On the global Hadamard parametrix in QFT and the signed squared geodesic distance defined in domains larger than convex normal neighborhoods

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Summary

1. Synge world function and Hadamard parametrix
   1.1 Relevance in aQFT in CS;
   1.2 Local definition of σ and need for a more global def.
   1.3 Issues with $\sigma(x,y)$ and $H_n(x,y)$
2. A topological argument to fix the problem
   2.1 Strongly convex neighborhoods
   2.2 A crucial property of paracompacntess
   2.3 σ and H_n defined around the diagonal of $M \times M$
3. A slight mathematical change in the geometric definition of Hadamard state
   3.1 Causal normal neighborhoods and parametrices
   3.2 Geometric definition of Hadamard state
   3.3 Relation with Radzikowski's micro-local definition
4. Conclusions and open issues: global locally-covariant parametrix?

1. Synge world function and Hadamard parametrix

1.1.1 Relevance in aQFT in CS

- Henceforth (M, g) 4-dim **globally hyperbolic spacetime**
- The Synge world function aka signed squared geodesical distance $\sigma(x, y)$ appears in the global Hadamard parametrix

$$H_n^T(x, y)_\epsilon := \frac{1}{(2\pi)^2}\left(\frac{\Delta(x, y)^{1/2}}{\sigma(x, y)_\epsilon^T} + v_n(x, y)\ln\sigma(x, y)_\epsilon^T\right)$$

$\sigma(x, y)_\epsilon^T := \sigma(x, y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2$, T global **time function**. $\Delta(x, y)$, $v_n(x, y)$ (recursively) constructed along the geodesic joining x and y. χ (below) smoothing hat function.

- Let Λ_ω be the two-point function of a KG Gaussian state ω,

$$\Lambda_\omega(x, y) - \chi(x, y)H_n^T(x, y)_{0^+} \textit{ of class } C^n \;\; \forall n \in \mathbb{N}\,,$$

is the geometric definition of Hadamard state made rigorous in [Kay-Wald91], but already introduced in [Fulling-Sweeny-Wald78], [Fulling-Narcowich-Wald81].

REMARKS

(1) Equivalent (analytic) microlocal definition in terms of WF set [Radzikowski96a] exploited in perturbative renormalization.

(2) At least **locally**, using smooth cutoffs, a **summed** parametrix H_∞ exists. The def above is equivalent [Radzikowski96b] to

$$\Lambda_\omega(x,y) - H^T_\infty(x,y)_{0^+} \text{ of class } C^\infty \quad \text{locally}$$

(3) The **geometric definition** and the parametrix H_n (local or global) has been used, in particular, to establish important theoretical features of Hadamard states [Verch94], in the explicit construction of locally-covariant Wick polynomials also time-ordered [Hollands-Wald01,02] including the stress-energy tensor [VM03,Hollands-Wald05], in quantum energy inequalities [Fewster-Smith08], rigorous treatements of Hawking radiation [Fredenhagen-Haag90][VM-Pinamonti12][Kurpicz-Verch-Pinamonti21], Semiclassical cosmological models [Meda-Pinamont-Siemssen21]. (...)

1.2 Local definition of σ and need for a more global def.

• In general contexts (e.g., **heat kernel theory**) $\sigma(x,y)$ is defined in a neighborhood of the **diagonal** of $M \times M$, where (M,g) is a Lorentzian (or Riemannian) manifold with any dimension .

• [Kay-Wald91] H_n defined around the diagonal of $M_\Sigma \times M_\Sigma$, M_Σ globally hyperbolic **sub**-spacetime around a **Cauchy surface** Σ of (M,g). (More details later.)

However

• $\sigma(x,y)$ **jointly smooth and well-defined** in **(geodesically) convex** (open) sets $C \ni x, y$

$$\sigma_C(x,y) := \pm \left(\int_0^1 \sqrt{\left| g\left(\dot{\gamma}^C_{xy}(t), \dot{\gamma}^C_{xy}(t)\right)\right|} dt \right)^2 \quad \text{for} \quad x,y \in C$$

• $\gamma^C_{xy} : [0,1] \to C$ **unique geodesic segment in** C joining x and y; $\pm$ fixed by the causal character of γ^C_{xy} if g is Lorentzian.

• $\sigma_C(x,y) \neq \sigma_{C'}(x,y)$ in general if $x, y \in C \cap C'$.

1.3.1 Issues with $\sigma(x,y)$ and $H_n(x,y)$

$\Longrightarrow$ in a neighborhoods $\mathcal{N}$ of $diag(M \times M) := \{(x,x) \mid x \in M\}$

$$\mathcal{N} = \bigcup_{p \in M} C_p \times C_p \,, \quad C_p \;\; \textit{convex neighborhood of } p$$

$\sigma(x,y) := \sigma_{C_p}(x,y)$ for $C_p \ni x,y \Rightarrow$ **multivalued in general!**
N.B. Same problem for Δ and v_n.

• If (M,g) is **Riemannian**, let every C_p in $\mathcal{C}$ be a **geodesic ball** centered on $p \in M$, **normal neighborhood** of p,
$\Longrightarrow$ $\sigma(x,y) := \sigma_{C_x}(x,y)$ is well defined, $= \sigma_{C_y}(x,y)$, jointly smooth on $\mathcal{C}$, and $\sigma(x,y) = dist_g(x,y)^2$
$(dist_g(x,y) := \inf\{L_g(\gamma) \mid \gamma \textit{ joins } x \textit{ and } y \textit{ in } M\})$
(PROOF: absolute length-minimizing property of geodesic in small geod balls, joint continuity of $dist_g$, smoothness of $(\exp_x)^{-1}$, and Sobolev's lemma)

N.B. No Lorentzian generalisation: analog of $dist_g$ different properties.

1.3.2 Issues with $\sigma(x, y)$ and $H_n(x, y)$

• [Kay-Wald91] σ (and thus H_n) defined in a causal normal neighborhood M_Σ of a **Cauchy surface** Σ of (M, g), i.e.,

(a) $(M_\Sigma, g|_{M_\Sigma})$ globally hyperbolic spacetime,

(b) $x, y \in M_\Sigma$ **causally related** in M if and only if the **causal double-cone** $J_M(x, y) \subset C_{x,y}$ **convex** neighborhood .

$\Longrightarrow$ If $x, y \in C_{x,y}, C'_{x,y}$ then every **causal** geodesic joining them stays in **both** convex neighborhoods and thus it is **unique**:

$$\sigma(x, y) := \sigma_{C_{x,y}}(x, y) \textit{ with } C_{x,y} \supset J_M(x, y) \quad \textit{well defined}$$

• However $\sigma(x, y)$, Δ, v_n appearing in $H_n(x, y)$ are assumed and used to be **well defined** and **smooth** in a **neighborhood** $\mathcal{O}$ of the subset of the **causally related** pairs in $M_\Sigma \times M_\Sigma$: there are also **causally separated** points in $\mathcal{O}$.

are $\sigma(x, y)$, Δ, v_n well defined and smooth?

N.B. Problem mainly of **mathematical** nature: **physics uses causal curves only**.

2. A topological argument to fix the problem

2.1 Strongly convex neighborhoods

N.B. Better to stay **as close as possible** to the original construction to preserve relevant consequences of that definition accumulated over the years.

BASIC IDEA [VM21] (Other ideas by Sánchez and Gérard) : A covering $\mathcal{C}$ of (M, g) **made of geodesical convex neighborhoods** is strongly convex if

$$C \cap C' \textit{ is geodesically convex for } C, C' \in \mathcal{C}$$

$\Longrightarrow$ If $\mathcal{C} \ni C, C' \ni x, y$ then $\sigma_C(x, y) = \sigma_{C'}(x, y)$, because there is **only one** geodesic joining x and y in $C \cap C'$.

$\Longrightarrow$ Take $\mathcal{C} := \{C_p\}_{p \in M}$ **strongly convex** and consider the neighborhood of $diag(M \times M)$ of the form $\mathcal{N} = \bigcup_{p \in M} C_p \times C_p$
$\Longrightarrow \sigma(x, y) := \sigma_{C_p}(x, y)$ *for* $C_p \ni x, y$ **well defined and smooth** (as locally smooth) on the whole $\mathcal{N}$ (same result for Δ and v_n).

QUESTION: **Do strongly convex coverings exist?**

2.2 A crucial property of paracompacntess

- A.H. Stone's result ('49)

THEOREM *A topological space X is* **Hausdorff** *and* **paracompact** *if and only if it is* **T1** *and every open covering $\mathcal{C}$ of X admits a *-refinement: another open covering $\mathcal{C}^*$ such that, if $V \in \mathcal{C}^*$,*

$$\bigcup \{V' \in \mathcal{C}^* \mid V' \cap V \neq \emptyset\} \subset U_V \quad \textit{for some } U_V \in \mathcal{C}.$$

- Useful consequence [VM21] since (M, g) is parcompact by definition.

PROPOSITION. *If $\mathcal{A}$ is an* **open covering** *of (M, g) (Riemannian or Lorentzian) there is a* **refinement** *$\mathcal{C}$ of $\mathcal{A}$ (i.e., if $C \in \mathcal{C}$, then $C \subset U_C \in \mathcal{A}$) that is a* **strongly convex** *covering of M.*

IDEA OF PROOF. $\mathcal{C}_0$ made of convex neighborhoods **subsets** of elements of $\mathcal{A}$. Pass to the $*$-refinement $\mathcal{C}_0^*$. $\mathcal{C}$ made of convex neighborhoods **subsets** of elements of $\mathcal{C}_0^*$. $C, C' \in \mathcal{C}$ convex and (if $C' \cap C \neq \emptyset$) $C' \cap C \subset C'' \in \mathcal{C}_0$ convex as well, $\Rightarrow$ $C \cap C'$ convex.

2.3 σ and H_n defined around the diagonal of $M \times M$

PROPOSITION [VM21] *Consider the neighborhood* $\mathcal{N}_{\mathcal{C}} = \bigcup_{p \in M} C_p \times C_p$ *of* $diag(M \times M)$ *where* $\mathcal{C} := \{C_p\}_{p \in M}$ *is s-convex.*

(a) *The extension*

$$\sigma_{\mathcal{C}}(x, y) := \sigma_{C_p}(x, y) \text{ for } C_p \ni x, y$$

is **well defined** *and* **smooth** *on* $\mathcal{N}_{\mathcal{C}}$. *The same is true for the associated* $H_n^{(\mathcal{C})}$.

(b) *If analogous definitions are given for another choice of the s-convex covering* $\mathcal{C}'$, *then there is a third s-convex covering* $\mathcal{C}''$ *such that* $\mathcal{N}_{\mathcal{C}} \cap \mathcal{N}_{\mathcal{C}'} \subset \mathcal{N}_{\mathcal{C}''}$ *and*

$$\sigma_{\mathcal{C}}|_{\mathcal{N}''_{\mathcal{C}}} = \sigma_{\mathcal{C}'}|_{\mathcal{N}''_{\mathcal{C}}}, \quad H_n^{T,\mathcal{C}}|_{\mathcal{N}''_{\mathcal{C}}} = H_n^{T,\mathcal{C}'}|_{\mathcal{N}''_{\mathcal{C}}}$$

($\mathcal{C}''$ made of the geodesically convex neighborhoods included in the intersections of the elements of $\mathcal{C}$ and $\mathcal{C}'$)

3. A slight mathematical change in the geometric definition of Hadamard state

3.1 Causal normal neighborhoods and parametrices

The definition of Hadamard state in [Kay-Wald91] can be fixed, **preserving all results established therein**, through a slight change in the definition of **causal normal neighborhood** of a Cauchy surface

DEFINITION A causal normal neighborhood $M_\Sigma^{\mathcal{C}}$ of a spacelike smooth Cauchy surface Σ of (M, g) subordinated to a strongly convex covering $\mathcal{C}$ of M is an open neighborhood of Σ such that

(a) $(M_\Sigma^{\mathcal{C}}, g|_{M_\Sigma})$ globally hyperbolic spacetime (with Σ itself as a Cauchy surface)

(b) $x, y \in M_\Sigma^{\mathcal{C}}$ **causally related** if and only if the **causal double-cone** $J_M(x, y) \subset C_{x,y} \in \mathcal{C}$.

N.B. Neighborhoods $M_\Sigma^{\mathcal{C}}$ **exist** for every Σ and $\mathcal{C}$: direct re-adaptation of the existence proof in [Kay-Wald91].

3.2.1 Geometric definition of Hadamard state

Let us review the **geometric definition of Hadamard** state according to the change above and following [Kay-Wald91] for a two-point function Λ_ω.

(1) Choose Σ, $\mathcal{C}$ and $M_\Sigma^{\mathcal{C}}$ as above.

(2) In $M_\Sigma^{\mathcal{C}}$, define $\sigma_{\mathcal{C}}$ and the associated parametrix $H_n^{T,\mathcal{C}}$ (under the choice of a global temporal function T).

(3) Choose a smooth function $\chi : M_\Sigma^{\mathcal{C}} \times M_\Sigma^{\mathcal{C}} \to [0,1]$ with $\chi = 1$ on the **causally connected pairs** (x,y) and which smoothly vanishes outside that set.

DEFINITION ω is **Hadamard** if

$$\Lambda_\omega(x,y) - \chi(x,y) H_n^{T,\mathcal{C}}(x,y)_{0+} \ \textit{is } C^n \ \forall n \in \mathbb{N}\,, x,y \in M_\Sigma^{\mathcal{C}}$$

The only change is the definition of M_Σ that now depends on $\mathcal{C}$.

REMARKS.

(1) If $H_n^{T,\mathcal{C}}$ and $H_n^{T,\mathcal{C}'}$ are two parametrices referred to the same Cauchy surface Σ, but **different** strongly convex coverings, in principle $\sigma_{\mathcal{C}} \neq \sigma_{\mathcal{C}'}$ by construction, however

$$\sigma_{\mathcal{C}}(x,y) = \sigma_{\mathcal{C}'}(x,y)\,, \quad v_n^{\mathcal{C}}(x,y) = v_n^{\mathcal{C}'}(x,y)$$

$$H_n^{T,\mathcal{C}}(x,y)_\epsilon = H_n^{T,\mathcal{C}'}(x,y)_\epsilon$$

when $(x,y) \in (M_\Sigma^{\mathcal{C}} \times M_\Sigma^{\mathcal{C}}) \cap (M_\Sigma^{\mathcal{C}'} \times M_\Sigma^{\mathcal{C}'})$ are **causally related** as physically espected.

That is because **causal geodesics** used to compute $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{C}'}$ are however the **same**, due to the definition of **causal normal neighborhood**.

(2) The definition of Hadamard state is again **independent** of the choices of Σ, $\mathcal{C}$, T, χ (same proofs as in [Kay-Wald91] with direct adaptations).

3.3.1 Relation with Radzikowski's micro-local definition

- Microlocal definition of Hadamard state: the 2-point function $\Lambda_\omega \in \mathcal{D}'(M \times M)$ satisfies the microlocal spectral condition
$WF(\Lambda_\omega) =$
$\{((x_1, k_1), (x_2, k_2)) \in T^*M \setminus 0 \times T^*M \setminus 0 \,|\, (x_1, k_1), (x_2, -k_2), k_1 \triangleright 0\}$

Radzikoski's crucial results [Radzikowski96a,b]:

- **THEOREM1** *Assume that, mod C^∞,*
(1) $\Lambda_\omega \in \mathcal{D}'(M \times M)$
(2) the antisymmetric part is $i/2$ *times the* **causal propagator**;
(3) Λ_ω *solves the* **KG equation**.
Then the **geometric** *definition of Hadamard state for* ω *and the* **microlocal** *one, are* **equivalent**.

- **THEOREM2** *Assuming (1) and (2), the microlocal condition is equivalent to the condition*

$$\Lambda_\omega(x, y) - H^T_\infty(x, y)_{0^+} \;\; \text{of class } C^\infty$$

in a **convex** *neighborhood of every point.*

3.3.2 Relation with Radzikowski's micro-local definition

QUESTION: Is Theorem 1 valid, taking the changes in the geometric definition of Hadamard state into account?

- Radzikowski's proof of Theorem1 relies upon three facts:

(a) standard structure of H_n;

(b) T- independence of the geometric definition;

(c) functions σ, Δ, v_n **well-defined** and **smooth** and, in a neighborhood of every point for **normal coordinates centered on** p, $\sigma(p, q) = -(x(q)^0)^2 + \sum_{k=1}^{3}(x^k(q))^2$

- These conditions are guaranteed, so Theorem 1 is true. Theorem 2 uses only the microlocal definition so that it is true as well.

4. Conclusions and open issues: global locally-covariant parametrix?

• As physically expected, the geometric definition of Hadamard state can be made completely consistent by means of a slight change in is te original definition, however based on a general topological result. (However other mathematical approaches possible.)

• The fundamental mathematical tool is the Hadamard parametrix that is well defined in a neighborhood of the diagonal of the spacetime and, in the very definition of Hadamard state, in a certain sub-spacetime constructed around any Cauchy surface.

• Dealing with these technical notions, also in view of **various and annoying technical subltelties** that one would like to fix once and for all, some natural questions pop up.

4. Conclusions and open issues: global locally-covariant parametrix?

Q1 Is it possible to extend parametrices to the **whole** spacetime?

Q2 Does an assigment of such extensions exists that **respects causal embeddings** of glob. hyp. spacetimes?

Regarding **states** (in place of parametrix) the answer to **Q2** is **negative**. It is well known that **no** canonical assigment of (Hadamard Gaussian) states is possible that **respects the causal embeddings** of globally hyperbolic spacetimes and **assigns the standard Minkowski vacuum to the Minkowski spacetime** when the **time-slice axiom** is assumed.
However parametrices are constructed with **local procedures** so that some hope of positive answers exist.

4. Conclusions and open issues: global locally-covariant parametrix?

I do not have a definite answer but just some clues. Disreagarding T and ϵ:

$$H_n(x,y) := \frac{1}{(2\pi)^2}\left(\frac{\Delta(x,y)^{1/2}}{\sigma(x,y)} + v_n(x,y)\ln\sigma(x,y)\right)$$

Two of the three objects, $\Delta(x,y)$, $\sigma(x,y)$, $v_n(x,y)$ are restrictions of **globally defined** objects when x and y are **causally related** (thus where physics matters).

4. Conclusions and open issues: global locally-covariant parametrix?

(1) In **strongly causal spacetimes** (thus globally hyperblic in particular) $\sigma(x,y)$ is the restriction to **convex neighborhoods** of the (continuous symmetric) Lorentzian distance

$$\tau(x,y) = -\sup\{L_g(\gamma)\,|\,\gamma\, \textit{is causal and joins x and y in any order}\}$$

(2) $v_n(x,y)$ is (up to constant universal factors) the "restriction" to **convex neighborhoods** of the kernel of the **causal propagator** $E(x,y)$. More precisely [Thm4.5.1:Friendlander75]

$$\frac{E(x,y) - v_n(x,y)}{\sigma(x,y)^n} \to 0 \quad \forall n \in \mathbb{N} \; \textit{and } x,y \textit{ timelike related}$$

and the **function** $E(x,y)$ computed for $\sigma(x,y) < 0$, continuously extends to a **smooth function** $E_0(x,y)$ up to $\sigma(x,y) = 0$.

4. Conclusions and open issues: global locally-covariant parametrix?

For **causally reated** x, y **in the whole spacetime** M, we can write

$$H_n^T(x,y)_\epsilon := \frac{1}{(2\pi)^2}\left(\frac{???(x,y)}{\tau^T(x,y)_\epsilon} + E_0(x,y)\ln \tau^T(x,y)_\epsilon\right)$$

Apparently, this attempt of global definition **respects causal embeddings of spacetimes** by construction (also T has to be handled).

However I do not know if a **global extesion** of Δ exists. A number of (also subtle) details need to be fixed, first of all how to deal with **spacelike related** arguments.

Many thanks for your attention!