Boundary conditions and infrared divergences

"Quantum effects in gravitational fields" - Leipzig 2023

Luca Sinibaldi

University of Pavia

August 30, 2023

General Comments

- Spacetimes with a time-like boundary require boundary conditions for both classical and quantum field theory.
- Different BCs $\implies$ different physics.
- The focus of the construction is the control of UV singularities.
- **Warning:** We must also check the infrared behaviour!

[0] L. Campos, C. Dappiaggi, L.S., arXiv:2308.01281

CONTENTS

General Construction

Standard Static Spacetimes

(M, g) globally hyperbolic spacetime with a **timelike boundary**[1].

Standard static $\implies$ isometric to $\mathbb{R} \times \Sigma$, $\partial\Sigma \neq \emptyset$, with line element

$$ds^2 = -\beta dt^2 + h.$$

Matter content: real scalar field $\Phi : M \to \mathbb{R}$

$$P\Phi = (\Box_g - V)\,\Phi = 0, \qquad V = m^2 + \xi R, \qquad \xi \in \mathbb{R}.$$

[1] L. Aké, J. L. Flores, M. Sánchez, Rev. Matem. Iberoamericana, Volume 37, Issue 1 (2021)

General Construction

Eigenvalue Problem

Static spacetime $\implies$ Fourier transform along the time direction

$$\widehat{\Phi}(\omega, x) \doteq \int_{\mathbb{R}} dt\, e^{i\omega t}\Phi(t, x),$$

$\implies$ eigenvalue problem for the operator K on $L^2(\Sigma)$

$$K\widehat{\Phi} = \omega^2\widehat{\Phi},$$

where

$$K = \beta\Delta_h - \frac{1}{2}h^{ij}(\partial_i\beta)\partial_j + \beta V.$$

General Construction
Boundary Conditions

Admissible BCs are into 1:1 correspondence with self-adjoint extensions of K on $L^2(\Sigma)$.

In particular *static* and of *Robin* type:

$$\Phi|_{\partial M} = \alpha\, \partial_n \Phi|_{\partial M}, \qquad \alpha \neq \alpha(t).$$

K_α self-adjoint extension associated to $\alpha \in \mathbb{R}$.

General Construction

Causal Propagator

$E_\alpha^\mp$ **unique** advanced and retarded Green's operator of P

Their integral kernels
$\mathcal{E}_\alpha^-(x,x') = \Theta(t-t')\mathcal{E}_\alpha(x,x')$ and $\mathcal{E}_\alpha^+(x,x') = -\Theta(t'-t)\mathcal{E}_\alpha(x,x')$,

$\mathcal{E}_\alpha \in \mathcal{D}'(M \times M)$ the **causal** propagator

$$\mathcal{E}_\alpha(f,f') = \int\limits_{\mathbb{R}^2} dtdt' \left(f(t), K_\alpha^{-\frac{1}{2}} \sin[K_\alpha^{\frac{1}{2}}(t-t')] f'(t) \right)_{L^2(\Sigma)},$$

$K_\alpha^{-\frac{1}{2}} \sin[K_\alpha^{\frac{1}{2}}(t-t')]$ is defined via functional calculus.

General Construction
Two-point Correlation function

We look for $\omega_{2,\alpha} \in \mathcal{D}'(M \times M)$ satisfying

$$\begin{cases} (P \otimes \mathbb{I})\,\omega_{2,\alpha} = 0, \quad (\mathbb{I} \otimes P)\,\omega_{2,\alpha} = 0, \\ \omega_{2,\alpha}(f, f') - \omega_{2,\alpha}(f', f) = i\mathcal{E}_\alpha(f, f') \quad \forall f, f' \in \mathcal{D}(M), \end{cases}$$

By construction,

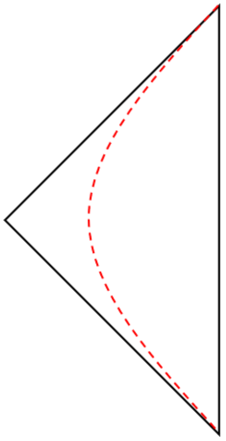
$$\omega_{2,\alpha}(f, f') = \int_{\mathbb{R}^2} dtdt' \left(f(t), K_\alpha^{-\frac{1}{2}} \exp[iK_\alpha^{\frac{1}{2}}(t - t')] f'(t) \right)_{L^2(\Sigma)} .$$

Its symmetric part is potentially ill-defined

$$\omega^S_{2,\alpha}(f, f') = \int_{\mathbb{R}^2} dtdt' \left(f(t), K_\alpha^{-\frac{1}{2}} \cos\left[K_\alpha^{\frac{1}{2}}(t - t') \right] f'(t) \right)_{L^2(\Sigma)} .$$

Bertotti-Robinson Spacetime

- Solution to the Einstein-Maxwell equations
- Near-Horizon limit of Reissner-Nordstrom black hole.
- Isometric to $CAdS_2 \times \mathbb{S}^2$



Bertotti-Robinson Spacetime

Coordinate Patches

Poincaré patch:

$$ds^2 = \frac{1}{r^2}(-d\tau^2 + dr^2 + r^2 d\Omega^2)$$

- $r = 0$ time-like boundary
- Manifestly conformal to Minkowski

"Horizon" patch[2]:

$$ds^2 = -(\rho^2 - 1)dt^2 + (\rho^2 - 1)^{-1} d\rho^2 + d\Omega^2(\theta, \varphi)$$

- $\rho = \infty$ time-like boundary
- Horizon at $\rho = 1$.

$$\tau = \frac{(\rho^2 - 1)^{1/2} \sinh t}{\rho + (\rho^2 - 1)^{1/2} \cosh t}, \qquad r = \frac{1}{\rho + (\rho^2 - 1)^{1/2} \cosh t}.$$

[2] A.C. Ottewill, P. Taylor, Phys.Rev.D 86 (2012)

Roadmap to the two-point correlation function

- Mode decomposition and *radial* equation.
- Classification of solutions by means of Sturm-Liouville theory.
- Radial Green function and Resolution of the identity
- Causal propagator
- Two-point function

Two-point correlation function for the ground state:

$$\widetilde{w}_{2,\gamma}(x,x') = \lim_{\varepsilon\to 0^+}\sum_{l=0}^{\infty}\int_0^{\infty} d\omega e^{-i\omega(\tau-\tau'-i\varepsilon)}\frac{2l+1}{4\pi}P_l(\cos\Gamma(\theta,\theta',\phi,\phi'))\tilde{R}_\gamma(\omega,r,r'),$$

Infrared behaviour enclosed by

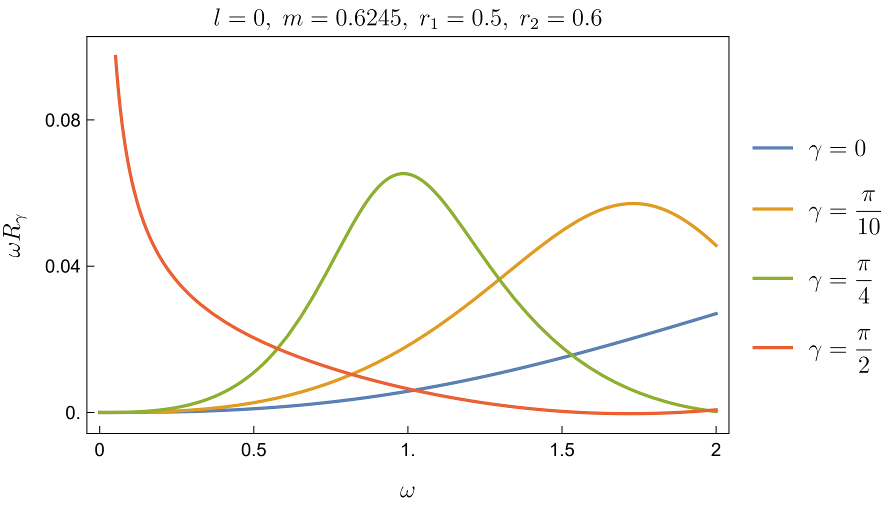
$$\tilde{R}_\gamma(\omega,r,r') = \frac{\sqrt{rr'}(\cos(\gamma)J_\eta(\omega r)+\sin(\gamma)\omega^{2\eta}J_{-\eta}(\omega r))(\cos(\gamma)J_\eta(\omega r')+\sin(\gamma)\omega^{2\eta}J_{-\eta}(\omega r'))}{2(\sin^2(\gamma)\omega^{4\eta}+\sin(2\gamma)\cos(\pi\eta)\omega^{2\eta}+\cos^2(\gamma))}.$$

$\eta = \frac{1}{2}\sqrt{1+4l(l+1)+4m^2}$

Computation in the Poincaré Patch

Infrared Divergences

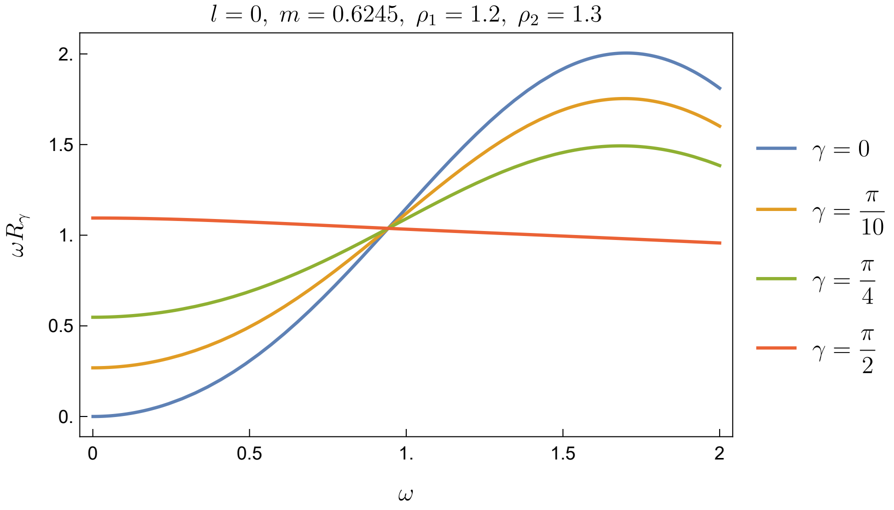
$$|\tilde{R}_\gamma(\omega, r, r')| \overset{\omega\to 0}{\sim} \omega^{2\eta}, \qquad |\tilde{R}_{\frac{\pi}{2}}(\omega, r, r')| \overset{\omega\to 0}{\sim} \omega^{-2\eta}$$



Computation in the Horizon Patch

Infrared Divergences

$$|R_0(\omega,\rho,\rho')| \overset{\omega\to 0}{\sim} \omega, \qquad |R_\gamma(\omega,\rho,\rho')| \overset{\omega\to 0}{\sim} \omega^{-1}$$



CONCLUSIONS

- Investigating the occurrence of infrared singularities with a more general class of boundary conditions, like Wentzell type and those associated to a dynamic wall.

- A comparison of the phenomenon with the freedom of choosing the secondary solution in a Sturm-Liouville problem[3].

[3] L. Campos, C. Dappiaggi, L.S., Gen.Rel.Grav. 55 (2023)