



UNIVERSITÄT
LEIPZIG

joint work with M. Soltani, M. Casals, and S. Hollands

The quantum scalar field on Kerr-de Sitter

August 30, 2023

Christiane Klein

Outline

Motivation

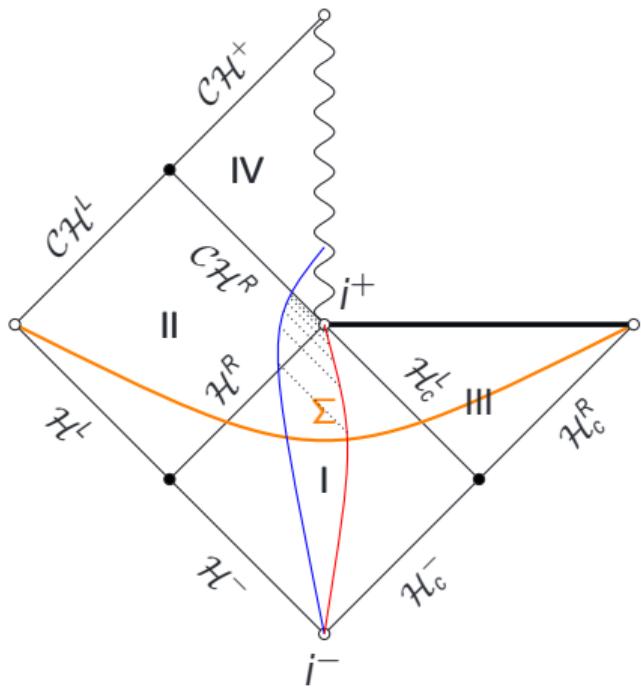
Kerr-de Sitter

The Unruh state

At the inner horizon

MOTIVATION

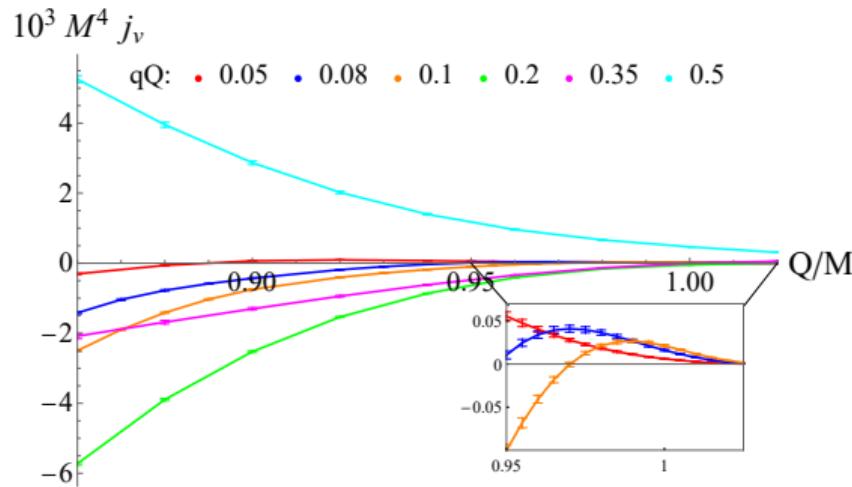
Strong cosmic censorship



- Cauchy horizon beyond which events not determined by initial data on Σ
- Signals reaching \mathcal{CH}^R infinitely blueshifted
[Penrose: 1974] \Leftrightarrow Cosmological redshift
- Strong cosmic censorship conjecture (sCC):
For generic initial data, metric is inextendible across \mathcal{CH}^R with certain regularity

[Christodoulou: 2008]

The charging of the RNdS inner horizon

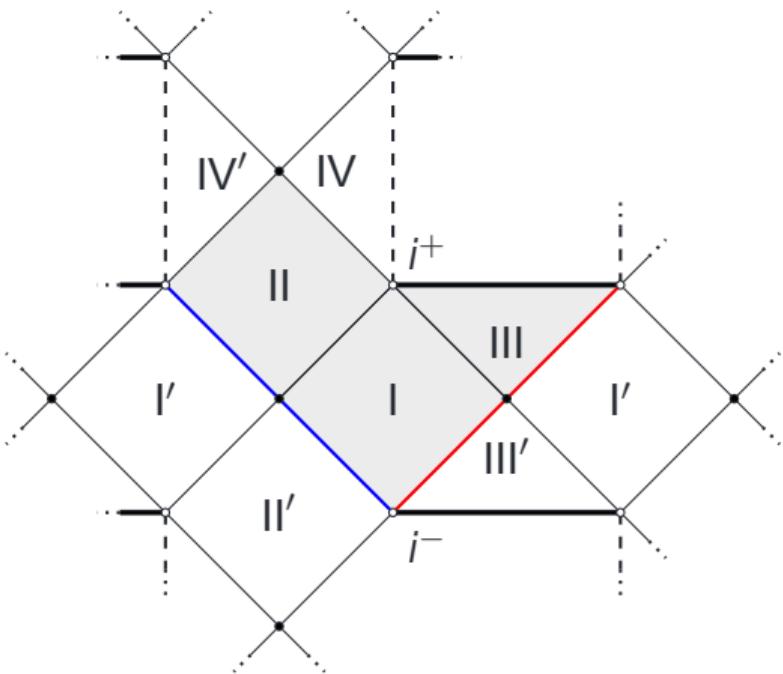


- Charge current can be negative
- ⇒ Charging of black hole
- Charge current positive near maximal Q
- ⇒ Cannot overcharge black hole

[CK, Zahn, Hollands: 2021]

KERR-DE SITTER

The Kerr-de Sitter spacetime



$$g = \frac{\Delta_\theta \sin^2 \theta}{\rho^2 \chi^2} (a dt - (r^2 + a^2) d\varphi)^2 + \frac{-\Delta_r}{\chi^2 \rho^2} (dt - a \sin^2 \theta d\varphi)^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right),$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

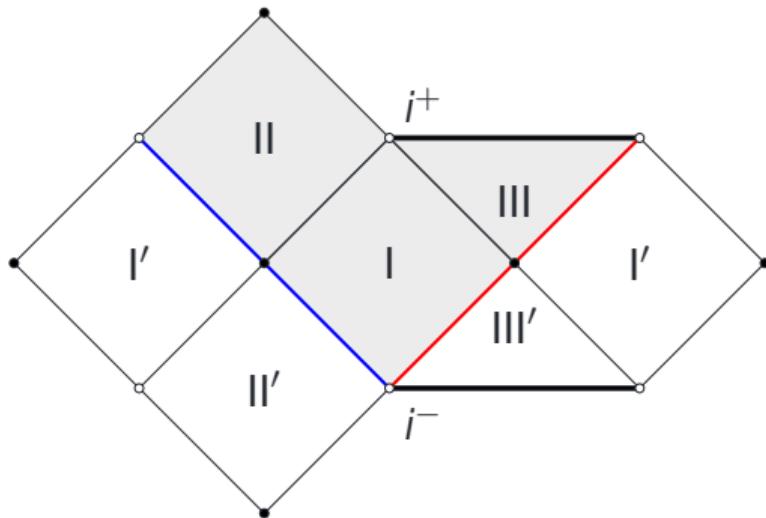
$$\chi = 1 + a^2 \Lambda / 3,$$

$$\Delta_\theta = 1 + a^2 \Lambda / 3 \cos^2 \theta,$$

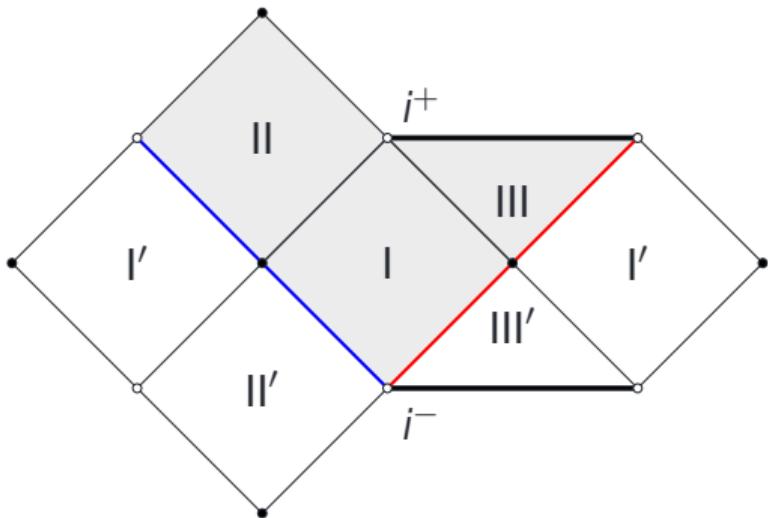
$$\Delta_r = (1 - \Lambda / 3r^2)(r^2 + a^2) - 2Mr$$

The Kerr-de Sitter spacetime - Coordinates

– Horizons: $r_- \sim \mathcal{CH}$, $r_+ \sim \mathcal{H}$ and $r_c \sim \mathcal{H}_c$

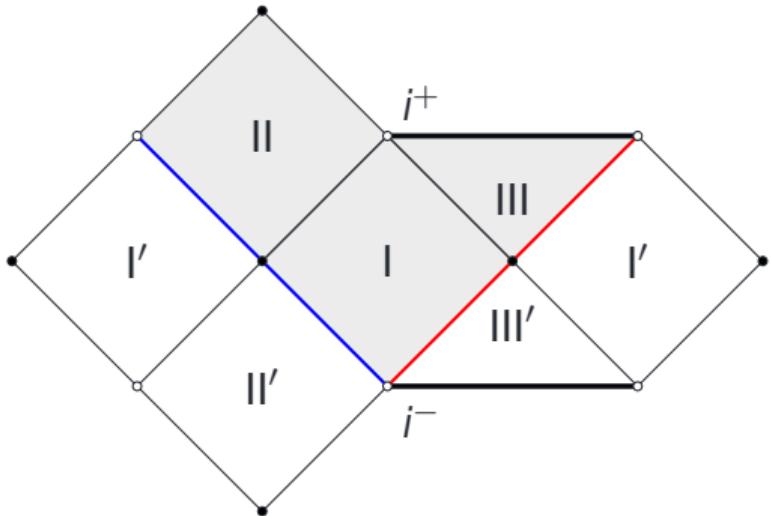


The Kerr-de Sitter spacetime - Coordinates



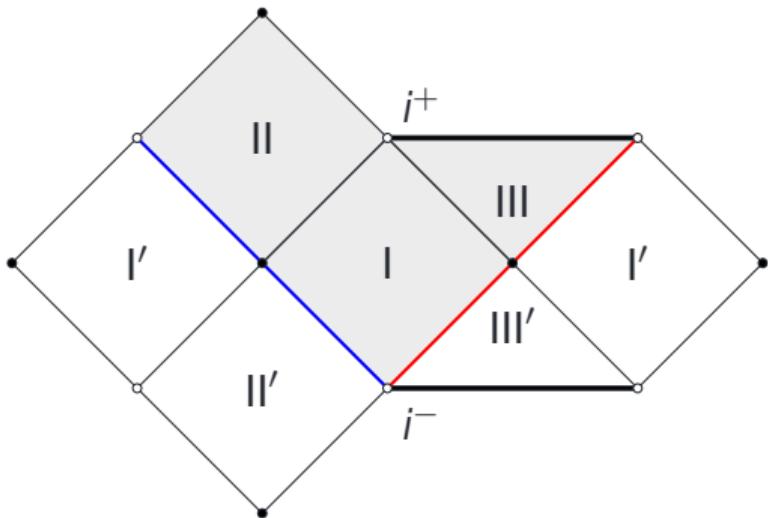
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with $\Omega_i = \frac{a}{r_i^2 + a^2}$, $i \in \{-, +, c\}$.

The Kerr-de Sitter spacetime - Coordinates



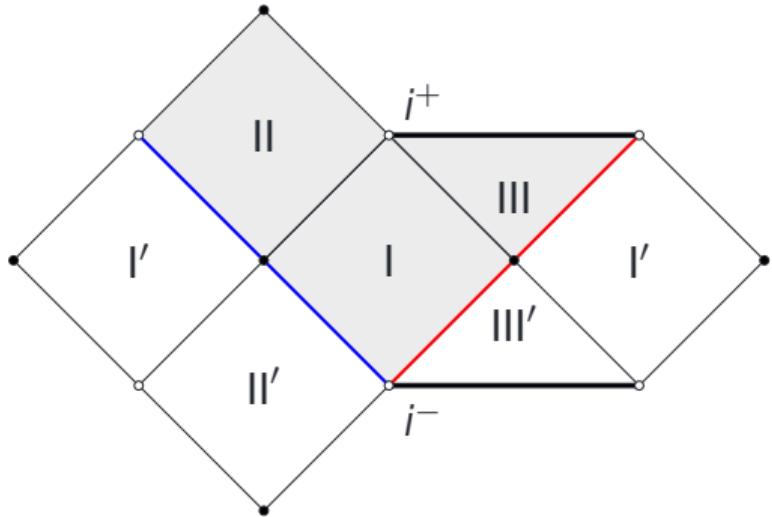
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- $v = t + r_*$, $u = t - r_*$,
with $dr_* = \frac{\chi(r^2 + a^2)}{\Delta_r} dr$

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- $v = t + r_*$, $u = t - r_*$,
with $dr_* = \frac{\chi(r^2 + a^2)}{\Delta_r} dr$
- $U_+ = -e^{-\kappa_+ u}$, $V_+ = e^{\kappa_+ v}$,
 $U_c = e^{\kappa_c u}$ and $V_c = -e^{-\kappa_c v}$,
with $\kappa_i = \frac{|\partial_r \Delta_r|_{r=r_i}}{2\chi(r_i^2 + a^2)}$

The Kerr-de Sitter spacetime - Some results



Spacetime M (gray) and its extension \tilde{M}

Lemma

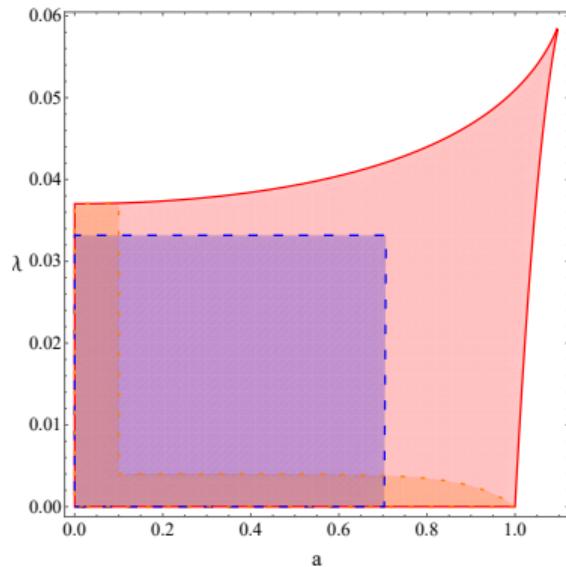
M and \tilde{M} are globally hyperbolic.

Lemma

If a, Λ sufficiently small, inextendible null geodesic not crossing \mathcal{H} or \mathcal{H}_c pass through region where $\partial_t + \Omega_+ \partial_\varphi$ and $\partial_t + \Omega_c \partial_\varphi$ are both timelike.

[Gérard, Häfner, Wrochna: 2020]

The Kerr-de Sitter spacetime - Some results



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THE UNRUH STATE

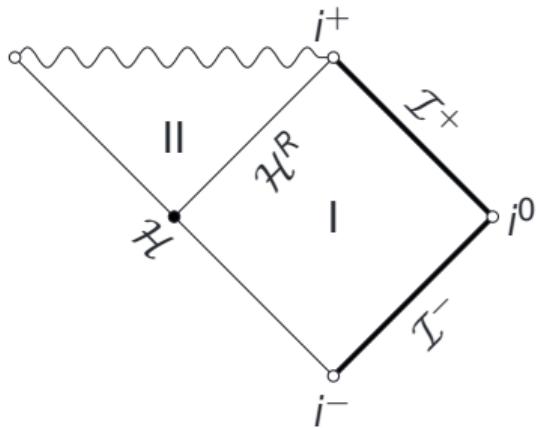
Quasi-free states for the real scalar

- Scalar field ϕ : $[\nabla_\nu \nabla^\nu - \mu^2] \phi = 0$
- Quasi-free state: determined by its two-point function $w(x, y) = \langle \phi(x) \phi(y) \rangle$, satisfying
 - Bi-distribution: $w \in \mathcal{D}'(M \times M)$
 - Bi-solution: $w([\nabla_\nu \nabla^\nu - \mu^2] f \otimes h) = w(f \otimes [\nabla_\nu \nabla^\nu - \mu^2] h) = 0$
 - Commutator property: $w(f \otimes h) - w(h \otimes f) = iE(f, h)$
 - Positivity: $w(\bar{f} \otimes f) \geq 0$

for all $f, h \in C_0^\infty(M)$

- Hadamard property: [Radzikowski:1996]
 $\text{WF}(w) = \{(x, k; y, -l) \in T^*(M \times M) \setminus o : (x, k) \sim (y, l) \text{ and } k \text{ future pointing}\}$

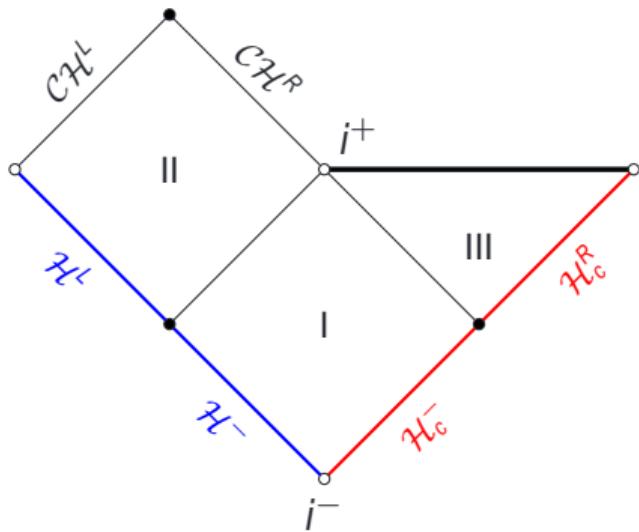
The Unruh state - Previous results



- Stationary, but non-equilibrium state
- Empty at \mathcal{I}^- , thermal energy flux at \mathcal{I}^+
- ⇒ Captures late-time behaviour in collapse
- Hadamard on
 - Schwarzschild [Dappiaggi, Moretti, Pinamonti: 2011]
 - Schwarzschild-de Sitter [Brum, Jorás: 2014]
 - Reissner-Nordström-de Sitter [Hollands, Wald, Zahn: 2019]
 - Kerr for massless free fermions

[Gérard, Häfner, Wrochna: 2020]

The Unruh state - Idea



— Expand:

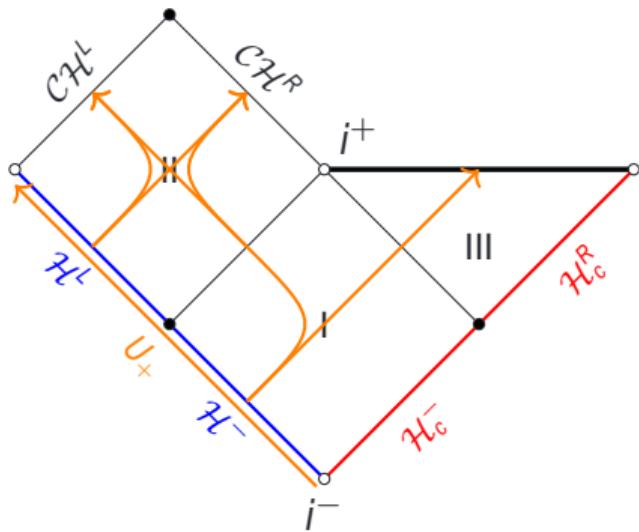
$$\phi(x) = \sum_J \int_0^\infty d\omega \psi_{\omega J}(x) a_{\omega J} + \bar{\psi}_{\omega J}(x) a_{\omega J}^\dagger$$

⇒ Unruh state: $a_{\omega J}|0\rangle = 0$

⇒ Two-point function:

$$\langle \phi(x)\phi(y) \rangle_U = \sum_J \int_0^\infty d\omega \psi_{\omega J}(x) \bar{\psi}_{\omega J}(y)$$

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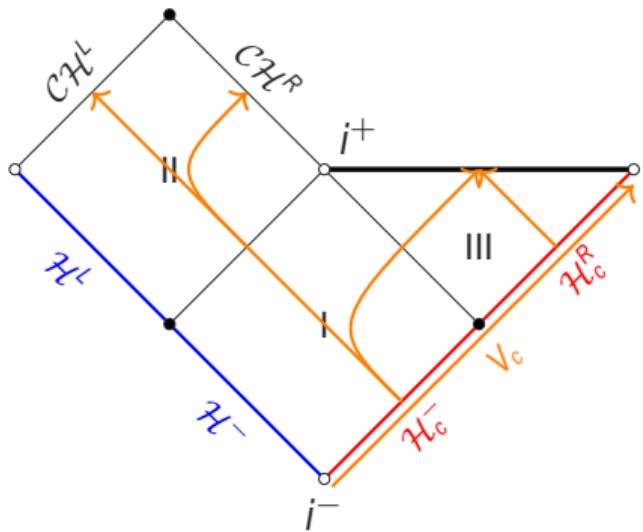
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- $\psi_{\omega J}$: Positive frequency modes w.r.t. affine parameter of null geodesics on \mathcal{H} and \mathcal{H}_c

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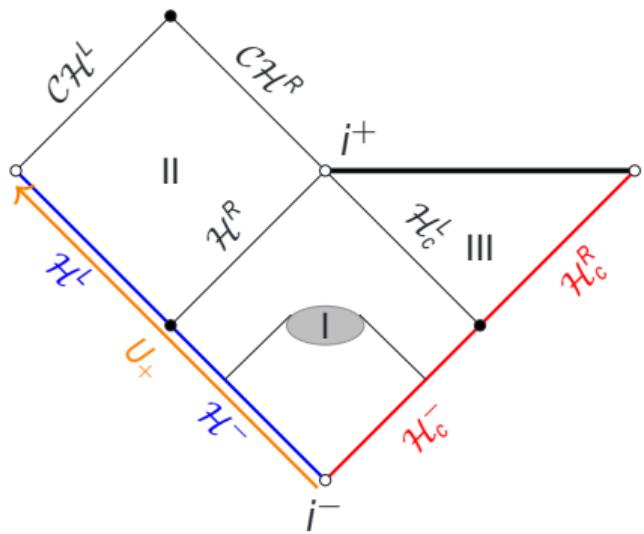
– $\psi_{\omega J}$: Positive frequency modes w.r.t. affine parameter of null geodesics on \mathcal{H} and \mathcal{H}_c

The Unruh state

- Consider $w(f, h)$ for test functions f, h
- Map to functional on solutions to $[\nabla_\nu \nabla^\nu - \mu^2] \phi = 0$ using E
- Map to functional on initial data on \mathcal{H} and \mathcal{H}_c (and i^-)
- Use Kay-Wald two-point function at \mathcal{H} and \mathcal{H}_c [Kay, Wald: 1988]

$$\begin{aligned} \Rightarrow w(f, h) = & - \lim_{\epsilon \rightarrow 0^+} \frac{r_+^2 + a^2}{\chi \pi} \int \frac{E(f)|_{\mathcal{H}}(U_+, \Omega_+) E(h)|_{\mathcal{H}}(U'_+, \Omega_+)}{(U_+ - U'_+ - i\epsilon)^2} dU_+ dU'_+ d^2\Omega_+ \\ & - \lim_{\epsilon \rightarrow 0^+} \frac{r_c^2 + a^2}{\chi \pi} \int \frac{E(f)|_{\mathcal{H}_c}(V_c, \Omega_c) E(h)|_{\mathcal{H}_c}(V'_c, \Omega_c)}{(V_c - V'_c - i\epsilon)^2} dV_c dV'_c d^2\Omega_c \end{aligned}$$

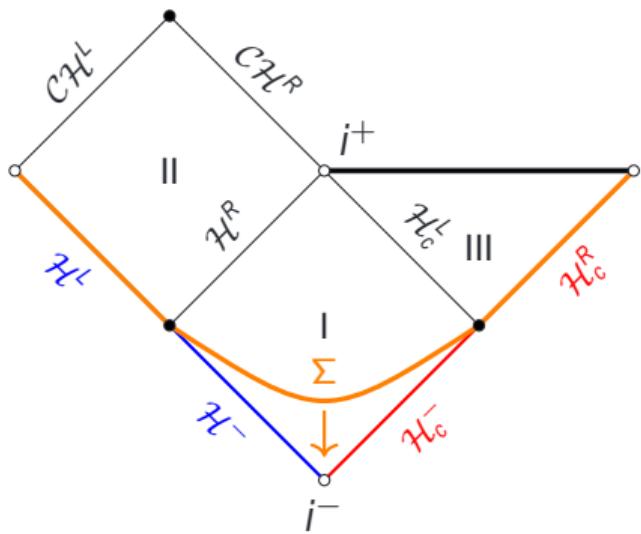
Well-definedness, Commutator property, Positivity



- Near i^- : $|E(f)|_{\mathcal{H}(U_+, \Omega_+)} \lesssim C|U_+|^{-\frac{\alpha}{\kappa_+}}$

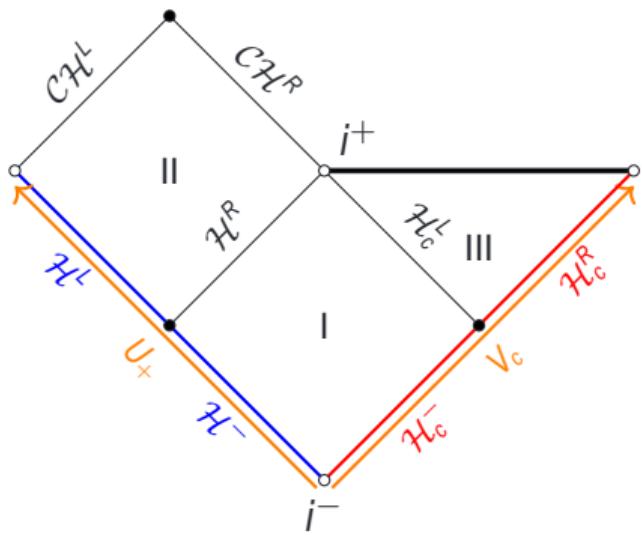
[Hintz, Vasy: 2017]

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- Commutator: $E(f, h) = \int_{\Sigma} E(f) \overleftrightarrow{\nabla}_n E(h) d\Sigma$
- ⇒ Take limit $\Sigma \rightarrow \mathcal{H} \cup \mathcal{H}_c$

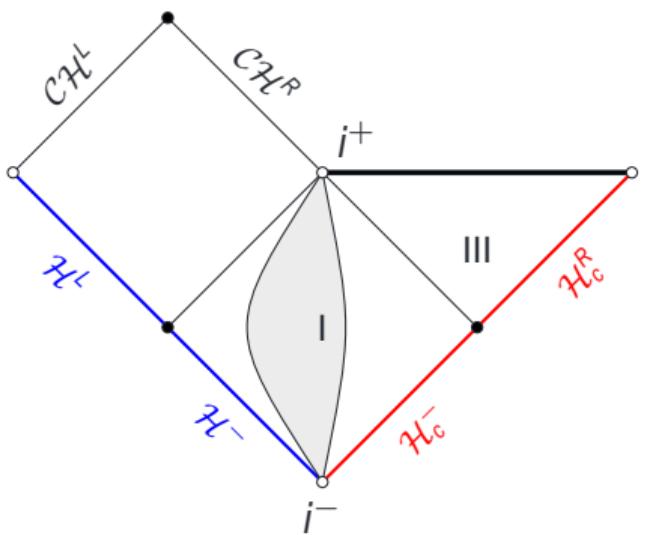
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- Commutator: $E(f, h) = \int_{\Sigma} E(f) \overleftrightarrow{\nabla} n E(h) d\Sigma$
- ⇒ Take limit $\Sigma \rightarrow \mathcal{H} \cup \mathcal{H}_c$
- $w(f, h) = \sum_{i=+, c} \left\langle K_i(\bar{f}), K_i(h) \right\rangle_{L^2(\mathbb{R}_+ \times \mathbb{S}^2; \nu_i)}$
- ⇒ $\nu_i = 2\eta(r_i^2 + a^2)\chi^{-1}d\eta d\Omega_i$
- ⇒ $K_i(f)(\eta, \Omega_i) = \mathcal{F}_i(E(f)|_{\mathcal{H}_i})|_{\eta \geq 0}(\eta, \Omega_i)$

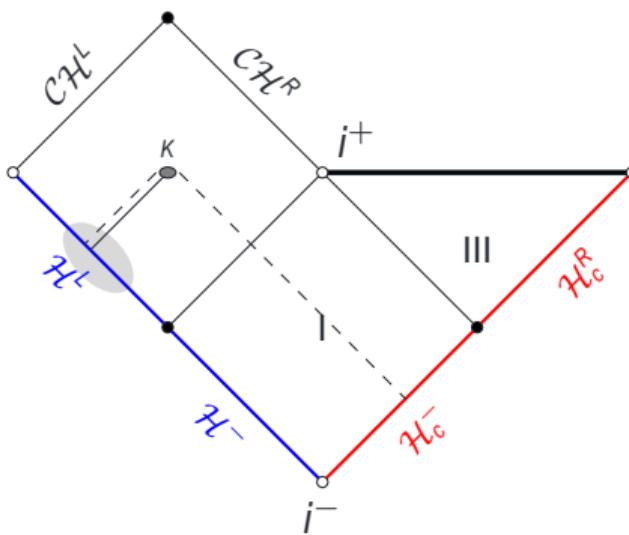
[Dappiaggi, Moretti, Pinamonti: 2009]

Hadamard property - Sketch of proof



- Propagation of singularities
 - In \mathcal{O} , $\partial_t + \Omega_{+}/c \partial_\varphi$ both timelike
- ⇒ Apply proof for passive states [Sahlmann, Verch: 2001]

Hadamard property - Sketch of proof



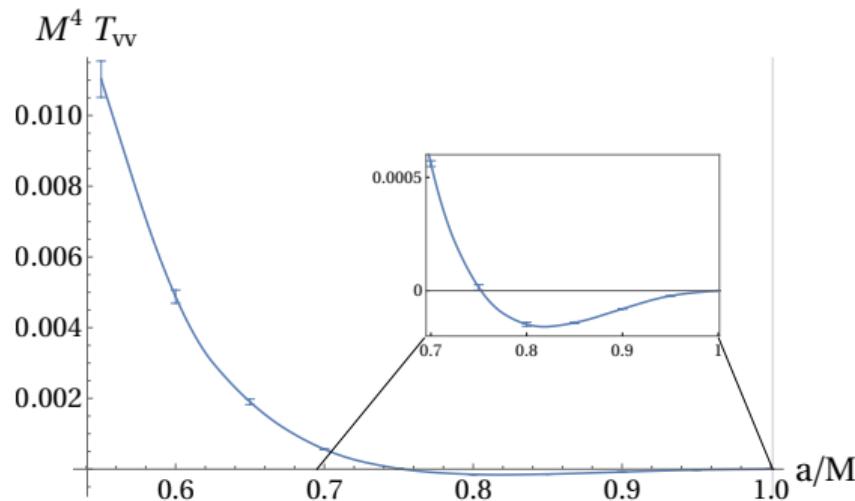
- Propagation of singularities
- In \mathcal{O} , $\partial_t + \Omega_{+/\circ} \partial_\varphi$ both timelike
- \Rightarrow Apply proof for passive states [Sahlmann, Verch: 2001]
- For geodesics intersecting horizon, split into relevant term + rest term
- \Rightarrow Direct computation for relevant term
- \Rightarrow Show that rest terms don't contribute

AT THE INNER HORIZON

The quantum field at the inner horizon

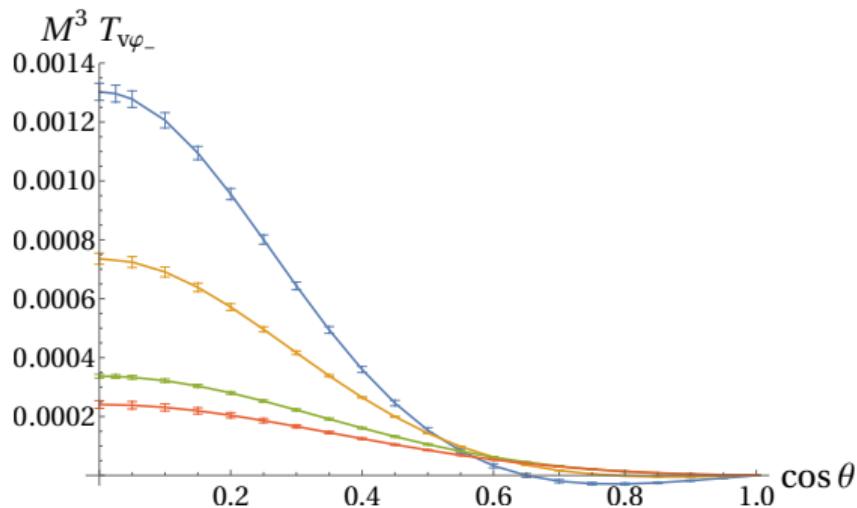
- Strong cosmic censorship in Kerr-de Sitter and the nature of the singularity at the Cauchy horizon
- ⇒ Leading divergence $\langle T_{\nu\nu} \rangle$
- Quasi-local angular momentum $J_{KdS} = Ma/\chi^2$
- ⇒ $J[S_1] - J[S_2] \sim \int \langle T_{\nu\varphi_-} \rangle$ at first order near \mathcal{CH}_-
- ⇒ Use mode-sum expression of the Unruh two-point function
- ⇒ Reduces to solving the Klein-Gordon equation on KdS

Preliminary results - leading divergence



The leading divergence of the stress-energy tensor on the axis of the inner horizon of a KdS spacetime for $\Lambda = 1/270$ as a function of a [Klein, Soltani, Casals, Hollands, in prep.]

Preliminary results - angular momentum current



The angular momentum current at the inner horizon as a function of $\cos \theta$ for different large values of a .

Summary

- Unruh-state of massive scalar on Kerr-de Sitter is Hadamard
 - restricted to region I, in-/outgoing modes thermally populated with temperature $T_{in/out} = \kappa_{c/+}(2\pi)^{-1}$ w.r.t. $\partial_t + \Omega_{c/+}\partial_\varphi$
 - Stress-energy tensor in Unruh state diverges quadratically at \mathcal{CH}^R
 - Quantum effects can (in)crease J near \mathcal{CH}^R
- ⇒ Are these results universal (i.e. state-independent)?
- ⇒ How do state-subtraction results compare to PMR-results?