Energy Inequalities in Integrable Quantum Field Theory

Jan Mandrysch (Leipzig University)
based on [arXiv 2302.00063]

30. August 2023

Outline of the talk

- **Quantum energy inequalities (QEI)**

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- Integrable Quantum Field Theory

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- One-Particle QEIs
- State-independent QEI

Energy Inequalities

Teaser: Negative energies

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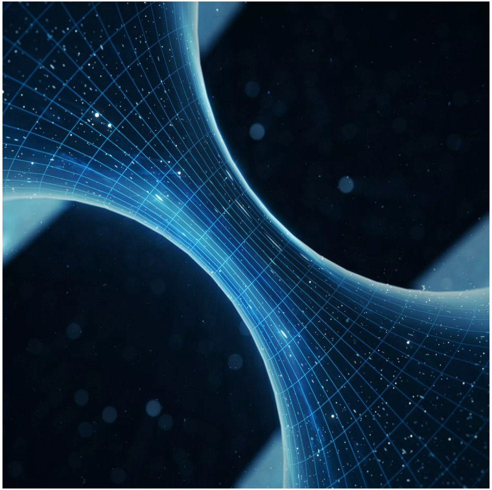

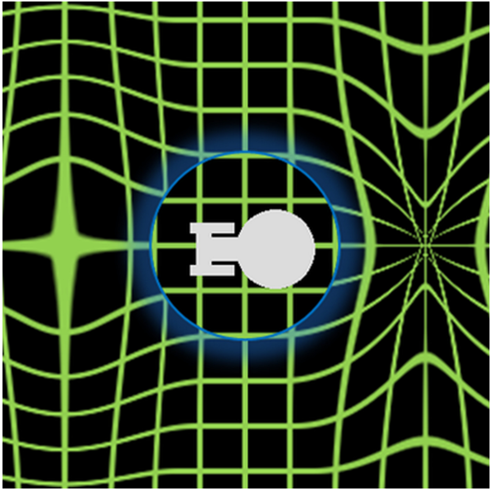

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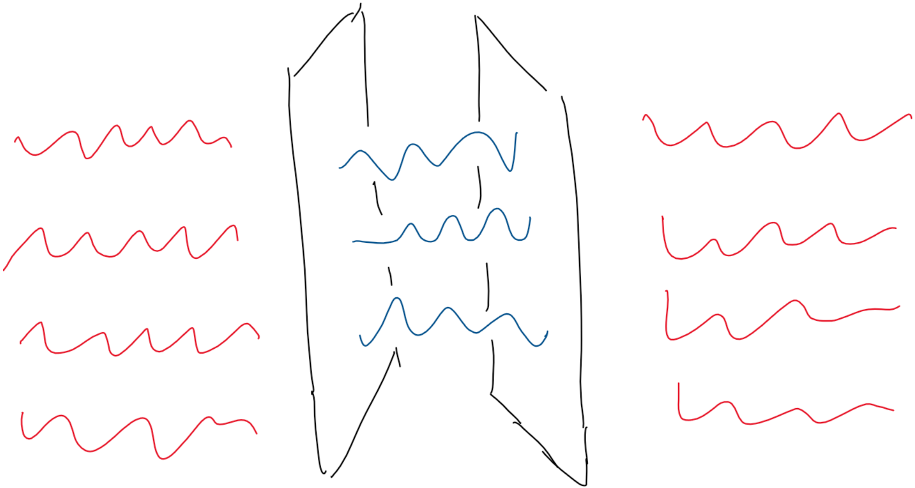

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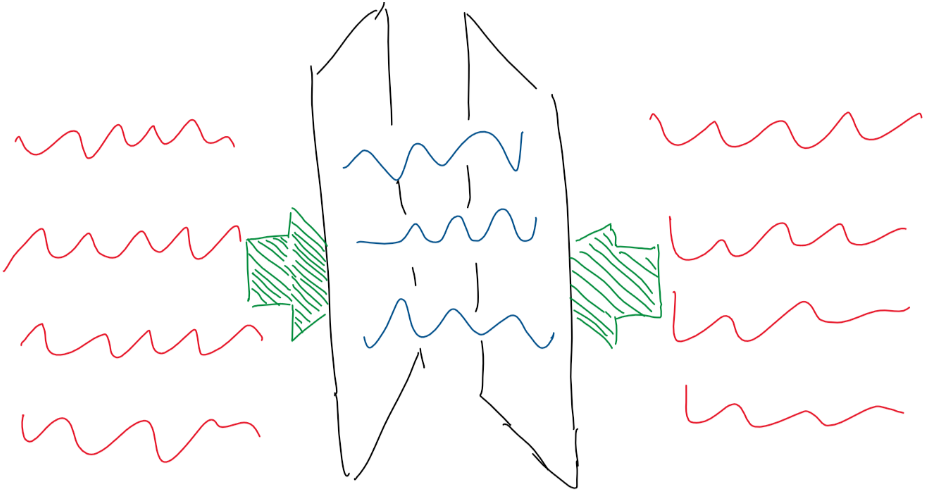

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- a QEI can take the form:

$$\langle \varphi | T(g^2, \gamma) | \varphi \rangle \geq -c_{g,\gamma},$$

holds for (sufficiently large) class of (normalized) states φ and $c_{g,\gamma} \geq 0$

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- integrable models of one scalar particle [Bostelmann/Cadamuro'16]

Integrability in QFT

Integrability and scattering theory

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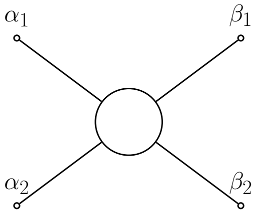
$\leftrightarrow$ existence of infinite set of (indep.) conserved charges



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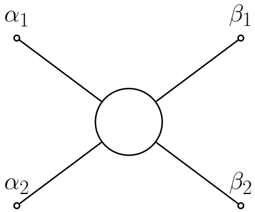

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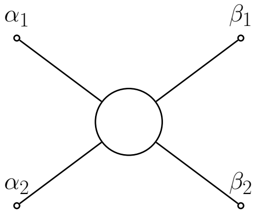

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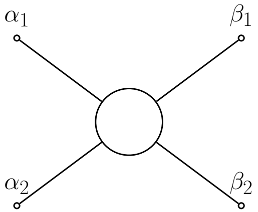

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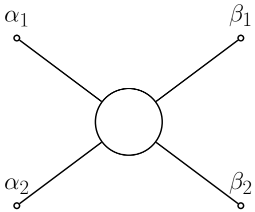

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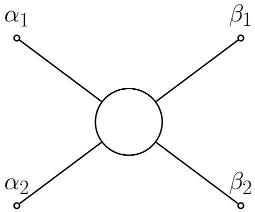

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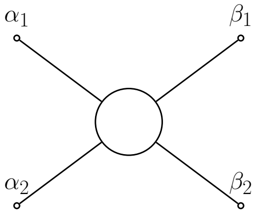

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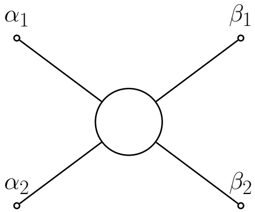

$\Rightarrow$ Construct integrable models via an inverse scattering approach:

- fix asymptotic data (particle spectrum and S-matrix)
- reconstruct local field content (form factor program)

Aim: QEI

For (sufficiently large) class of (normalized) states φ,

$$\langle\varphi|T(g^2,\gamma)|\varphi\rangle \geq -c_{g,\gamma}$$

The Stress-Energy Tensor

The stress tensor

$T^{\mu\nu}(x)$ is expected to be

- a local conserved generator of the translations,

$$\int dx^1\, T^{0\mu}(0,x^1) = P^\mu, \qquad\qquad \partial_\mu T^{\mu\nu} = 0,$$

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QEIs at One-particle level

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Proposition (General form at one-particle level)

$$\varphi \in \mathcal{H}_1, \quad \langle \varphi, T_{\mu\nu}(x)\varphi \rangle = \int d\theta d\eta \, T^{\text{free},1}_{\mu\nu}(\theta,\eta) e^{i(P(\theta)-P(\eta)).x} \overline{\varphi_\alpha(\theta)} F^\alpha_\beta(\theta-\eta) \varphi^\beta(\eta)$$

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Proposition (Go)

If for some $c < \frac{1}{4}$, $$\|F(\zeta)\|_{\mathcal{B}(\mathcal{K})} \lesssim c \exp|\operatorname{Re}\zeta|, \quad |\operatorname{Re}\zeta| \to \infty$$

then there exists $c_{g,\gamma} > 0$ *such that* $\forall \varphi \in \mathcal{D}(\mathbb{R},\mathcal{K}), ||\varphi|| = 1$:

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Proposition (No-Go)

If for some $c > \frac{1}{4}$, $\quad \|F(\zeta)\|_{\mathcal{B}(\mathcal{K})} \gtrsim c \exp|\operatorname{Re}\zeta|, \quad |\operatorname{Re}\zeta| \to \infty,$

then there exists a sequence $(\varphi_j) \subset \mathcal{D}(\mathbb{R},\mathcal{K}), ||\varphi_j|| = 1$ *such that*

$$\langle \varphi_j, T(g^2,\gamma)\varphi_j\rangle \overset{j\to\infty}{\to} -\infty.$$

Freedom of choice for the (one-particle) stress tensor

Integrable model	**Stress tensor family**
free, sinh-Gordon	one-parameter family [Bostelmann/Cadamuro'15]
Ising	unique ($q = 1$) [Bostelmann/Cadamuro'15]
Bullough-Dodd	two-parameter family
$O(N)$-nonlinear-sigma	unique ($q = 1$)
Federbush	unique for parity-invariant part ($q_1 = q_2 = 1$)

State-Independent QEI

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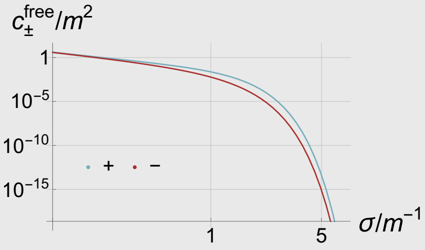
Theorem

For constant "diagonal" S-matrix S and suff. regular normalized states $\Psi \in \mathcal{H}$

$$\langle \Psi, T(g^2, \gamma)\Psi \rangle \geq -\operatorname{tr}_{\mathcal{K}^{\otimes 2}_{\text{diag}}} (c_+^{\text{free}} P_+ + c_-^{\text{free}} P_-),$$

with constants $c_\pm^{\text{free}}$ representing bounds for the free scalar bosonic/fermionic field depending only on m, g and γ.

Plot for g Gaussian; averaging scale σ



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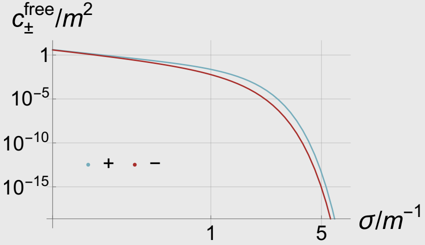

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- Includes new models: Fermionic Ising model, Federbush model, generalizations

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- I have also conducted a numerical analysis of the two-particle level for the sinh-Gordon and $O(n)$-NLS model where similar bounds seem to hold
- Analytic results for higher particle numbers would be desirable

Thanks for your attention!

Optional slides

$O(n)$-nonlinear σ (NLS) model

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- Large n-expansion correspondence:

$$S_{NLS}(\zeta) = (b(\zeta)\mathbb{1} + c(\zeta)\mathbb{F} + d(\zeta)\mathbb{K})\mathbb{F},$$

with

- $\mathbb{1}^{\gamma\delta}_{\alpha\beta} = \delta^\delta_\alpha \delta^\gamma_\beta, \quad \mathbb{F}^{\gamma\delta}_{\alpha\beta} = \delta^\gamma_\alpha \delta^\delta_\beta, \quad \mathbb{K}^{\gamma\delta}_{\alpha\beta} = \delta^{\gamma\delta}\delta_{\alpha\beta}$
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One-particle QEI for NLS model

NLS model: $n \geq 3$ massive spinless particles, global $O(n)$-symmetry, no bound states

Lemma

The two-particle form factor of the stress tensor is of the form

$$F_{\alpha\beta}(\theta-\eta) = q(-\operatorname{ch}(\theta-\eta))\, f_0^{\min}(\theta-\eta)\delta_{\alpha\beta},$$

where q is a real polynomial with $q(1)=1$ and $f_0^{\min}$ is the minimal solution wrt S_0.

The case $q=1$ corresponds to the quantized classical stress tensor.

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Theorem

A one-particle QEI in the NLS model holds iff $q=1$.

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- $c(\zeta) = -i\pi\nu\zeta^{-1}b(\zeta), \quad d(\zeta) = -i\pi\nu(i\pi - \zeta)^{-1})b(\zeta),$
- $b(\zeta) = q(\zeta)q(i\pi - \zeta)$
- $q(\zeta) = \frac{\Gamma\left(\frac{\nu}{2}+\frac{\zeta}{2\pi i}\right)\Gamma\left(\frac{1}{2}+\frac{\zeta}{2\pi i}\right)}{\Gamma\left(\frac{1+\nu}{2}+\frac{\zeta}{2\pi i}\right)\Gamma\left(\frac{\zeta}{2\pi i}\right)}$
- $\nu = \frac{2}{N-2}$

Minimal solution and characteristic function

- $$S_0(\theta) = b(\theta) + c(\theta) + Nd(\theta) = -\frac{\Gamma\left(1+\frac{\nu}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{3}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\theta}{2\pi i}\right)\Gamma\left(1-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(1+\frac{\theta}{2\pi i}\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{3}{2}-\frac{\theta}{2\pi i}\right)}.$$

Lemma

$$\exists c > 0 : |F_0^{min}(\zeta)| \lesssim c\,|\operatorname{Re}\zeta|^{-(1+\frac{\nu}{2})}\exp\left(\tfrac{1}{2}|\operatorname{Re}\zeta|\right), \quad |\operatorname{Re}\zeta| \to \infty$$

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- Malmstèn's formula: For $\mathrm{Re}\, z > 0$
  $\log\Gamma(z) = \int_0^\infty \left(z - 1 - \frac{1-e^{-(z-1)t}}{1-e^{-t}}\right)\frac{e^{-t}}{t}dt$

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- $f[S_0](t) = 1 + \frac{e^{-t}+e^{-\nu t}}{e^t+1} = 2 - (1+\frac{\nu}{2})t + \mathcal{O}(t^2), \quad t \to 0.$

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One-particle stress tensor (NLS model)

- $F(\theta)$ must be a Hermitian $O(N)$-invariant tensor
  $\Rightarrow F(\theta) = F_0(\theta) 1_{(\mathbb{C}^n)^{\otimes 2}}$

Lemma

$$F_2^{\mu\nu}(\theta, \eta + i\pi) = G_{\mathit{free}}^{\mu\nu}\left(\frac{\theta+\eta}{2}\right) Q(\mathrm{ch}(\theta - \eta)) F_0^{\mathit{min}}(\theta - \eta + i\pi) 1_{(\mathbb{C}^n)^{\otimes 2}},$$

where Q is a real polynomial with $Q(1) = 1$.

One-particle stress tensor (NLS model)

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- Uniqueness of minimal solution implies
  $F_0(\theta) = Q(\operatorname{ch}\theta) F_0^{\mathrm{min}}(\theta + i\pi)$

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Example: Bullough-Dodd model

BD model: one massive spinless particle which fuses to itself

- Lagrangian density:

$$\mathcal{L}_{\mathrm{BD}} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{6g^2}\left(2e^{g\varphi} + e^{-2g\varphi}\right)$$

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- Perturbation theoretic correspondence: $b = \frac{g^2}{2\pi}(1+\frac{g^2}{4\pi})^{-1}$ yields S-matrix

$$S_{\mathrm{BD}}(\zeta) = s(\zeta; -\tfrac{2}{3})s(\zeta; \tfrac{b}{3})s(\zeta; \tfrac{2-b}{3}), \quad s(\zeta;b) = \frac{\operatorname{sh}\zeta - i\sin\pi b}{\operatorname{sh}\zeta + i\sin\pi b}.$$

One-particle QEI for Bullough-Dodd model

BD model: one massive spinless particle which fuses to itself

Lemma

The 1particle stress tensor is of the form

$$F_2^{\mu\nu}(\theta,\eta;x) = T^{\mu\nu}_{\text{free},1}(\theta,\eta;x)q(-\operatorname{ch}(\theta-\eta))(2\operatorname{ch}(\theta-\eta)-1)^{-1}f_{\text{bd}}^{\min}(\theta-\eta),$$

where q is a real polynomial with $q(1)=1$ and $f_{\text{bd}}^{\min}$ is the minimal solution wrt S_{bd}.

Lemma

$\exists c>0 : |f_{\text{bd}}^{\min}(\zeta)| \lesssim c\exp\{-|\operatorname{Re}\zeta|\}, \quad |\operatorname{Re}\zeta| \to \infty$

Theorem

A 1particle QEI in the BD model holds if $\deg q \leq 2$ and cannot hold if $\deg q \geq 4$. For $\deg q = 3$ there is a threshold for the leading coefficient of q which decides the QEI.

Asymptotic growth estimate

Lemma

Let $f \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ *be exponentially decaying. Let further* $f_0 := f(0), f_1 := f'(0)$. *Then there is a constant* $c > 0$ *such that with* $\operatorname{Im}\zeta \in [0, 2\pi]$

$$F[f](\zeta) \sim c\,|\operatorname{Re}\zeta|^{f_1} \exp\left(\tfrac{f_0}{2}|\operatorname{Re}\zeta|\right), \quad |\operatorname{Re}\zeta| \to \infty.$$

Sketch of proof 1

Lemma

$F[f](\zeta) \sim c\,|\operatorname{Re}\zeta|^{f_1} \exp\left(\frac{f_0}{2}|\operatorname{Re}\zeta|\right), \quad |\operatorname{Re}\zeta| \to \infty.$

Proof.

Let $x, y \in \mathbb{R}$, $|y| \le \frac{1}{2}$, then for $|x| \to \infty$

$$\begin{aligned}
I(x+iy) &:= \operatorname{Re}\log F[f](2\pi(x+iy)+i\pi) \\
&= 2\int_0^\infty \frac{f(t)}{t\operatorname{sh} t}\operatorname{Re}\sin^2((x+iy)t)dt \\
&= \int_0^\infty \frac{f(t)}{t\operatorname{sh} t}(1-\cos 2xt\operatorname{ch} 2yt)dt \\
&\sim \int_0^1 \frac{f(t)}{t\operatorname{sh} t}(1-\cos 2xt\operatorname{ch} 2yt)dt + \mathcal{O}(1)
\end{aligned}$$

□

Sketch of proof 2

Lemma

$$F[f](\zeta) \sim c\,|\operatorname{Re}\zeta|^{f_1} \exp\Big(\tfrac{f_0}{2}|\operatorname{Re}\zeta|\Big), \qquad |\operatorname{Re}\zeta| \to \infty.$$

Proof.

$$\begin{aligned}
I(x+iy) &\sim \int_0^1 \frac{f(t)}{t\,\operatorname{sh} t}(1-\cos 2xt\,\operatorname{ch} 2yt)dt + \mathcal{O}(1)\\
&\sim \int_0^1 \frac{f_0 + f_1 t + \mathcal{O}(t^2)}{t^2}(1-\cos 2xt)dt + \mathcal{O}(1)\\
&\sim f_0 \int_0^1 \frac{1-\cos 2xt}{t^2}dt + f_1 \int_0^1 \frac{1-\cos 2xt}{t}dt + \mathcal{O}(1)\\
&\sim f_0 \pi |x| + f_1 \ln|x| + \mathcal{O}(1), \qquad |x| \to \infty.
\end{aligned}$$

$\square$

Construction of QFT from scattering function: Setup

S-deformed Fock space construction

- start with adequate one-particle space $\mathcal{H}_1 = L^2(\mathbb{R})$
- define n-particle state space $\mathcal{H}_n$ as the subspace of S-symmetric functions of $L^2(\mathbb{R}^n)$
- define full state space $\mathcal{H}$ as direct sum of n-particle subspaces
- unitary representation of the Poincaré group as for ordinary Fock space $(U(x,\lambda)\Phi)_n(\boldsymbol{\theta}) := e^{ip(\boldsymbol{\theta})x}\Phi_n(\boldsymbol{\theta} - \boldsymbol{\lambda})$
- creation and annihilation operators are given by

$$(z^\dagger(f)\Phi)_n := \sqrt{n}P_n^S(f \otimes \Phi_{n-1}) \quad \text{and} \quad (z(f)\Phi)_n := \sqrt{n}\Phi_{n+1}(f, \_).$$

- one may define an (auxillary) wedge-local field

$$\varphi(f) = z^\dagger(f^+) + z(f^-), \quad f^\pm(\theta) = \tilde{f}(p(\pm\theta))$$

(sharing many properties with the free scalar field)

Construction of QFT from scattering function: Observables

- creation and annihilation operators fulfill ZF-relations (S-deformed CCR):

$$[z,z]_S = [z^\dagger, z^\dagger]_S = 0$$
$$[z,z^\dagger]_S(\theta,\eta) = \delta(\theta-\eta)\cdot 1_{\mathcal{H}}$$

  where $[a,b]_S(\theta,\eta) = a(\theta)b(\eta) - S(\theta-\eta)b(\theta)a(\eta)$ is the S-deformed commutator.
- an observable $\mathcal{O}$ is a quadratic form on a dense subspace of regularized states $\mathcal{D}$ of $\mathcal{H}$
- Araki-Haag expansion

$$\mathcal{O} = \sum_{m,n=0}^{\infty} \int \frac{d\boldsymbol{\theta} d\boldsymbol{\eta}}{m!n!} F_{m+n}^{[\mathcal{O}]}(\boldsymbol{\theta}+i\mathbf{0}, \boldsymbol{\eta}+i\boldsymbol{\pi}-i\mathbf{0}) z^\dagger(\theta_1)...z^\dagger(\theta_m)z(\eta_1)...z(\eta_n)$$