Relative entropy in de Sitter

Markus B. Fröb

Institut für Theoretische Physik, Universität Leipzig

Quantum Effects in Gravitational Fields Conference
30. August 2023, Universität Leipzig

Based on arXiv:2308.14797 and work in progress with Albert Much (U Leipzig) and Kyriakos Papadopoulos (U Kuwait)

Entanglement entropy in QM and QFT

Entanglement entropy in QM

- Entropy quantifies the amount of information an observer has access to
- For a system in a pure state $|\Psi\rangle$, everything is known $\Leftrightarrow$ entropy is zero
- For a generic state described by density matrix ρ, define von Neumann entropy $S_{\mathrm{vN}}(\rho) = -\operatorname{tr}(\rho \ln \rho) \geq 0$ (since $0 \leq \rho \leq 1$)
- For a system in a pure state $|\Psi\rangle$, but with the observer having access only to degrees of freedom within some region A, define density matrix $\rho_A = \operatorname{tr}_{A^\perp} |\Psi\rangle\langle\Psi|$ (trace over degrees of freedom of the complement region $A^\perp$)
- Entanglement entropy: $S(A) = S_{\mathrm{vN}}(\rho_A) = -\operatorname{tr}(\rho_A \ln \rho_A)$
- Other entropy measures: Tsallis entropy $S^{\mathrm{T}}_q(\rho) = \frac{1}{q-1}(1 - \operatorname{tr}\rho^q)$, Rényi entropy $S^{\mathrm{R}}_\alpha(\rho) = \frac{1}{1-\alpha} \ln \operatorname{tr}\rho^\alpha$, and $\lim_{q\to 1} S^{\mathrm{T}}_q(\rho) = S_{\mathrm{vN}}(\rho) = \lim_{\alpha\to 1} S^{\mathrm{R}}_\alpha(\rho)$
- Thermal density matrix $\rho = \frac{1}{Z} \exp(-\beta H)$ with inverse temperature β, $Z = \exp(-\beta F)$ with free energy F gives $S_{\mathrm{vN}} = \beta(\langle H\rangle - F) = S$ (thermodynamic entropy)

Entanglement entropy in QFT

- In QFT in $d+1$ dimensions, density matrices only exist formally and the trace is infinite, both due to the infinite number of degrees of freedom
- Compute regularised entanglement entropies with UV cutoff ϵ: $S(A) = g_{d-1}[\partial A]\epsilon^{-(d-1)} + \cdots + g_1[\partial A]\epsilon^{-1} + g_0[\partial A]\ln\epsilon + S_0(A) + \mathcal{O}(\epsilon)$, where the g_i are homogeneous functions depending on the boundary ∂A (area law in 4D)
- Source of divergences: high-energy vacuum fluctuations of the fields
- $\Rightarrow$ differences in entropies between different states are finite
- Relative entropy (Kullback–Leibler divergence): $S(\rho\|\sigma) = \operatorname{tr}(\rho\ln\rho - \rho\ln\sigma)$
- Relative Rényi entropy: $S^{\mathrm{R}}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\ln\operatorname{tr}\left(\rho^\alpha\sigma^{1-\alpha}\right)$
- α-z-Rényi entropy (generalized quantum Rényi div.): $S^{\mathrm{RG}}_{\alpha,z}(\rho\|\sigma) = \frac{z}{\alpha-1}\ln\operatorname{tr}\left(\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{z}}\right)$ and $\lim_{\alpha\to1} S^{\mathrm{R}}_\alpha(\rho\|\sigma) = S(\rho\|\sigma) = \lim_{\alpha,z\to1} S^{\mathrm{RG}}_{\alpha,z}(\rho\|\sigma)$

Tomita–Takesaki theory

Tomita–Takesaki theory

- Mathematically, difference between QM and QFT is the type of (factors of) von Neumann algebra of operators (I, II, III)
- Tomita–Takesaki theory gives information on structure of vN algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ acting on Hilbert space $\mathcal{H}$, given a cyclic and separating vector $\Omega \in \mathcal{H}$
- Tomita operator S is the closure of the map $S_0 \colon a\Omega \to a^\dagger\Omega$ for $a \in \mathfrak{A}$
- Polar decomposition $S = J\Delta^{\frac{1}{2}}$ gives positive modular operator $\Delta = S^\dagger S \geq 0$ and antilinear modular conjugation J
- Modular flow $\sigma_s(a) = \Delta^{\mathrm{i}s} a \, \Delta^{-\mathrm{i}s} \in \mathfrak{A}$ for $a \in \mathfrak{A}$
- State ω defined by Ω is a thermal (KMS) state: $\omega(\sigma_s(a)b) = (\Omega, \sigma_s(a)b\Omega)$ satisfies $\omega(\sigma_{s-\mathrm{i}}(a)b) = \omega(b\,\sigma_s(a))$, with inverse temperature normalised to $\beta = 1$
- Both J and $\Delta^{\frac{1}{2}}$ map $\mathfrak{A}$ to commutant $\mathfrak{A}'$

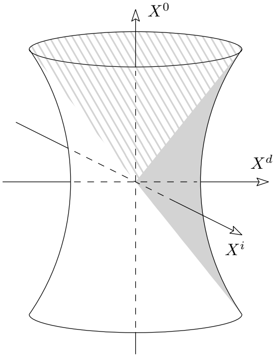
Tomita–Takesaki theory

- Relative Tomita operator $S_{\Phi|\Psi}$ is closure of map $a\Phi \mapsto a^\dagger\Psi$ for $a \in \mathfrak{A}$ and cyclic and separating vectors Φ, $\Psi \in \mathcal{H}$, relative modular operator $\Delta_{\Phi|\Psi}$ and relative modular conjugation $J_{\Phi|\Psi}$ defined by polar decomposition $S_{\Phi|\Psi} = J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2}$
- Araki formula relates relative modular Hamiltonian $\ln \Delta_{\Phi|\Psi}$ to relative entropy: $S(\Phi\|\Psi) = -\left(\Phi, \ln \Delta_{\Phi|\Psi}\Phi\right)$ (well-defined and finite)
- Important case: $\Phi = uu'\Omega$ and $\Psi = vv'\Omega$ for unitary operators $u, v \in \mathfrak{A}$ and $u', v' \in \mathfrak{A}'$ commuting with u and v
- $\Rightarrow \Delta_{\Phi|\Psi} = u'v\Delta_\Omega v^\dagger(u')^\dagger$ and $S(\Phi\|\Psi) = -\left(v^\dagger u\Omega, \ln \Delta_\Omega v^\dagger u\Omega\right)$
- Relative entropy between two "excited" states relative to a "vacuum" state Ω can be computed using only the modular Hamiltonian $\ln \Delta_\Omega$ of the "vacuum" state, e.g., for coherent state with $u = u' = v' = \mathbb{1}$ and $v = \exp[\mathrm{i}\phi(f)]$

De Sitter spacetime and relative entropy

De Sitter spacetime

- d-dimensional de Sitter is embedded in $(d+1)$-dimensional Minkowski space as hyperboloid $\eta_{AB}X^A X^B = H^{-2}$
- Expanding half of dS (Poincaré patch) with metric $\mathrm{d}s^2 = \eta_{AB}\,\mathrm{d}X^A\,\mathrm{d}X^B = -\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\,\mathrm{d}\mathbf{x}^2$ describes primordial inflation and current accelerated expansion of our universe
- Maximally symmetric solution of Einstein's equations with cosmological constant $\Lambda = (d-1)H^2$
- Generator of boosts is tangent to hyperboloid: $M_{0j} = X_0\partial_{X^j} - X_j\partial_{X^0} = -\frac{1}{2H}\left(H^2\mathbf{x}^2 - \mathrm{e}^{-2Ht} + 1\right)\partial_j - \mathbf{x}_j\partial_t + H\mathbf{x}_j\mathbf{x}^i\partial_i$
- Modular Hamiltonian is known for dS vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to intersection of hyperboloid and wedge $W_1 = \{X^A\colon X^1 \geq |X^0|\}$: $\ln\Delta_\Omega = \mathrm{i}M_{01}$ (Borchers/Buchholz, Global properties of vacuum states in de Sitter space 1999)



Relative entropy in de Sitter spacetime

- Compute relative entropy $S(\Omega\|\Psi_f)$ for coherent state $\Psi_f = \mathrm{e}^{\mathrm{i}\phi(f)}\Omega$ via Araki formula ($=$ relative entanglement entropy, with entanglement region A the intersection of hyperboloid and wedge $A = \{(t,\mathbf{x})\colon 2H\mathbf{x}^1 \geq \left|1 - \mathrm{e}^{-2Ht} + H^2\mathbf{x}^2\right|\}$, and $\operatorname{supp} f \subset A$)
- $S(\Omega\|\Psi_f) = \mathrm{i}\pi\langle\Delta(M_{01}f), \Delta f\rangle$ with commutator function $\Delta(x,y) = \mathrm{i}[\phi(x),\phi(y)]$ and symplectic product $\langle f,g\rangle = \mathrm{i}\int \left[f^*(t,\mathbf{x})\dot{g}(t,\mathbf{x}) - g(t,\mathbf{x})\dot{f}^*(t,\mathbf{x})\right]_{t=0} \mathrm{d}^{d-1}\mathbf{x}$
- Further manipulations:
  $S(\Omega\|\Psi_f) = 2\pi\int\left[\mathbf{x}^1\mathcal{H}(\hat{f},x) + \frac{1}{2}H\mathbf{x}^2\partial_1\hat{f}\partial_t\hat{f} - H\mathbf{x}^1\mathbf{x}^i\partial_i\hat{f}\partial_t\hat{f}\right]_{t=0}\mathrm{d}^{d-1}\mathbf{x}$ with $\hat{f}(x) = \int\Delta(x,y)f(y)\sqrt{-g}\,\mathrm{d}^d y$ and $\mathcal{H}(g,x) = \frac{1}{2}\left(\dot{g}^2 + \mathrm{e}^{-2Ht}\partial_i g\partial^i g + m^2 g^2\right)$ the Hamiltonian density $\Rightarrow$ not manifestly positive!
- However, correct flat-space limit $H \to 0$

Relative entropy in de Sitter spacetime

- Solution: evaluate symplectic product on different Cauchy surface $\Sigma = \{(t, \mathbf{x})\colon 2Ht + \ln(1 + H^2\mathbf{x}^2) = 0\}$ instead of just $t = 0$
- $S(\Omega \| \Psi_f) = 2\pi \int \mathcal{Q}(\hat{f})\left(1 + H^2\mathbf{x}^2\right)^{-\frac{d}{2}} \mathrm{d}^{d-1}\mathbf{x}$ with $\mathcal{Q}(h) = \frac{\mathbf{x}^1}{2\sqrt{1+H^2\mathbf{x}^2}}\left[\partial_n h \partial_n h + \left[(1 + H^2\mathbf{x}^2)\delta^{kl} - H^2\mathbf{x}^k\mathbf{x}^l\right]\hat{\partial}_k h \hat{\partial}_l h + m^2 h^2\right]_\Sigma \geq 0$
- $\mathcal{Q}(h) = -n_\mu \xi^{(1)}_\nu T^{\mu\nu}(h)$ with n_μ normal to Σ, $\xi^{(1)}_\nu$ Killing vector associated to boosts: $M_{01} = \xi^\mu_{(1)}\partial_\mu$, and $T_{\mu\nu}$ canonical stress tensor
- $\mathcal{Q}$ is a Noether charge, compare Wald (Black hole entropy is the Noether charge 1993) and Floerchinger (Lectures on quantum fields and information theory)
- Relative entropy is also convex: $\lambda S(\Omega\|\Psi_f) + (1-\lambda)S(\Omega\|\Psi_g) \geq S(\Omega\|\Psi_{\lambda g + (1-\lambda)h})$ for $\operatorname{supp} f, g \subset A$ and $\lambda \in [0, 1]$

Outlook

Outlook

- Modular Hamiltonian is also known in some other cases:
- De Sitter vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to static patch: $\mathrm{i} \ln \Delta_\Omega = \frac{2\pi}{H} \partial_T$ (static patch appears thermal with inverse temperature $\beta = 2\pi/H$)
- Recent result for conformal theories in small diamonds of size ℓ in the static patch: $\mathrm{i} \ln \Delta_\Omega = \frac{2\pi}{H} \partial_T - \frac{\pi}{H^2 \ell} \frac{\mathrm{e}^{-HT}}{\sqrt{1-H^2R^2}} \left[\partial_T - H(1 - H^2 R^2) \left(R \partial_R + \frac{d-2}{2} \right) \right]$
- $\Rightarrow$ compute relative entropy for small diamonds
- Other measures: mutual information $I(A, B) = S(A) + S(B) - S(A \cup B)$ for $A \cap B = \emptyset$

Thank you for your attention

Questions?

This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — project no. 396692871 within the Emmy Noether grant CA1850/1-1 and project no. 406116891 within the Research Training Group RTG 2522/1.

References: arXiv:2308.14797, arXiv:2309.XXXXX