Graviton Couplings on Hilbert space

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Quantum Effects in Gravitational Fields
Leipzig
August 30, 2023

based on *Quantum General Covariance*, Class. Quant. Grav. 2023.

Introduction

Scharf's approach to perturbative QFT

- Do not assume interaction Lagrangian as given (from a classical theory) but fix it perturbatively by quantum consistency conditions, mainly BRST.
- Start at lowest order in the coupling constant with terms of smallest possible UV dimension and work your way up order by order, implementing the consistency conditions.
- Implementation of the Epstein-Glaser scheme.

But: Gauge theories with massless gauge fields are formulated on a large indefinite metric space with many "unphysical" d.o.f. (gauge d.o.f., ghost fields, . . .).

A proper quantum theory should be formulated *on a Hilbert space*.

String-localized fields

S-loc fields can be defined by acting with the following operation on other fields:

Definition

Let $H := \{e \in \mathbb{R}^{1+3} \mid e^2 < 0\}$ and $c \in C_c^\infty(H)$ (with some technical assumptions). Define

$$I_c^\mu f(x) := \int_H d^4e\, c(e) e^\mu \int_0^\infty ds\, f(x+se). \tag{1}$$

Translate Scharf's approach to s-loc QFT:

Main consistency condition is locality of observables
(despite the non-locality of certain fields).

Why should one pay the price of non-local fields? S-loc fields . . .

- . . . allow for a covariant formulation of PT with massless tensor fields on Hilbert space.
- . . . allow for a more economic construction of models by avoiding unphysical d.o.f.

The massless free *field strength* $F_{[\mu\kappa][\nu\lambda]}(x)$ is defined on the Wigner-Fock space corresponding to the $(m = 0, |h| = 2)$ rep. of $\mathcal{P}$. It has 2 degrees of freedom.

The only s-loc fields in this talk will be the graviton potential and its string variation:

Definition

The *string-localized graviton potential* is defined via $h_{\mu\nu}(x; c) := I_c^\kappa I_c^\lambda F_{[\mu\kappa][\nu\lambda]}(x)$.

- It also has only the two physical degrees of freedom.
- Its string variation is $\delta_c h_{\mu\nu}(x; c) = \partial_\mu w_\nu(x; c) + \partial_\nu w_\mu(x; c)$.

The formal power series for the $\mathbb{S}$-matrix in s-loc QFT then depends on c via the s-loc fields:

$$\mathbb{S}[g; c] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \, g(x_1) \dots g(x_n) \, S_n(x_1, \dots, x_n; c), \tag{2}$$

where $g \in \mathcal{S}(\mathbb{R}^d)$ and the S_n are time-ordered products of an interaction density.

Implementing locality: the string independence (SI) condition

Main consistency condition: $\mathbb{S}[g;c]$ must be local and string independent in the adiabatic limit:

$$\lim_{g\uparrow\text{const}} \delta_c\big(\mathbb{S}[g;c]\big) \stackrel{!}{=} 0 \quad \Leftarrow \quad \delta_c\big(S_n(x_1,\dots,x_n;c)\big) = i^n \sum_{k=1}^{n} \partial_{x_k,\mu}\, Q_n^\mu(x_1,\dots,x_n;c).$$

- $\Rightarrow$ Lowest order of interaction density $L_1 = -iS_1$ must satisfy L-Q condition $\delta_c L_1 = \partial_\mu Q^\mu$.
- $\Rightarrow$ 2nd order: S_2 is the T-product of S_1, possibly plus *local* terms (all strings emanate from the same point):

$$S_2(x_1,x_2;c) = i^2\, T_{\text{ren}}[L_1(x_1;c)L_1(x_2;c)] + iL_2(x_1;c)\delta(x_1-x_2), \tag{3}$$

  with the 2nd order contribution to the interaction density L_2.
- $\Rightarrow$ Continue to higher orders: $L_{\text{tot}} = \sum_{n=1}^{\infty} \frac{g^n}{n!} L_n$.

We currently restrict ourselves to *tree graphs*, s.t. 2nd order SI becomes

$$\delta_c\, \underbrace{L_1 L_1'} \stackrel{!}{=} \partial_\mu Q^\mu_{2,\text{tree}} + i\delta(x-x')\, \delta_c L_{2,\text{tree}}. \tag{4}$$

Graviton self-coupling

- $O(\kappa)$ (cubic self-coupling, 1st order in perturbation theory): L-Q condition selects

$$L_1 \overset{\mathrm{div}}{=} h^{\mu\nu}\big[\tfrac{1}{2}(\!(\partial_\mu h \partial_\nu h)\!) + \partial^\kappa h_{\mu\lambda}\partial^\lambda h_{\nu\kappa} - (\partial_\kappa h \partial^\kappa h)_{\mu\nu}\big]. \tag{5}$$

- $O(\kappa^2)$ (2nd order in perturbation theory): tree graph is

$$\underbrace{L_1 L_1'} \overset{\mathsf{div}}{=} U^{\mu\nu}\langle\!\langle T_{\mathsf{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle U'^{\rho\sigma}. \tag{6}$$

with $U^{\mu\nu}$ a quadratic Wick polynomial, $\langle\!\langle T_{\mathsf{ren}} \bullet \ \bullet' \rangle\!\rangle$ the renormalized propagator.

Theorem

The SI condition at tree level determines a unique (up to div*) 2nd order self-interaction*

$$L_2 \overset{\mathrm{div}}{=} -2(\partial_\nu h h h \partial_\mu h)^{\mu\nu} - \tfrac{1}{2}(\!(h\partial_\mu h)\!)(\!(h\partial^\mu h)\!) - 2(\!(h\partial_\mu h \partial_\nu h)\!)h^{\mu\nu}. \tag{7}$$

Brute force proof.

1. Fix the pertinent properties of the renormalized propagator.
2. List all possible independent candidate monomials for L_2.
3. Apply δ_c to $U^{\mu\nu}\langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle U'^{\rho\sigma}$ and to the candidates.
4. Perform a cascade of integrations by parts to remove all derivatives from w_κ.
5. Remove further linear dependencies (up to div).
6. Compare coefficients.

$\square$

Renormalized graviton propagators

- Epstein-Glaser: possible ambiguities in defining time-ordered products.
- In s-loc case: ambiguities are certain string-deltas.
- Important constraints: $\eta^{\mu\nu}\langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle = 0$, scaling behavior, SI condition.
- $\langle\!\langle T_{\text{ren}} \partial_\kappa h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle = \partial_\kappa \langle\!\langle T_{\text{ren}} h_{\mu\nu} h'_{\rho\sigma}\rangle\!\rangle$.

There are shorter proofs but with the brute force method, one obtains an interesting corollary:

Corollary

Any interaction density with field and derivative content $(\partial, \partial, h, h, h, h)$, *which is string-independent on its own, must itself be a total divergence.*

Graviton coupling to matter

We consider the coupling of an s-loc $h_{\mu\nu}(x;c)$ to Maxwell, Dirac and scalar "matter".

1st order

L-Q condition implies that

$$L_{1,\mathrm{mat}} \overset{\text{div}}{=} \tfrac{1}{2} h_{\mu\nu}(x;c)\Theta^{\mu\nu}_{\mathrm{mat}}(x) \tag{8}$$

with the symmetric and conserved matter stress energy tensors

$\Theta^{\mu\nu}_{\phi} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\left(\partial_\kappa\phi\partial^\kappa\phi - m^2\phi^2\right)$ of a scalar field $\phi(x)$;

$\Theta^{\mu\nu}_{F} = -F^{\mu\kappa}F^{\nu}{}_{\kappa} + \frac{1}{4}\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda}$ of the Maxwell field strength $F_{\mu\nu}(x)$;

$\Theta^{\mu\nu}_{\psi} = \frac{i}{4}\left(\overline{\psi}\gamma^\mu \overset{\leftrightarrow}{\partial^\nu}\psi + \overline{\psi}\gamma^\nu \overset{\leftrightarrow}{\partial^\mu}\psi\right)$ of a Dirac field $\psi(x)$.

2nd order

For all three cases, one finds

$$\delta_c\Big(L_{1,\mathrm{mat}}L'_{1,\mathrm{mat}}\Big) = \partial_\mu Q^\mu_{2,\mathrm{mat,tree}}(x,x') + i\delta_c\big(L_{2,1,\mathrm{mat}}\big)\delta(x-x') + \mathcal{O}_{2,\mathrm{mat}}(x,x'), \tag{9}$$

with a *non-resolvable universal obstruction* to SI at second order tree level

$$\mathcal{O}_{2,\mathrm{mat}}(x,x') = -i\,\Theta^{\mu\nu}_{\mathrm{mat}}\,w^\kappa\big(\partial_\kappa h_{\mu\nu} - \partial_\mu h_{\kappa\nu} - \partial_\nu h_{\kappa\mu}\big)\,\delta(x-x'). \tag{10}$$

$\mathcal{O}_{2,\mathrm{mat}}(x,x')$ only appears due to the violation of a matter (i.e., point-localized) Ward identity:

$$\partial_\mu \sum_{\varphi,\chi'} \frac{d\Theta^{\mu\nu}_{\mathsf{mat}}}{d\varphi}\langle\!\langle T_{\mathsf{ren}}\varphi\chi'\rangle\!\rangle\frac{d\Theta'^{\rho\sigma}_{\mathsf{mat}}}{d\chi'} \neq 0. \tag{11}$$

The s-loc field does not contribute to this obstruction!

On its own, the graviton-matter coupling is inconsistent.

Theorem

The non-resolvable obstruction to SI at 2nd order of the graviton-matter interaction is exactly cancelled if the graviton self-coupling is taken into account.

Proof.

The proof is independent of the type of matter. We only use that $L_{1,\mathrm{mat}} = \frac{1}{2} h_{\mu\nu} \Theta^{\mu\nu}_{\mathrm{mat}}$ with $\Theta^{\mu\nu}_{\mathrm{mat}}$ point-localized, symmetric and conserved and the universal form of $\mathcal{O}_{2,\mathrm{mat}}$.

Write $\tilde{L}_1 = L_1 + L_{1,\mathrm{mat}}$. Then

$$\delta_c\Big(\underbracket{\tilde{L}_1 \tilde{L}_1'}\Big) \overset{\text{div}}{=} i\delta_c\big(L_{2,1,\mathrm{mat}} + L_2\big)\delta(x-x') + \mathcal{O}_{2,\mathrm{mat}}(x,x') + \delta_c\Big(\underbracket{L_1 L'}_{1,\mathrm{mat}} + \underbracket{L_{1,\mathrm{mat}} L_1'}\Big).$$

Interference term cancels $\mathcal{O}_{2,\mathrm{mat}}(x,x')$ and gives a contribution $i\delta_c\big(L_{2,2,\mathrm{mat}}\big)\delta(x-x')$.

The required cancellation of $\mathcal{O}_{2,\mathrm{mat}}$ fixes the relative prefactor between L_1 and $L_{1,\mathrm{mat}}$.

We define the full induced matter Lagrangian $L_{2,\mathrm{mat}} := L_{2,1,\mathrm{mat}} + L_{2,2,\mathrm{mat}}$. $\square$

Matter-matter obstructions for several free fields & obstructions of the interference term are additive. Hence

Corollary

The result of the above Theorem is true for the coupling to any number of scalar, Maxwell and Dirac fields, with the respective sum of induced matter interactions.

Compare to the expansion of the classical Lagrangians (with the s-loc field inserted):

Theorem

The induced terms of the graviton matter and self-interaction as a (Wick)polynomial of the s-loc quantum field $h^{\mu\nu}(x;c)$ and the quantum matter fields coincide with the respective second-order terms of the classical generally covariant Lagrangians as functionals of the classical metric deviation field $h_{\mu\nu}(x)$ and matter fields, provided the matter renormalization constants are chosen to be zero.

Main message

Quantum consistency conditions imply a *"lock-key scenario"*: the graviton coupling to matter is only consistent if the self-interaction is taken into account as well.

Further questions

- So far only considered up to second order tree level: What happens at higher orders?
- What is the contribution of loop graphs?
- For what other couplings can we find similar statements?
  1. higher-helicity s-loc fields?
  2. self-interacting fields?
  3. higher half-integer spin fields?
  4. . . . ?

Thank you for your attention!

Expansion of the classical Lagrangians

The Goldberg variable is

$$\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}, \quad \tilde{g}_{\mu\nu} = \tfrac{1}{\sqrt{-g}} g_{\mu\nu} = \eta_{\mu\nu} - \kappa h_{\mu\nu} + \kappa^2 (hh)_{\mu\nu} + \dots, \tag{12}$$

where $h_{\mu\nu}$ tends to zero at infinity. We have (neglecting $(\!(h)\!)$- and $\partial^\mu h_{\mu\nu}$-terms)

$$\begin{aligned} L_{g,\phi} &= \frac{\sqrt{-g}}{2}\big(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - m^2\chi^2\big) \qquad (13)\\ &= L_{0,\phi} + \tfrac{1}{2}\kappa(\!(h\Theta_\phi)\!) + \tfrac{\kappa^2}{2}\Big[\tfrac{1}{2}(\!(hh)\!)L_{0,\phi} + \tfrac{1}{4}(\!(hh)\!)(\!(\Theta_\phi)\!)\Big] + \dots\\ L_{g,F} &= \sqrt{-g}\big(-\tfrac{1}{4}g^{\mu\nu}g^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\big) \qquad (14)\\ &= L_{0,F} + \tfrac{1}{2}\kappa(\!(h\Theta_F)\!) + \tfrac{\kappa^2}{2}\big[\tfrac{1}{2}(\!(hh)\!)L_{0,F} - \tfrac{1}{2}h^{\mu\nu}h^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\big] + \dots\\ L_{g,\psi} &= \sqrt{-g}\,\overline{\psi}\big(\tfrac{i}{2}(\gamma_g^\mu D_\mu - \overleftarrow{D}_\mu\gamma_g^\mu) - m\big)\psi \qquad (15)\\ &= \tfrac{1}{2}\kappa(\!(h\Theta_\psi)\!) + \tfrac{\kappa^2}{8}\big((\!(hh)\!)(\!(\Theta_\psi)\!) - (\!(hh\Theta_\psi)\!)\big)\\ &\quad - \tfrac{i\kappa^2}{16}\big((hh)_{\mu\nu}\partial^\mu j^\nu + (h\partial_\nu h)_{\beta\alpha}\overline{\psi}\gamma^\nu\gamma^\beta\gamma^\alpha\psi\big) + \dots \end{aligned}$$