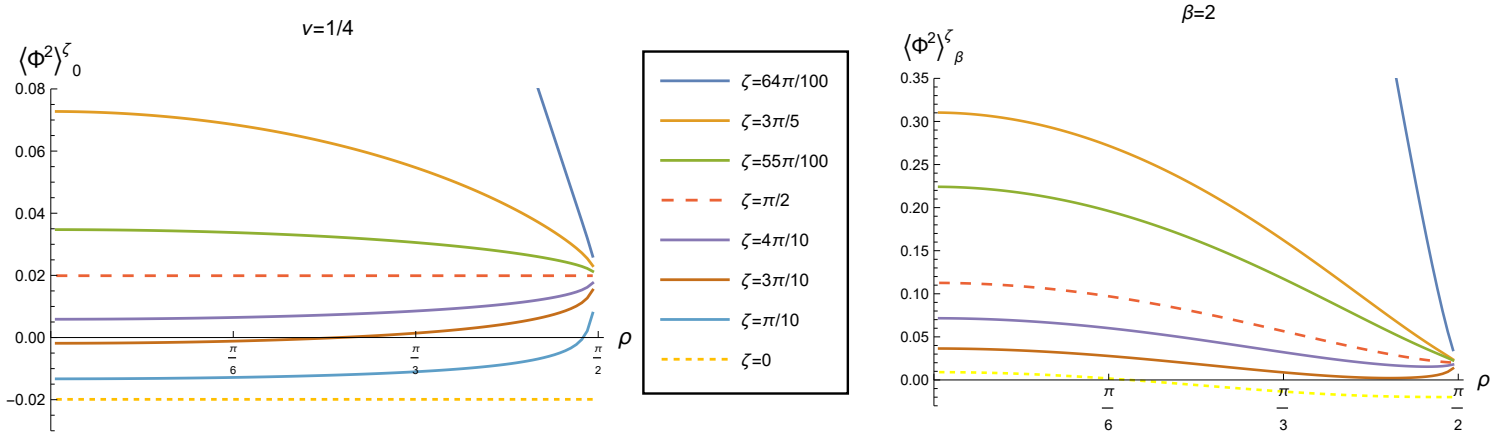
- $\langle \hat{T}^{\rho}_{\rho} \rangle - \langle \hat{T}^{\theta}_{\theta} \rangle$
- $\Pi_0^{D/N} = 0$
- Π depends on ζ and β
- $\Pi^{\zeta}_{0,\beta} = 0$ at $\rho = 0, \pi/2$

Conclusions

- v.e.v. of $\langle \hat{T}^{\nu}_{\mu} \rangle$ is a constant for both Dirichlet and Neumann boundary conditions.
- Temperature breaks the underlying symmetry of adS
- Robin boundary conditions also break the underlying symmetry for both vacuum and thermal states for all ζ
- Both v.e.v. and t.e.v. of $\langle \hat{T}^{\nu}_{\mu} \rangle$ converge to the common vacuum Dirichlet/Neumann result at the space-time boundary for all Robin boundary conditions.

Future work ...

- VP in adS3 for general m and ξ [8]



- v.e.v and t.e.v of $\langle \hat{T}^{\nu}_{\mu} \rangle$ for scalar field with general mass and ξ in adS3 with Robin b.c.
- $\langle \hat{T}^{\nu}_{\mu} \rangle^{D}_{0}$ and $\langle \hat{T}^{\nu}_{\mu} \rangle^{N}_{0}$ are different
- $\langle \hat{T}^{\nu}_{\mu} \rangle^{\zeta}_{0,\beta} \rightarrow$ Neumann result at space-time boundary ?

[8] Namasivayam and Winstanley, *GRG*; 2023

Thank You

....any questions?

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