Linear Stability of Semiclassical Theories of Gravity

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Introduction (1/2)

QFT vs. GR

How to describe quantum matter and gravity **interplay**? In first approximation,

- **QFT on curved spacetimes**: quantum matter field ϕ on a physical state ω propagating over classical globally hyperbolic spacetimes $(\mathcal{M}, g)$
- The formulation of interacting quantum field theories on curved spacetimes can be achieved in a local and covariant way in the algebraic approach to perturbative quantum field theory (**pAQFT**)
- **Semiclassical gravity**: backreaction on the background geometry

$$G_{ab} = 8\pi G \langle :T_{ab}: \rangle_\omega \, .$$

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- **Physical applications**:
  1. *Black Hole Physics*: Hawking effect, evaporation.
  2. *Cosmology*: Inflationary Universe.
- **Cosmological scenario**: recent developments have been made in cosmological spacetimes on the existence of **local and global solutions**[1]

[1] See N. Pinamonti and N. Rothe's talks

The issue of runaway solutions

- Semiclassical theories of gravity seem to admit unstable, exponentially growing solutions in time (**runaway solutions**)
  - *G. T. Horowitz and R. M. Wald. "Dynamics of Einstein's equation modified by a higher-order derivative term", PRD* **17***, 414–416 (1978).*
  - *G. T. Horowitz. "Semiclassical relativity: The weak-field limit", PRD* **21***, 1445–1461 (1980).*
  - *E. E. Flanagan and R. M. Wald. "Does back reaction enforce the averaged null energy condition in semiclassical gravity?", PRD* **36***, 6233–6283 (1996).*
- Runaway solutions might invalidate the research of **complete global solutions** of the Semiclassical Einstein Equations
- Perturbative expansion around background solution (**linearization**):

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab} + o(\varepsilon^2), \qquad h_{ab} = \partial_\varepsilon g^{(\varepsilon)}_{ab}(\varepsilon = 0)$$

  The stability of the background solution cannot be guaranteed if the linear perturbation becomes dominant at large times $t > 0$.
- Investigate the problem of stability using a **semiclassical toy model**.

The toy model

Semiclassical toy model

- **Quantum massive scalar field** ϕ + **classical scalar field** ψ in flat spacetime

$$\begin{cases} \Box\phi - m^2\phi = \lambda\psi\phi, & \lambda \in \mathbb{R} \\ g_2\Box\psi - g_1\psi = \lambda_1 \left\langle :\phi^2: \right\rangle_\omega - \lambda_2 \Box \left\langle :\phi^2: \right\rangle_\omega, & \lambda_1, \lambda_2, g_1, g_2 \in \mathbb{R} \end{cases}$$

- **Linearization**: $\psi = \psi_0 + \psi_1$ pAQFT
  1. Quantization of ϕ is performed "on the **background field**" ψ_0.
  2. Formulate an interacting theory for the **classical perturbation** ψ_1.
  3. To simplify the analysis, choose $\psi_0 \in \mathbb{R}$ and the *vacuum state* $|0\rangle$.

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- **Linearized equation** for ψ_1

$$(g_2\Box - g_1)\,\psi_1 = (\lambda_1 - \lambda_2\Box) \left\langle :\phi^2: \right\rangle_0^{(\mathrm{lin})} , \qquad \left\langle :\phi^2: \right\rangle_0^{(\mathrm{lin})} = \hbar\lambda\, \mathcal{K}_a(\psi_1),$$

where

$$\mathcal{K}_a : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$$

$$\mathcal{K}_a(x-y) = (\Box + a) \int_{4m^2}^\infty \mathrm{d}M^2 \varrho(M^2) \frac{1}{M^2 + a} \Delta_R(x-y, M^2),$$

$$\varrho(M^2) = \frac{1}{16\pi^2}\sqrt{1 - \frac{4m^2}{M^2}}, \qquad -4m^2 < a < 0, \qquad (\Box - M^2)\Delta_R(x, M^2) = \delta_x$$

Linear stability (1/2)

Study the following **fourth-order differential equation** in ψ_1

$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1) + (g_2\Box - g_1)\psi_1 = f, \qquad f \in \mathcal{D}(\mathcal{M}), \qquad \mathcal{K}_a \approx (\Box + a)\Delta_R.$$

1. Show that **past compact solutions** ψ_1 respect causality:
$$\mathrm{supp}(\psi_1) \subset J^+(\mathrm{supp} f).$$
2. Construct the **retarded fundamental solution** $D_R : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$, such that past compact solutions
$$\psi_1 = D_R(f)$$
decay at zero for large $t > 0$. ▸ sol
3. Prove that
$$(g_2\Box - g_1)\,\psi_1 = (\lambda_1 - \lambda_2\Box)\,\langle{:}\phi^2{:}\rangle_0^{(\mathrm{lin})}$$
$$\updownarrow$$
$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1) + (g_2\Box - g_1)\psi_1 = 0$$
has a well-posed **initial-value problem** with initial data $\psi_1^{(0,j)}(0,\mathbf{x})$, with $j \in \{0,1\}$ or $j \in \{0,1,2,3\}$, and for wide ranges of values of $(a, g_1, g_2, \lambda, \lambda_1, \lambda_2)$.

Linear stability (2/2)

Stability of the linearized backreacted system

- **Well-posed initial-value problem**: if one considers
  1. spatially compact perturbations;
  2. massive quantum fields;
  3. compactly-supported initial data/source;

  then there are several choices of renormalization constants such that linearized semiclassical solutions decay as $1/t^{3/2}$ for large $t > 0$ (**no runaway solutions**).

$$\psi_1(t, \mathbf{x}) = \sum_{j=0,1} \int_{\mathbb{R}^3} \left(C_+^j(\mathbf{p}) \mathrm{e}^{+i\omega_j t} + C_-^j(\mathbf{p}) \mathrm{e}^{-i\omega_j t} \right) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \omega_j = \sqrt{|\mathbf{p}|^2 + \mu_j^2}.$$

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$$\psi_1(t,\mathbf{x}) = \sum_{j=0,1} \int_{\mathbb{R}^3} \left(C^j_+(\mathbf{p}) \mathrm{e}^{+i\omega_j t} + C^j_-(\mathbf{p}) \mathrm{e}^{-i\omega_j t} \right) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \omega_j = \sqrt{|\mathbf{p}|^2 + \mu_j^2}.$$

Applications in cosmological models

- **Formal correspondence** with the *Semiclassical Einstein Equations* having a classical source $\tau_{\mu\nu}[g^{(0)}]$

$$-R = 8\pi G \, \langle :T_\rho{}^\rho: \rangle^{(\mathrm{lin})}_\omega \quad \leftrightarrow \quad (g_2 \Box - g_1)\, \psi_1 = (\lambda_1 - \lambda_2 \Box) \left\langle :\phi^2: \right\rangle^{(\mathrm{lin})}_0$$

$$-R = 8\pi G \, \langle :T_\rho{}^\rho: \rangle^{(\mathrm{lin})}_\omega + \tau_\rho{}^\rho \quad \leftrightarrow \quad (g_2 \Box - g_1)\, \psi_1 = (\lambda_1 - \lambda_2 \Box) \left\langle :\phi^2: \right\rangle^{(\mathrm{lin})}_0 + f$$

with

$$\lambda \leftrightarrow \xi, \qquad g_1 \leftrightarrow -(8\pi G)^{-1}, \qquad \lambda_1 \leftrightarrow m^2, \qquad \text{etc.}$$

Linear Stability of Minkowski spacetime

- **Backreaction** of a **massive** quantum scalar field ϕ, with $m^2 > 0$, $0 \leq \xi < 1/6$, over Minkowski spacetime $(\mathcal{M}, \eta)$.
- **Steps of the work:**

1. Show that $(\mathcal{M}, \eta)$ is solution of the **zeroth-order Semiclassical Einstein Equations** using the Minkowski vacuum state ω_0
$$G_{ab}^{(0)}[\eta] = 8\pi G \left\langle :T_{ab}[\phi, \eta]: \right\rangle_{\omega_0}^{(0)} \tag{0}$$
2. Study stability of the **linearized Semiclassical Einstein Equations**
$$G_{ab}^{(1)}[\eta, h] = 8\pi G \left\langle :T_{ab}[\phi, \eta, h]: \right\rangle_{\omega^{(1)}} \tag{1}$$
constructing $\omega^{(1)}$ using perturbation theory in *Hollands and Wald's* axiomatic framework.
3. Show that **classical gravitational waves** are solutions of Eq. (1), and runaway solutions are ruled out.

Current work

Linear Stability of Minkowski spacetime

- **Backreaction** of a **massive** quantum scalar field ϕ, with $m^2 > 0$, $0 \leq \xi < 1/6$, over Minkowski spacetime $(\mathcal{M}, \eta)$.
- **Steps of the work:**
  1. Show that $(\mathcal{M}, \eta)$ is solution of the **zeroth-order Semiclassical Einstein Equations** using the Minkowski vacuum state ω_0
  $$G^{(0)}_{ab}[\eta] = 8\pi G \left\langle :T_{ab}[\phi, \eta]: \right\rangle^{(0)}_{\omega_0} \tag{0}$$
  2. Study stability of the **linearized Semiclassical Einstein Equations**
  $$G^{(1)}_{ab}[\eta, h] = 8\pi G \left\langle :T_{ab}[\phi, \eta, h]: \right\rangle_{\omega^{(1)}} \tag{1}$$
  constructing $\omega^{(1)}$ using perturbation theory in *Hollands and Wald's* axiomatic framework.
  3. Show that **classical gravitational waves** are solutions of Eq. (1), and runaway solutions are ruled out.
- **Preliminar results** show stabilty for large times! The preprint[2] should appear soon... **Stay tuned!**

[1]P. Meda, S. Murro, and N. Pinamonti. "Linear stability of Minkowski Spacetime in Semiclassical Gravity" (2023). In preparation.

Thanks for the attention!

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Perturbative construction

- **Perturbation theory**

$$V=\int_{\mathcal{M}}\mathcal{L}_I(x)f(x)\mathrm{d}^4x=-\frac{\lambda}{2}\int_{\mathcal{M}}\phi^2(x)\psi_1(x)f(x)\mathrm{d}^4x,\qquad f\in\mathcal{D}(\mathcal{M}),$$

$$R_V(\phi^2)=S(V)^{-1}T(S(V)\phi^2),\qquad S(V)=T\left(\exp\left(\frac{i}{\hbar}V\right)\right).$$

- The **Bogoliubov map** R_V allows to obtain a perturbative expansion of the interacting ϕ^2 as formal power series in λ

$$\left\langle {:}\phi^2{:}\right\rangle_\omega=\omega(R_V(\phi^2))=\left\langle {:}\phi^2{:}\right\rangle_\omega^{(\mathrm{bac})}+\left\langle {:}\phi^2{:}\right\rangle_\omega^{(\mathrm{lin})}+\dots,$$

$$\left\langle {:}\phi^2{:}\right\rangle_\omega^{(\mathrm{bac})}=\omega(\phi^2)\overset{|0\rangle}{=}0,\qquad \left\langle {:}\phi^2{:}\right\rangle_\omega^{(\mathrm{lin})}=\frac{i}{\hbar}\left(\omega(T\left(V\phi^2\right))-\omega(V\phi^2)\right).$$

- The state for the interacting theory is constructed as $\omega\circ R_V$ by means of the **free state**, and it is fixed once and forever.
- **Linearized expectation value** of the Wick square in the adiabatic limit ($f=1$)

$$\left\langle {:}\phi^2{:}\right\rangle_\omega^{(\mathrm{lin})}(x)=-i\hbar\lambda\int_{\mathcal{M}}\left(\Delta^2_{F,\omega}(y-x)-\Delta^2_{+,\omega}(y-x)\right)\psi_1(y)\mathrm{d}y,$$

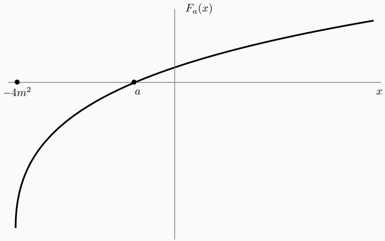
where $\Delta_{F,\omega}(y,x)=\hbar^{-1}\langle T\left(\phi(y)\phi(x)\right)\rangle_\omega$ and $\Delta_{+,\omega}(y,x)=\hbar^{-1}\langle\phi(y)\phi(x)\rangle_\omega$.

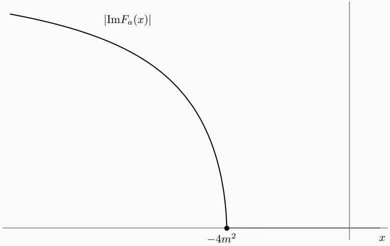
Fourier transform of the Wick square

$$\mathcal{F}\{\langle{:}\phi^2{:}\rangle_0^{(\mathrm{lin})}\}(p_0,\mathbf{p}) = \lim_{\epsilon\to 0^+} \frac{\lambda\hbar}{16\pi^2} F_a(-(p_0 - i\epsilon)^2 + |\mathbf{p}|^2)\hat{\psi}_1(p_0,\mathbf{p}),$$

$$F_a(z) = \int_{4m^2}^{\infty} \sqrt{1 - \frac{4m^2}{M^2}} \left(\frac{1}{M^2 + a} - \frac{1}{M^2 + z}\right) \mathrm{d}M^2 \qquad z = -(p_0 - i\epsilon)^2 + |\mathbf{p}|^2.$$

$F_a(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, -4m^2]$, and has a branch cut on $z \in (-\infty, -4m^2)$.





In the massless case [Hor80,FW96]

$$F_a(-p_0^2 + |\mathbf{p}|^2) = \log\left(\frac{-p_0^2 + |\mathbf{p}|^2}{a}\right), \qquad -p_0^2 + |\mathbf{p}|^2 > 0,\ a > 0.$$

Stable linearized solutions

- **Linearized equation in Fourier space**

$$S(-(p_0 - i0^+)^2 + |\mathbf{p}|^2)\hat{\psi}_1(p_0, \mathbf{p}) = \hat{f}(p_0, \mathbf{p}),$$

  where

$$S(z) = -(\lambda_1 + \lambda_2 z)\frac{\lambda\hbar}{16\pi^2}F_a(z) - (g_1 + g_2 z), \qquad z = -(p_0 - i\epsilon)^2 + |\mathbf{p}|^2.$$

- **Retarded fundamental solution**. Let $\mathcal{S}$ be the set of points $z \in \mathbb{C}$ in which $S(z) = 0$: if $\mathcal{S}$ contains only $s < 0$ for choices of $(a, \lambda, \lambda_1, \lambda_2, g_1, g_2)$, then

$$\hat{D}_R(p_0, \mathbf{p}) = \frac{1}{S(-(p_0 - i0^+)^2 + |\mathbf{p}|^2)},$$

  and hence for $s \in (-4m^2, \infty) \cup \{-\lambda_1/\lambda_2\}$ in $\mathcal{S}$

$$D_R(x) = -\sum_{s\in\mathcal{S}} \frac{1}{S'(s)}\Delta_R(x, s) - \frac{\lambda\hbar}{16\pi^2}\int_{4m^2}^{\infty}\sqrt{1 - \frac{4m^2}{M^2}}\frac{(\lambda_2 M^2 - \lambda_1)}{|S(-M)|^2}\Delta_R(x, M^2)\mathrm{d}M^2,$$

- **Past compact solution** $\psi_1 = D_R(f)$, $f \in \mathcal{D}(\mathcal{M})$, of the form

$$\psi_1(x) = \psi_1^O(x) + \psi_1^C(x).$$

  Branch cuts do not contribute to the homogeneous equation $S(z)\hat{\psi}_1 = 0$.