# Adibatic Ground States in Non-smooth Spacetimes

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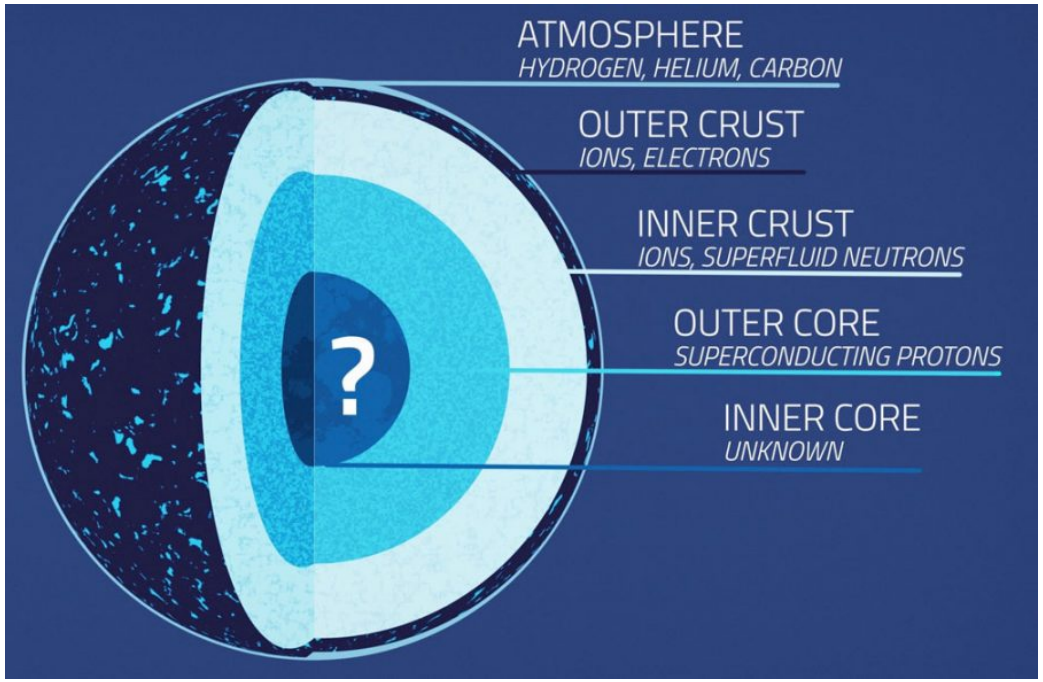
Quantum Effects in Gravitational Fields, Leipzig, 23

# Why **non-smooth spacetimes?**

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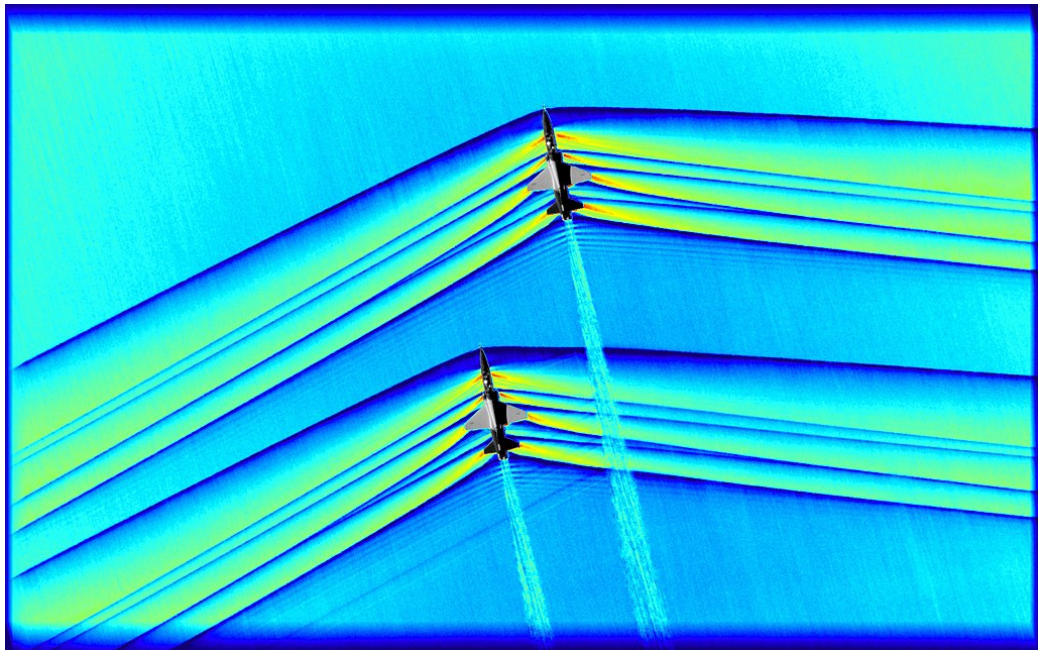

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[1] NASA's Goddard Space Flight Center / Conceptual Image Lab
[2] NASA

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THÉORÈME D'EXISTENCE POUR CERTAINS SYSTÈMES D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES NON LINÉAIRES.

Par

Y. FOURÈS-BRUHAT.

*Self-Gravitating Relativistic Fluids: A Two-Phase Model*

DEMETRIOS CHRISTODOULOU

[3] George M. Bergman, Oberwolfach Photo Collection

[4] George M. Bergman, Oberwolfach Photo Collection

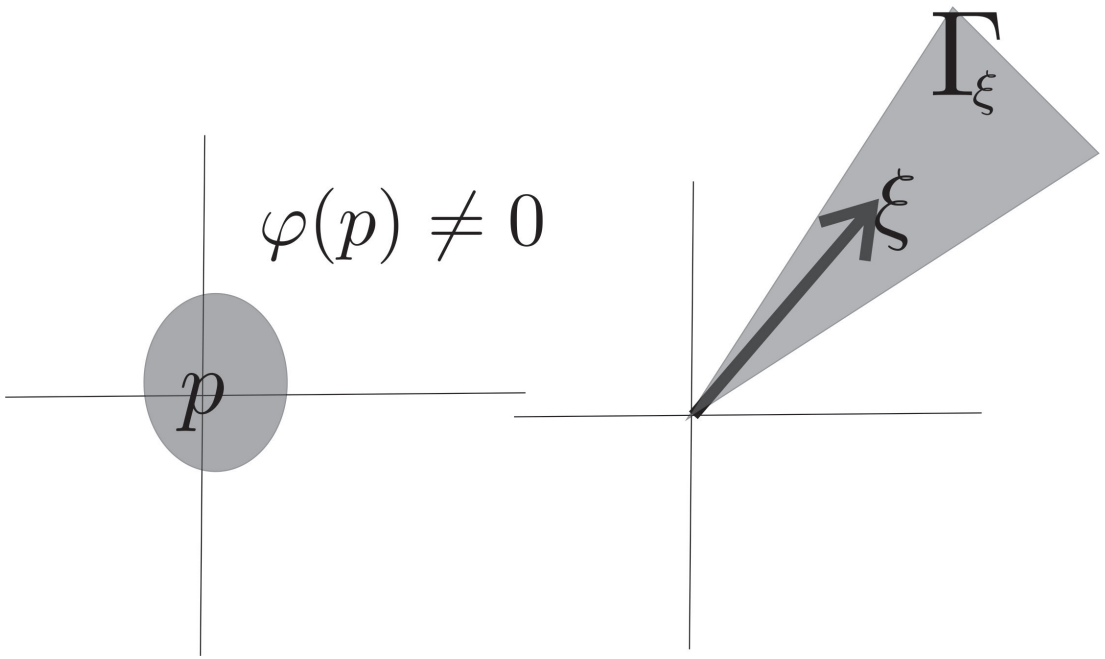
## QFT in Non-smooth Spacetimes - general regularity assumptions

- (Dereziński, Siemssen; 18,19)
  - Constructions of inverses and bisolutions.
  - $L^\infty_{loc} \cap H^1_{loc}$ regularity + local in time assumptions.
- (Vickers,-; 18)
  - Construction of causal propagator based on well-posedness.
  - $C^{1,1}$ regularity.
- (Hörmann, Spreitzer, Vickers,-; 19)
  - Construction of quantization functors.
  - Haag-Kastler Axioms.
  - $C^{1,1}$ regularity.
- (Schrohe,-; 22)
  - Adiabatic ground states.
  - Microlocal structure of causal propagator.
- Future Work:
  - Adiabatic states in non-smooth globally hyperbolic spacetimes.
  - Improvement of adiabatic order.
  - Physical signatures.

# Sobolev Wavefront Set and Non-Smooth $\Psi$DOs.

## $H^s$ – wavefront set :

$$(p,\xi) \notin WF^s(u) \iff \int_{\Gamma_\xi} (1+|\chi|^2)^s |\widehat{\varphi u}(\chi)|^2 d^n\chi < \infty$$




$$(p,\xi) \notin WF^s(u) \iff u \in H^s_{mcl}(p,\xi)$$

Properties:

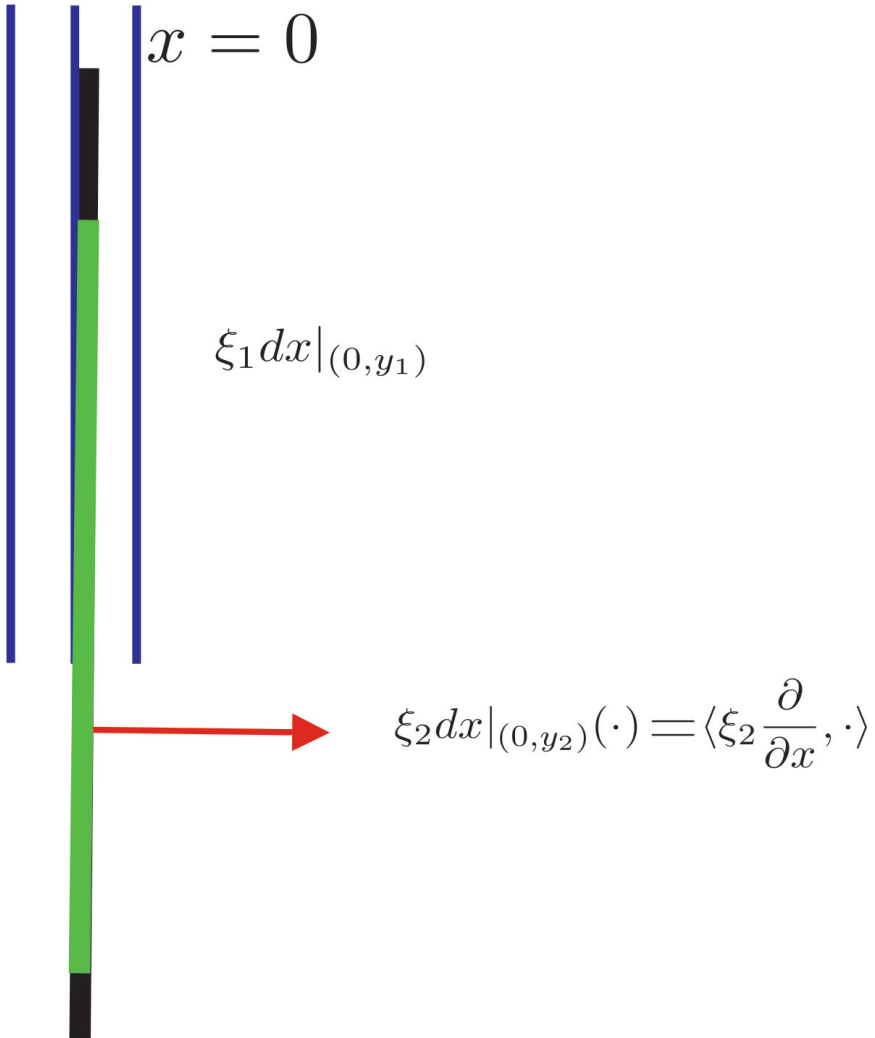
-$WF^s(u) \subset T^*(M)\backslash\{0\}$

- $WF^s(u) = \emptyset \iff u \in H^s_{loc}$

- for $s_1 < s_2 \quad WF^{s_1}(u) \subset WF^{s_2}(u) \subset WF(u)$

$\varphi\delta_{x=0}(\phi) = \int_{\mathbb{R}} \varphi(0,y)\phi(0,y)dy, \varphi \in C_c^\infty$
$WF^s(\varphi\delta_{x=0}) =$

$$\begin{cases} \emptyset & s < -\frac{1}{2} \\ \{(0,y) \in \mathbb{R}^2 \cap \mathrm{supp}(\varphi)\} \times \{(\xi,0) \in \mathbb{R}^2 \backslash \{0\}\} & s \geq -\frac{1}{2} \end{cases}$$



## Non-smooth symbols

A **symbol** $p(x,\xi) \in C^{\tau} S^{m}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if,

$$||D^{\alpha}_{\xi} p(\cdot,\xi)||_{C^{\tau}} \leq C_{\alpha} \langle \xi \rangle^{m-|\alpha|+\tau\delta} \text{ for } |\alpha| \geq 0.$$

$$|f|_{C^{0,\beta}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x-y\|^{\beta}}, \beta \in (0,1)$$

$$\|f\|_{C^{\tau=k+\beta}} = \|f\|_{C^k} + \max_{|\rho|=k} |D^{\rho} f|_{C^{0,\beta}}$$

**Non-smooth** $\Psi DO$

The $\Psi$**DO** $\mathbf{p(x, D)}$ associated to $p(x,\xi) \in C^{\tau} S^{m}_{1,\delta}$ is

$$p(x,D)u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi$$

Mapping properties:

$$p(x,D) : H^{s+m} \to H^{s}$$

for $-\tau(1-\delta) < s < \tau$.

**Symbol decomposition:** $p(x,\xi) \in C^{\tau} S^{m}_{1,0}$

$$p(x,\xi) = \underbrace{p^{\#}(x,\xi)}_{\text{smooth}} \quad + \quad \underbrace{p^{b}(x,\xi)}_{\text{non-smooth, but lower order}}$$

$$p^{\#}(x,\xi) \in S^{m}_{1,\delta}$$

$$p^{b}(x,\xi) \in C^{\tau} S^{m-\tau\delta}_{1,\delta}; \delta \in (0,1)$$

Example:The Klein-Gordon Operator

$p(t,x,\xi_0,\xi) = (-{\xi_0}^2 + h^{ij}(x)\xi_i\xi_j) + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j + m^2$

$h^{ij} \in C^\tau, p(t,x,\xi_0,\xi) \in C^{\tau-1}S^2_{1,0}$

$p(t,x,D) = \partial_{tt} - \Delta_h + m^2 := P_{KG}$

Example: The Klein-Gordon Operator

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$p(t,x,D) = \partial_{tt} - \Delta_h + m^2 := P_{KG}$

$\mathrm{Char}(P_{KG}) := \{(t,x,\xi_0,\xi) \in T^*M\backslash\{0\} : \mathcal{H}(t,x,\xi_0,\xi) = 0\}$

$X_{\mathcal{H}} = (\partial_{\xi_0}\mathcal{H}, \partial_\xi\mathcal{H}, -\partial_t\mathcal{H}, -\partial_x\mathcal{H})$

A single flow line in $\mathrm{Char}(P_{KG})$ is called a **null bicharacteristic strip**.

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A single flow line in $\mathrm{Char}(P_{KG})$ is called a **null bicharacteristic strip**.

The **bicharacteristic relation** $C$ is defined as:

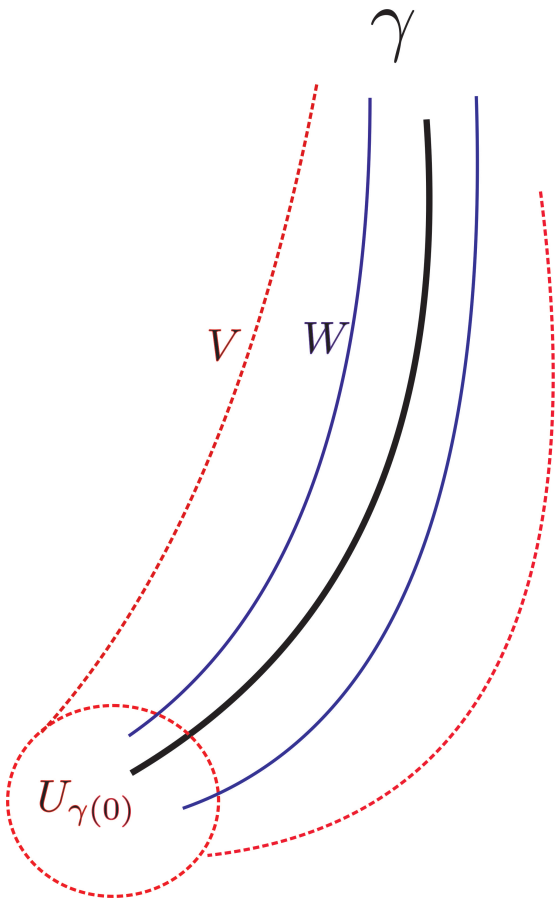
$$C = \{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG}), \quad (1)$$
$$(\tilde{x},\tilde{\xi}) \text{ and } (\tilde{y},\tilde{\eta}) \text{ lie on the same null bicharacteristic strip}\}$$

where $\tilde{x} = (t,x), \tilde{\xi} = (\xi_0,\xi), \tilde{y} = (s,y), \tilde{\eta} = (\eta_0,\eta)$.

**Propagation of singularities.** (Taylor)
Let $P_{KG}(x,\xi) \in C^1 S^2_{1,0}$, $\gamma$ a null bicharacteristic strip.
If $v \in D'(X)$ solves

$$P_{KG} v = f \iff P^{\#}_{KG} v = g := f - P^{b}_{KG} v,$$

$g \in H^{\sigma}_{mcl}(V)$ and $v \in H^{1+\sigma}_{mcl}(U_{\gamma(0)})$ then $v \in H^{1+\sigma}_{mcl}(W)$.



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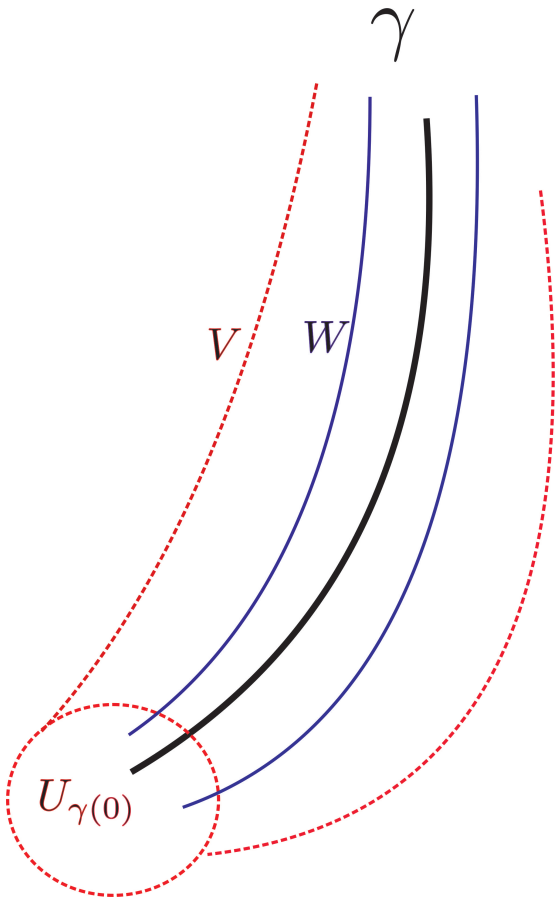

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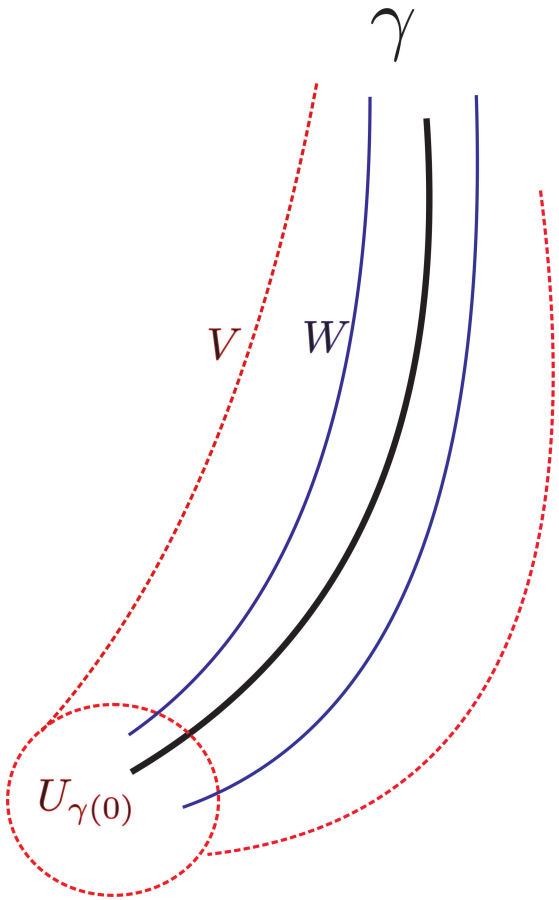

# Adiabatic Ground States

A quasifree state $\omega_N$ is an **adiabatic state** of order $N$ if its two-point function $\Lambda_{2N}$ satisfies for all $s \leq N + \frac{3}{2}$

$$WF'^{s}(\Lambda_{2N}) \subset C^{+}$$

$$C^{+} = \left\{ (\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in C; \tilde{\xi}^{0} \geq 0, \tilde{\eta}^{0} \geq 0 \right\},$$

where
$WF'(\Lambda_{2N}) := \{(\tilde{x}, \tilde{\xi}; \tilde{y}, -\tilde{\eta}) \in T^{*}(M \times M); (\tilde{x}, \tilde{\xi}; \tilde{y}, \tilde{\eta}) \in WF(\Lambda_{2N})\}$

## Ground State

Ultrastatic setting: $M = \mathbb{R} \times \Sigma$, $\Sigma$ compact

$ds^2 = dt^2 - h_{ij}(x)dx^i dx^j$

Klein-Gordon operator: $\partial_{tt} - \Delta_h + m^2$

The **ground state**, $\omega_g$, is completely determined by its two-point function

$$\omega_g^{(2)}(t, x; s, y) = \sum_{l \in \mathbb{N}} \frac{e^{i\lambda_l(t-s)}\phi_l(x)\phi_l(y)}{\lambda_l}$$

The eigenvalues of $-\Delta_h \phi + m^2$ are $\{\lambda_j^2\}_{j \in \mathbb{N}}$ and the set of eigenvectors $\{\phi_l\}$.

**Theorem**

*(Schrohe,- '22) Let $(M, g)$ be a $C^\tau$ ultrastatic spacetime with $\tau > 2$, dim $M = 4$ and $\omega_g^{(2)}$ the two-point function of the ground state. Then $WF'^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(\omega_g^{(2)}) \subset C^+$ for every $\tilde{\epsilon} > 0$*

If the spacetime is smooth, then the ground states are adiabatic states of infinite order (Hadamard states).

## Lemma

$\omega_g^{(2)} \in H_{loc}^{-\frac{1}{2}-\epsilon}(M \times M)$ *for* $\epsilon > 0$

For

$u(t,s,x,y) = \sum_{j,k} u_{jk}(t,s)\phi_j(x)\phi_k(y) \quad \text{with } u_{jk} = \langle u, \phi_j \otimes \phi_k \rangle$

$$u \in H^s(\mathbb{R}^2 \times \Sigma^2)$$
$$\iff \sum_{j,k} \int_{\mathbb{R}^2} (\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2)^s |(\mathcal{F} u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty\}.$$

$$\omega_g^{(2)}(t,x;s,y) = \sum_{l \in \mathbb{N}} \frac{e^{i\lambda_l(t-s)}\phi_l(x)\phi_l(y)}{\lambda_l}$$

$$\|\omega_g^{(2)}\|^2_{H_{loc}^{-\frac{1}{2}-\epsilon}(M\times M)} \leq \sum_l \frac{C}{\lambda_l^{3+\frac{2}{3}\epsilon}} \underbrace{\leq}_{\text{Weyl's law}} \sum_l \frac{C'}{l^{1+\epsilon}} < \infty$$

## Lemma

*For any* $\tilde{\epsilon} > 0$

$$WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) \subset \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG}).$$

$$P_{(t,x)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) = (-\xi^{0^2} + h^{ij}(x)\xi_i\xi_j) + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j + m^2$$

$$P_{(t,x)}\omega_G^{(2)} = 0 \iff P^{\#}_{(t,x)}\omega_G^{(2)} = -P^{b}_{(t,x)}\omega_G^{(2)}$$

## Lemma

*For any* $\tilde{\epsilon} > 0$

$$WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) \subset \mathrm{Char}(P) \times \mathrm{Char}(P).$$

$$P_{(t,x)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) = (-\xi^{0^2} + h^{ij}(x)\xi_i\xi_j) + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j + m^2$$

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$$P^{b}_{(t,x)} : \underbrace{H^{s+2-\tau\delta \sim s+2-\tau+\tilde{\epsilon}}}_{H^{-\frac{1}{2}-\epsilon}} \to H^s$$

$$\delta \sim 1, 0^- < s < \tau, s = -\frac{5}{2} + \tau - \tilde{\epsilon}$$

$$P^{\#}_{(t,x)}\omega_G^{(2)} = -P^{b}_{(t,x)}\omega_G^{(2)} =: g \in H^{s=-\frac{5}{2}+\tau-\tilde{\epsilon}}$$

## Lemma

*For any* $\tilde{\epsilon} > 0$

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$WF^{s+2=-\frac{1}{2}+\tau-\tilde{\epsilon}}(\omega_G^{(2)}) \subset \mathrm{Char}(P^{\#}_{(t,x)}) \cup \underbrace{WF^s(g)}_{\emptyset}$

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$$WF^{s+2=-\frac{1}{2}+\tau-\tilde{\epsilon}}(\omega_G^{(2)}) \subset \mathrm{Char}(P^{\#}_{(t,x)}) \cup \underbrace{WF^s(g)}_{\emptyset} = \mathrm{Char}(P_{(t,x)})$$

$$\mathrm{Char}(P_{(t,x)}) = (\mathrm{Char}(P_{KG}) \times T^*M) \cup \{(\tilde{x}, \tilde{y}, \tilde{\xi}, \tilde{\eta}); \tilde{\xi} = 0, \tilde{\eta} \neq 0\}$$

## Lemma

*For any* $\tilde{\epsilon} > 0$

$$WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) \subset \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG}).$$

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$$P_{(s,y)}(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) = (-{\eta_0}^2 + h^{ij}(y)\eta_i\eta_j) + i\frac{1}{\sqrt{h}}\partial_{y^i}(h^{ij}\sqrt{h}(y))\eta_j + m^2$$

$$WF^{-\frac{1}{2}+\tau-\tilde{\epsilon}}(\omega_G^{(2)}) \subset \mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)})$$

$$\mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)}) = \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG})$$
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*For any* $\tilde{\epsilon} > 0$

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$$(\partial_t + \partial_s)\omega_G^{(2)} = 0 \Rightarrow WF^l(\omega_G^{(2)}) \subset \mathrm{Char}(\partial_t + \partial_s) =$$
$$\{(\tilde{x},\xi_0,\xi,\tilde{y},\eta_0,\eta) \in T^*(M \times M)\backslash\{0\}; \xi_0 + \eta_0 = 0\}$$
$$\tilde{\xi} = 0 \implies \xi_0 = 0 \implies \eta_0 = 0 \implies \eta_0^2 = h^{ij}(y)\eta_i\eta_j = 0 \implies$$
$$\tilde{\eta} = 0$$

## Lemma

*(Positivity) For all* $s \in \mathbb{R}$,
$WF^s(\omega_g^{(2)}) \subset \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in T^*(M \times M); \tilde{\xi}^0 > 0\}$

We define $F : \mathbb{R} + i\,]0, \epsilon_1[\, \subset \mathbb{C} \to \mathcal{D}'(\Sigma \times M)$ for $\epsilon_1 > 0$ by

$$F(z) := F(t, \epsilon) = \sum_j e^{i(t+i\epsilon)\lambda_j} e^{-is\lambda_j} \phi_j(x)\phi_j(y)$$

Then, $F$ is holomorphic and $\lim_{\epsilon \to 0} F = \omega_g^{(2)}$.

## Lemma

*Let $(\tilde{x}, \tilde{y}) \in M \times M$ be such that $\tilde{x}$ and $\tilde{y}$ are not causally related, i.e. $\tilde{x} \notin J(\tilde{y})$. Then $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \notin WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_g^{(2)})$.*

$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset N_+ \times N_-$,
$N_\pm := \{(t, x, \xi_0, \xi) \in \mathrm{Char}(P_{KG}); \pm\xi_0 > 0\}$
$\omega_G^{(2)}|_{\mathcal{Q}}$, $\mathcal{Q}$ the set of pairs of causally separated points.
$\omega_G^{(2)} = \omega^+ + iK_G$. Then $\omega_G^{(2)}|_{\mathcal{Q}} = \omega^+|_{\mathcal{Q}}$.

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$\omega_G^{(2)} = \omega^+ + iK_G$. Then $\omega_G^{(2)}|_{\mathcal{Q}} = \omega^+|_{\mathcal{Q}}$ .
"flip" map $\rho(\tilde{x}, \tilde{y}) = (\tilde{y}, \tilde{x})$, $\rho^*\omega_G^{(2)}|_{\mathcal{Q}} = \omega_G^{(2)}|_{\mathcal{Q}}$.

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) = WF^{-\frac{1}{2}-\epsilon+\tau}(\rho^*\omega_G^{(2)}|_{\mathcal{Q}})$$

$$\subset \rho^* WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) \subset \rho^*(N_+ \times N_-) = N_- \times N_+$$

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$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) = WF^{-\frac{1}{2}-\epsilon+\tau}(\rho^*\omega_G^{(2)}|_{\mathcal{Q}})$$

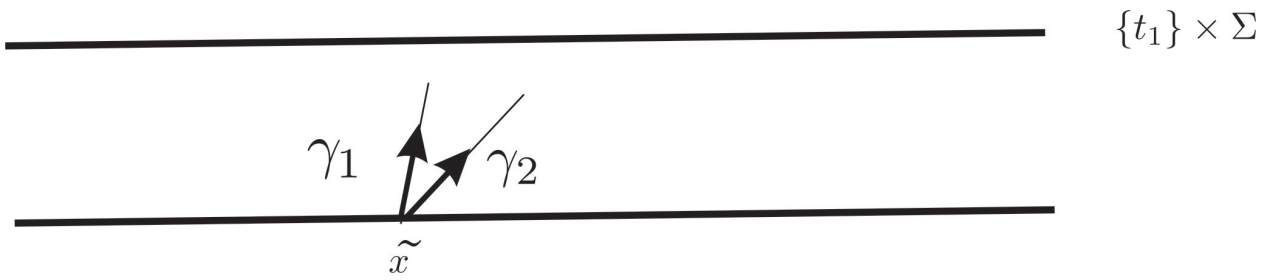
$$\subset \rho^* WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) \subset \rho^*(N_+ \times N_-) = N_- \times N_+$$

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) \subset (N_+ \times N_-) \cap (N_- \times N_+) = \emptyset.$$

## Lemma

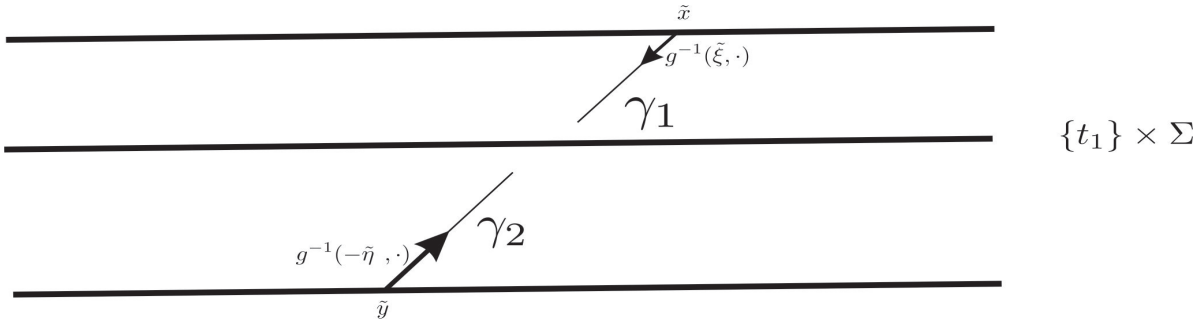
*If* $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(\omega_g^{(2)})$ *for* $\tilde{\epsilon} > 0$. *Then* $\tilde{\eta} = -\tilde{\xi}$.

Let $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ with $\tilde{\eta} \neq \lambda\tilde{\xi}$ then $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{x}, \tilde{\eta})) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ $(P^b\omega_G^{(2)} \in H^{-\frac{5}{2}-\epsilon+\tau=\sigma})$.




Exists $\{t_1\} \times \Sigma$ where $\Pi\gamma(\tilde{x}, \tilde{\xi}) = (t_1, w_1), \Pi\gamma(\tilde{x}, \tilde{\eta}) = (t_1, w_2)$ are causally separated !
Then $\tilde{\eta} = \lambda\tilde{\xi}, \lambda \in \mathbb{R}$.
From $(\partial_t + \partial_s)\omega_G^{(2)} = 0$ we have $\lambda = -1$.

Proof of the Theorem: Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ then $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ and $(t_1, w_1, \chi, t_1, w_1, -\chi) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$




Define

$$\tilde{\gamma}(\lambda) = \begin{cases} \gamma_1(\lambda) := \Pi(\gamma(\tilde{x}, \tilde{\xi}))(\lambda) & \lambda \in (-\infty, t_1] \\ -\gamma_2(\lambda) := -\Pi(\gamma(\tilde{y}, -\tilde{\eta})(\lambda)) = \Pi(\gamma(\tilde{y}, \eta)(\lambda)) & \lambda \in (t_1, \infty) \end{cases}$$

where $-\gamma_2$ denotes the curve with opposite orientation.
Then $\exists a, b \in \mathbb{R}$
$\tilde{\gamma}(a) = \tilde{x}, \tilde{\gamma}(b) = \tilde{y}; g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{x}}M} = \tilde{\xi}, g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{y}}M} = \tilde{\eta}$,
i.e., $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in C$.
This gives $WF'^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C$. Combining with positivity.

$$WF'^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C^+$$

Let $\omega_A \in \mathcal{D}'(M \times M)$ be the bidistribution given by

$$\omega_A := -\sum_j \lambda_j^{-2} e^{i\lambda_j(t-s)} \phi_j(x)\phi_j(y)$$

Then,

$$\omega_A \in H_{loc}^{\frac{1}{2}-\epsilon}(M \times M) \text{ for every } \epsilon > 0.$$

Properties: $\partial_t \omega_A = i\omega_G^{(2)}$
Idea: Estimates for $\omega_A$ and
$WF^s(\omega_G^{(2)}) = WF^s(\partial_t \omega_A) \subset WF^{s+1}(\omega_A)$

Let $\{\psi_j; j = 0, 1, \ldots\}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, i.e., a partition of unity $1 = \sum_{j=0}^{\infty} \psi_j$, where $\psi_0 \equiv 1$ for $|\xi| \leq 1$ and $\psi_0 \equiv 0$ for $|\xi| \geq 2$ and $\psi_j(\xi) = \psi_0(2^j\xi) - \psi_0(2^{1-j}\xi)$. The support of $\psi_j$, $j \geq 1$, then lies in an annulus around the origin of interior radius $2^j$ and exterior radius $2^{1+j}$.
Given $p(x, \xi) \in C^\tau S^m_{1,0}$ and $\gamma \in (0, 1)$ let

$$p^{\#}(x, \xi) = \sum_{j=0}^{\infty} J_{\epsilon_j} p(x, \xi)\psi_j(\xi). \tag{2}$$

Here $J_\epsilon$ is the smoothing operator given by $(J_\epsilon f)(x) = (\phi(\epsilon D)f)(x)$ with $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(\xi) = 1$ for $|\xi| \leq 1$, and we take $\epsilon_j = 2^{-j\gamma}$.