Relative entropy and dynamical black holes

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Outline

- Black hole mechanics and thermodynamics
- Microscopic origin of black hole entropy
- Dynamical black holes
- Algebraic quantisation
- Relative entropy
- Results

A cup of tea with black holes

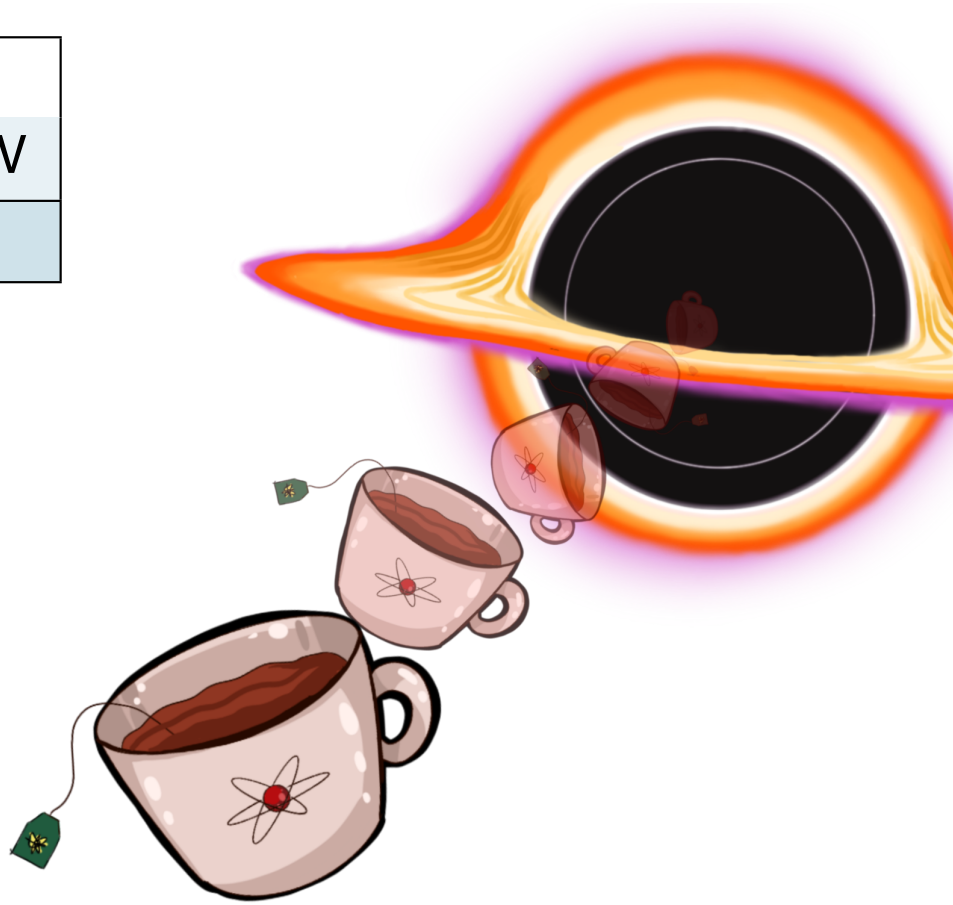
Bardeen, Carter, Hawking; Bekenstein (1973)

Black hole mechanics	Thermodynamics
$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J$	$dE = T\,dS + p\,dV$
$\delta A \geq 0$	$\delta S \geq 0$

But: No-hair theorem

Hawking (1975)

$$T_H = \frac{\kappa}{2\pi} \Rightarrow S_{BH} = \frac{A}{4\pi}$$

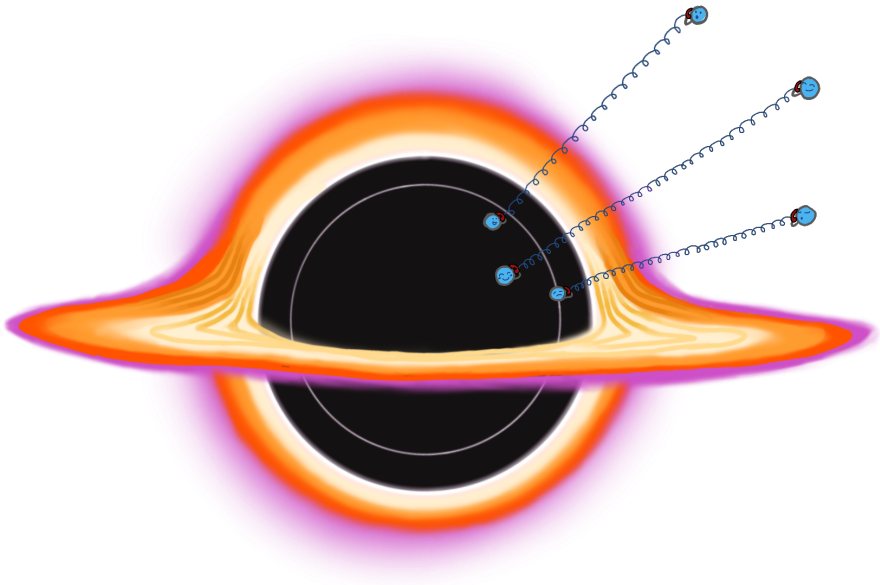


Microscopic origin of BH entropy

Black hole hair:

- QG approaches: LQG, String Theory, path integrals...
- Symmetry-based approaches: Wald entropy, CFT,...
- Matter based approaches: Entanglement entropy

Entanglement entropy



Bombelli et al. PRD (1986)

- Pure quantum state $\rho = |\psi\rangle\langle\psi|$
- Trace outside the BH horizon: $\rho_{BH} = \mathrm{Tr}_{BH'}\, \rho$
- Entanglement entropy: $\mathcal{S}_{BH} = -\mathrm{Tr}\, \rho_{BH} \log \rho_{BH}$

$$\mathcal{S}_{BH} \simeq \frac{A}{\epsilon^{d-2}}$$

$\Rightarrow$ Algebras in QFT are Type III (Yngvason Rept. Math. Phys. 2004, Witten Rev. Mod. Phys. 2018)

Relative entropy

Hollands & Ishibashi *Class. Quantum Grav.* (2019)

Relative entropy

Differences:

- Well-defined for continuum theories
- Measure the distinguishability between two states

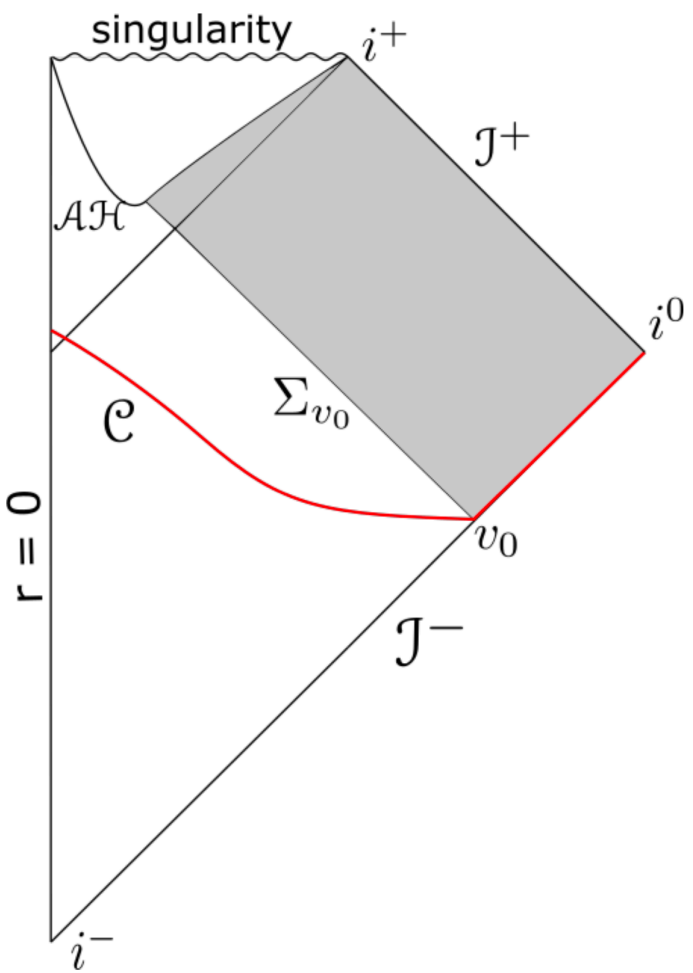
The model: **Relative entropy** between coherent states on **dynamical** black holes

Dynamical black holes

- Generalisation to spherically symmetric, asymptotically flat, dynamical black holes
- Entropy between a Unruh-like state and a coherent perturbation

Tools

- **Geometric** toolkit: Kodama conservation law
- **Quantum** toolkit: Algebraic quantisation, Tomita-Takesaki theory



Geometry

- Globally hyperbolic spacetime $(\mathcal{M}, g)$ with metric:

$$\mathrm{d}\, s^2 = -e^{2\gamma(v,r)} f(v,r)\, \mathrm{d}\, v^2 + 2e^{\gamma(v,r)}\, \mathrm{d}\, v\, \mathrm{d}\, r + r^2\, \mathrm{d}\, \Omega^2$$

- Misner-Sharp mass: $m = \frac{r}{2}\left(1 - \nabla^a r \nabla_a r\right)\,, \quad f = 1 - \frac{2m}{r}$
- Apparent horizon: $2m(r_{\mathrm{AH}}, v) = r_{\mathrm{AH}}$ (Hayward *Class. and Quantum Grav. (1998)*)

Kodama miracle

- Kodama vector: $k := \epsilon^{ij} \nabla_j r$, *not a Killing field!*
- Kodama conservation law: $\mathrm{F} := \oint \mathrm{J}_a\, \mathrm{d}\, \Sigma^a = \oint \mathrm{T}_{ab} k^b\, \mathrm{d}\, \Sigma^a = 0$

Perturbations

Perturbative (semi-classical) corrections:

$$T_{tot} = T + \lambda^2 T^\phi \,, \quad G_{tot} = G + \lambda^2 G^\phi$$

$$m \to M = m + \delta^2 m \,, \quad \gamma \to \Gamma = \gamma + \delta^2 \gamma$$

T^ϕ = SET of a coherent wave from past infinity

Kodama conservation law

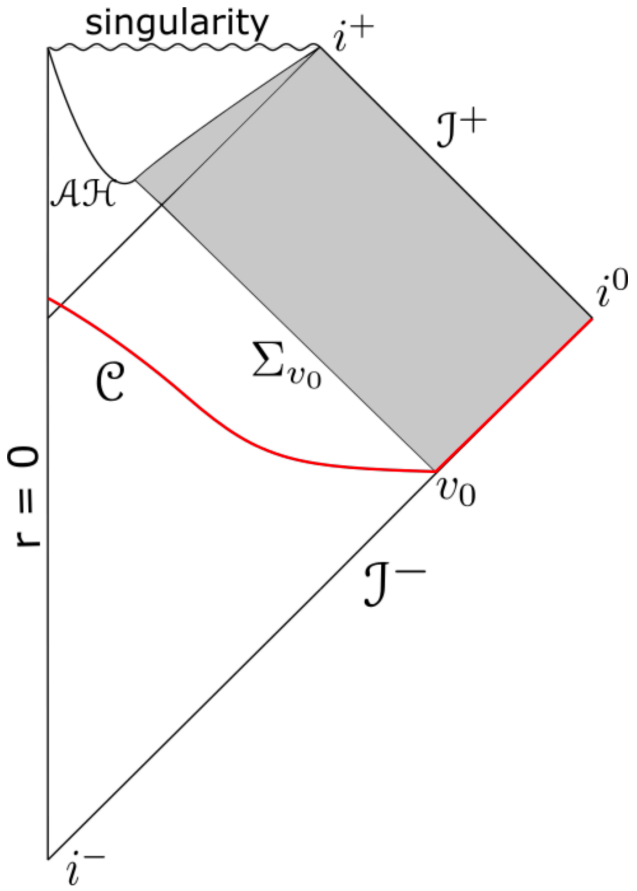
Kodama flux

$$\mathcal{F}_\Sigma = \int_\Sigma T^\phi_{ab} k^a \, d\Sigma^b$$

Kodama conservation law

$$\mathcal{F}_{AH} + \mathcal{F}_{I^+} = \mathcal{F}_{I^-}$$

$$\mathcal{F}_{\Sigma v_0} = 0 \text{ for causality}$$



Area law

On the **perturbed** apparent horizon

$$\mathcal{F}_{AH} = \delta^2 m_{AH}(v_0) - \delta^2 m_{i^+}$$

On the other hand, $\delta^2 \frac{dA}{dv}\Big|_{v_0} = 8\pi\delta^2 m_{AH}(v_0) \Rightarrow$

$$\frac{1}{4}\delta^2 A'(v_0) = 2\pi\left(\mathcal{F}_{AH} + \mathcal{F}_{i^+}\right)$$

It remains to show that $\mathcal{F}_{\mathcal{J}^-} = S'_{\omega,\phi}$

Algebraic quantisation

Classical theory

- Smooth solutions of $\Box\phi = 0$
- Advanced and retarded propagators:

$$\Box E^{\pm}(x,y) = \delta(x,y)\,, \quad \mathrm{supp}E^{\pm} = \{(y,x) \in M \times M | x \in J^{\pm}(y)\}$$

- Causal propagator: $E = E^{-} - E^{+}$
- Symplectic space of solutions (Sol, σ) with

$$\sigma(\phi_f, \phi_g) = \int_C \left(\phi_g \nabla_a \phi_f - \phi_f \nabla_a \phi_g\right) \mathrm{d}\,\Sigma^a$$

$$\phi_f := Ef\,, \quad \sigma(\phi_f, \phi_g) = E(f,g)$$

Weyl algebra

Weyl algebra $\mathcal{A}$:

1. **Generators:**
   - $\mathbb{1}$,
   - $W(f)$ " = "$e^{i\sigma(\phi,f)}$

2. **Weyl relations:**
   - $W(0) = \mathbb{1}$
   - $W(f)^* = W(-f) = W(f)^{-1}$
   - $W(f)W(g) = e^{-\frac{i}{2}\sigma(\phi_f,\phi_g)}W(f+g)$

GNS representation

Quasi-free state $\omega : \mathcal{A} \to \mathbb{C}$, $\omega(W(f)) = e^{-\frac{1}{2}\omega_2(f,f)} \Rightarrow \omega(\phi_f) = 0$
Class of quasifree Hadamard states ω

$$\omega_2(f,g) = \frac{1}{\pi}\int_{\mathcal{J}^-(v_0)} \frac{f(v_1)g(v_2)}{(v_2 - v_1 - i\epsilon)^2}\, \mathrm{d}\, v_1\, \mathrm{d}\, v_2\, \mathrm{d}\, \Omega + \omega_{2,\mathcal{C}\setminus\mathcal{J}^-(v_0)}$$

Unruh-like: vacuum with respect to modes coming from past infinity

GNS representation: $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ s.t. $\omega(A) = \langle\Omega_\omega|\, \pi_\omega(A)\, |\Omega_\omega\rangle$

Tomita-Takesaki theory

- von Neumann algebra $A = \pi_\omega(\mathcal{A})''$ and Hilbert space $\mathcal{H}$
- Two cyclic and separating vectors $|\Omega\rangle$, $|\Phi\rangle$
- Relative Tomita operator

$$T_{\Omega,\Phi} A\,|\Phi\rangle = A^*\,|\Omega\rangle\ , \quad T_{\Omega,\Phi} = J_{\Omega,\Phi}\Delta_{\Omega,\Phi}^{1/2}$$

Araki formula for the relative entropy:

$$S_{\Omega,\Phi} = \langle\Omega|\log\Delta_{\Omega,\Phi}\,|\Omega\rangle$$

Relative entropy for coherent states

Entropy between vacuum $|\Omega\rangle$ rep. of ω and a coherent excitation $|\Phi\rangle = W(f)\,|\Omega\rangle$:

$$S_{\omega,\phi} = \frac{1}{2}\sigma\left(\frac{d}{dt}\alpha_t(\phi_f)_{t=0}\,,\,\phi_f\right)$$

Modular flow: $\alpha_t(A) = \Delta_\Omega^{it} A \Delta_\Omega^{-it}$

Longo (2019), Ciolli Longo Ruzzi CMP (2020), Casini Grillo Pontello PRD (2019)

Generalisation of Bisognano-Wichmann theorem to spaces with an invariant "wedge": modular flow = geometric action

Brunetti Guido Longo Rev. Math. Phys. (2002)

Relative entropy on dynamical black holes

The space of initial data $f \in C_0^\infty(\mathcal{J}^-(v_0) \times S^2)$ forms a boundary algebra (Dappiaggi Moretti Pinamonti Rev. Math. Phys. 2006) where

$$\alpha_t f(v) = f(v_0 + e^{-2\pi t}(v - v_0)) \Rightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\alpha_t(\phi_f)_{t=0} = -\pi(v - v_0)k^a\partial_a(r\phi)|_{\mathcal{J}^-(v_0)}$$

$$S_{\omega,\phi} = 2\pi\int_{I^-(v_0)}(v - v_0)T_{ab}n^a_{I^-}k^b\,\mathrm{d}\,v\,\mathrm{d}\,\Omega \Rightarrow -2\pi\mathcal{F}_{\mathcal{J}^-} = S'_{\omega,\phi}$$

Result

$$\left(S_{\omega,\phi} + \frac{1}{4}A\right)'(v_0) = 2\pi\left(\mathcal{F}_{\mathcal{J}^+} + \mathcal{F}_{i^+}\right)$$

Conclusions and outlook

Conclusions

Direct semi-classical computation of the black hole entropy

Further investigations

- **Geometry:** axisymmetric black holes, cosmological constant, modified theories of gravity
- **Quantum:** different states, different matter content, higher-order perturbations
- Study in perturbative gravity
- Connection with asymptotic symmetries

Thank you for your attention!

Apparent horizons

Hayward PRD (1996)
Future, outer, trapping horizon:

$$\nabla_a r \nabla^a r = 0 \,, \quad \Box r > 0$$

- Quasi-local, causal concept
- Coincide with Killing horizons
- $r_{\mathrm{H}} = 2m(v, r)$
- Can be space-like or time-like

Kodama miracle

Kodama vector

$$k := \epsilon^{ij} \nabla_j r$$

Kodama conservation law

$$J_a = T_{ab} k^b \quad \text{is conserved}$$

Misner-Sharp mass

$$m(q) - m(p) = \int_p^q d\tau \int_{S^2} j^\phi \, d\Omega_2$$

News versus information

Hollands Ishibashi (2019) In Schwarzschild, good decay properties of gravitational perturbations ⇒

$$\left(S_{\omega,\phi} + \frac{1}{4}A\right)' = -\frac{1}{8}\int_{\mathcal{J}^+} N_{ab}N^{ab}\, d\,u\, d\,\Omega$$