Black hole simulations with the FO-CCZ4 formulation of the Einstein equations and ADER discontinuous Galerkin schemes

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From Dumbser, Guercilena, Köppel, Rezzolla, and Zanotti (2018)

Motivation

In numerical simulations, often a balance must be sought between accuracy and performance.

"Accuracy" is quantified *e.g.* as:

- shock capturing abilities
  hydrodynamic equations
- constraint preservation
  dynamical spacetimes
  Maxwell equations
- handling of stiff terms
  resistive MHD
  neutrino transport (M1)
- well-balancedness
  balance laws (*e.g.* relativistic hydro)

Performance improvements include:

- stability for large timesteps
- optimal single CPU use
  multi-threading
  vectorization
- reducing communication overhead between CPUs

The ADER approach

The ADER (Arbitrary Accuracy DERivative Riemann problem) [1] can address many of the previous points:

High order DG expansion	$\Rightarrow$	uniformly high order in both space and time
WENO reconstruction	$\Rightarrow$	shock handling
Implicit timestepping	$\Rightarrow$	large Δt and handling of stiff terms
Fluxes and sources on equal footing	$\Rightarrow$	well balanced scheme
Most operations element local	$\Rightarrow$	small communication overhead

[1] *e.g.* (Dumbser et al., 2008; Dumbser and Toro, 2011; Dumbser et al., 2013, 2014; Dumbser et al., 2017)

DG schemes are typically formulated for 1st order (often flux-conservative) systems[2]. 3+1 Einstein equations are a 2nd order (non-conservative) system of equations.

Cast the 3+1 Efe in 1st order form

$\Downarrow$

Promote $\partial_i A = A_i$ to an independently evolved variable:

$\partial_t A_i = \ldots$

$\rightarrow$ *Increase in the number of evolved fields*

A first order system is not enough.

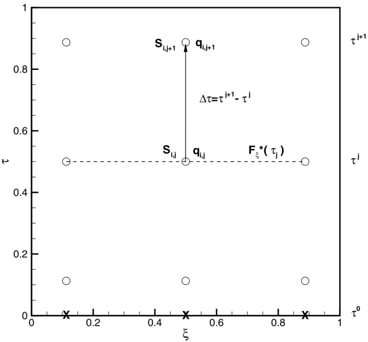
- need to ensure hyperbolicity
- constraint violation damping is a desirable property *(Gundlach et al., 2005; Brodbeck et al., 1999)*

[2] an exception: *Miller and Schnetter (2017)*

The ADER-DG method

Finite element decomposition of the domain $\Omega = \bigcup_k \Omega_k$;

Nodal DG representation of the solution in each element $u(t^n, x) = \sum_{i=0}^{N} u_i^n \, \ell_i$



Placement of the predictor collocation nodes in a 1D reference element. *Figure from Hidalgo and Dumbser (2011).*

Given a hyperbolic PDE
$\partial_t u + B(u)\partial_x u = S(u)$
we have to solve the equation:

$$\left(\int_{\Omega_k} \ell_j \ell_i dV\right)(u_{i,k}^{n+1} - u_{i,k}^{n}) + \int_{t^n}^{t^{n+1}} \int_{\Omega_k^\circ} \ell_j (B(q) \cdot \nabla q) dV dt + \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_k} \ell_j D(q^-, q^+) \cdot n dS dt = \int_{t^n}^{t^{n+1}} \int_{\Omega_k} \ell_j S(q) dV dt$$

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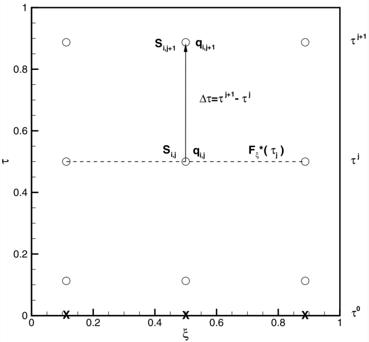

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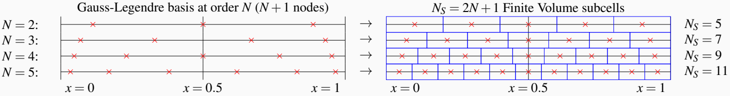
The ADER-DG approach

A few more details:

- Path-conservative approach[3] for the jump terms

$$\mathrm{D}(\mathrm{q}^{-},\mathrm{q}^{+}) = \frac{1}{2}\left(\int_0^1 \mathrm{B}(\boldsymbol{\psi})\cdot \mathrm{n}\mathrm{d}s\right)(\mathrm{q}^{+}-\mathrm{q}^{-}) - \frac{1}{2}\boldsymbol{\Theta}(\mathrm{q}^{+}-\mathrm{q}^{-})$$

- Switch to a WENO FV scheme in troubled cells



Placement of the predictor collocation nodes in

- (Fixed) mesh refinement for BH spacetimes

[3] *Pares (2006)*

The FO-CCZ4 formulation of Einstein's equations

Starting from the vacuum Efe

$$R_{\mu\nu} = 0\,,$$

switch from $g_{\mu\nu}$ to γ_{ij}, $K_{ij} \sim \partial_t \gamma_{ij}$, α, β^i
$\Rightarrow$ ADM formulation.

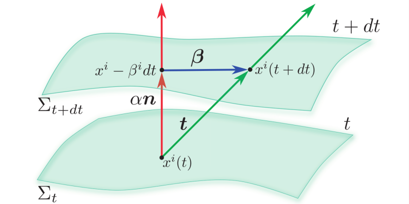
Split $\phi = (\det(\gamma_{ij}))^{-1/6}$, $\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij}$
$K = K_{ij}\gamma^{ij}$, $\tilde{A}_{ij} = \phi^2 (K_{ij} - \frac{1}{3} K \gamma_{ij})$
and also introduce $\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk}$
$\Rightarrow$ BSSN formulation *(Shibata and Nakamura, 1995; Baumgarte and Shapiro, 1999; Nakamura et al., 1987; Brown, 2009)*

Starting from the Z4 system:

$$R_{\mu\nu} + \nabla_{(\mu} Z_{\nu)} + $$
$$+ k_1 (n_{(\mu} Z_{\nu)} - (1 - k_2) g_{\mu\nu} n_\alpha Z^\alpha) = 0\,,$$
$$Z_\mu = 0$$

$\Rightarrow$ CCZ4 formulation *(Alic et al., 2012)*

Note however the Z4c formulation of Bernuzzi and Hilditch (2010)



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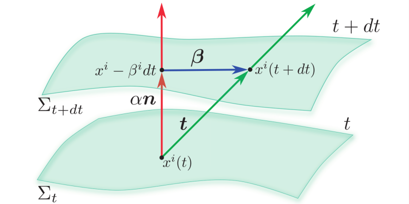


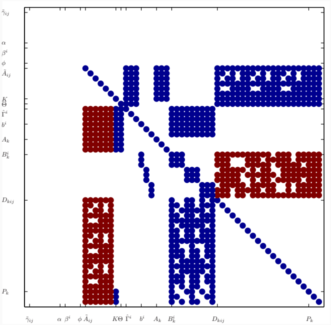
✓) Constraint damping

FO-CCZ4 is based on CCZ4, which includes a constraint-damping mechanism

✓) 1st order formulation

Achieved by introducing:

$$A_i = \partial_i \log\alpha\,, \quad B^i{}_k = \partial_k \beta^i$$
$$D_{ikj} = \frac{1}{2}\partial_k \tilde{\gamma}_{ij}\,, \quad P_i = \partial_i \log\phi$$



✓) Hyperbolicity

Achieved via the use of

- a fully non-conservative formulation: $\partial_t u + A^i \cdot \partial_i u = S(u)$
- appropriate recombinations of the second order ordering constraints

$$\mathcal{A}_{ik} = \partial_k A_i - \partial_i A_k = 0$$

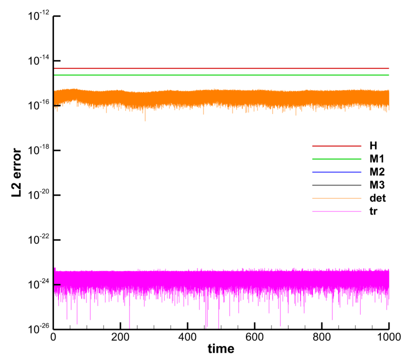
The FO-CCZ4 formulation

Properties of FO-CCZ4:

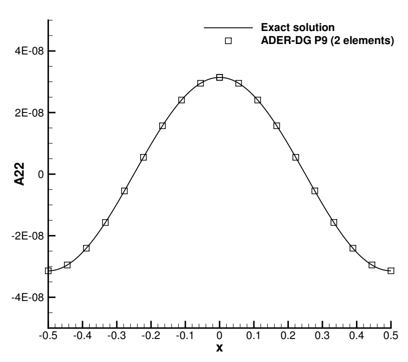
- 1st order, fully non-conservative
- manifestly linearly degenerate
- proven hyperbolic for all gauges (full set of eigenvectors and eigenvalues)
- constraint damping
- adjustable constraint propagation speed
- $\alpha, \phi > 0$ guaranteed by evolving $\log \alpha$ and $\log \phi$

Tests and applications

$$ds^2 = -dt^2 + dx^2 + [1 + \epsilon \sin(2\pi(x - t))]dy^2 + [1 - \epsilon \sin(2\pi(x - t))]dz^2 , \quad \epsilon = 10^{-8}$$



Constraints violations as function of time.



Solution profile of the $\tilde{A}_{yy}$ component of the traceless extrinsic curvature at $t = 1000$.

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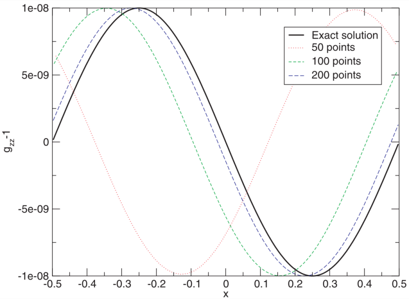
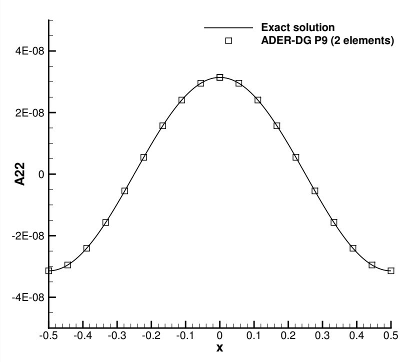
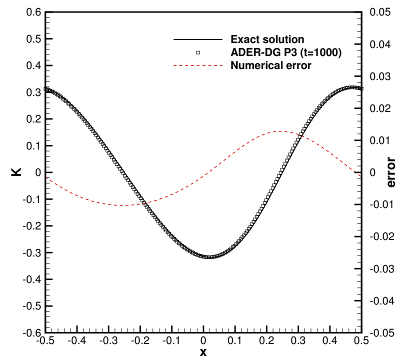


Solution profile of the γ_{zz} component of the metric at $t = 500$ for a FD code. *Figure from (Alcubierre et al., 2004).*

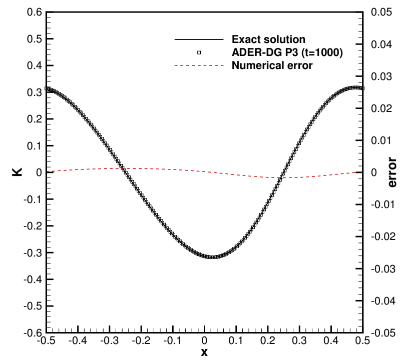


Solution profile of the $\tilde{A}_{yy}$ component of the traceless extrinsic curvature at $t = 1000$.

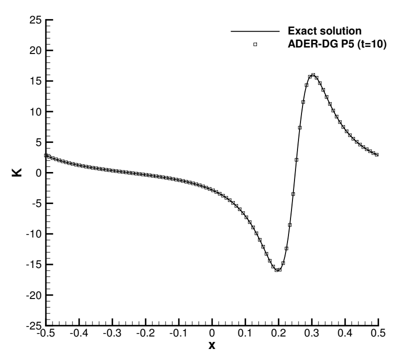
$$ds^2 = -H\,dt^2 + H\,dx^2 + dy^2 + dz^2\,, \quad H(t,x) = 1 - A\sin(2\pi(x-t))$$



Solution profile of the trace of the extrinsic curvature at $t = 1000$ for $A = 0.1$ and $e = 1$.



Solution profile of the trace of the extrinsic curvature at $t = 1000$ for $A = 0.1$ and $e = 2$.

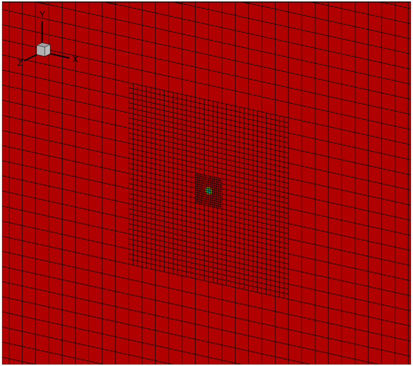


Solution profile of the trace of the extrinsic curvature at $t = 10$ for $A = 0.9$ and $e = 2$.

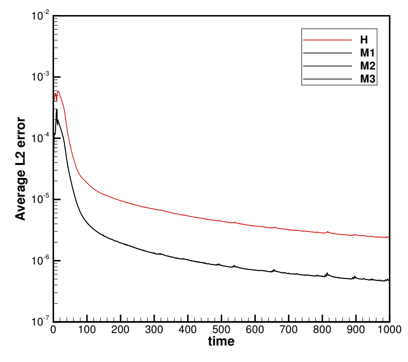
Resolution	Convergence order
	$N = 4$
60	
80	5.1
100	5.2
120	5.2
	$N = 7$
30	
40	8.4
60	8.0
80	8.8

Convergence order at $t = 10$ for $A = 0.9$ and $e = 2$.

Long term evolution of a puncture BH

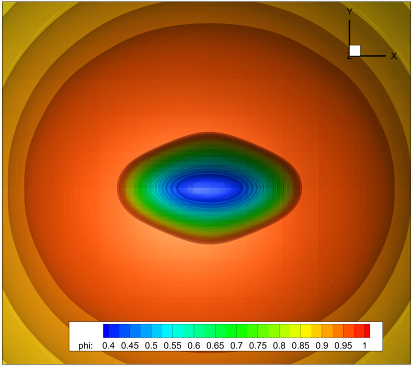


Solution profile of the lapse and grid setup

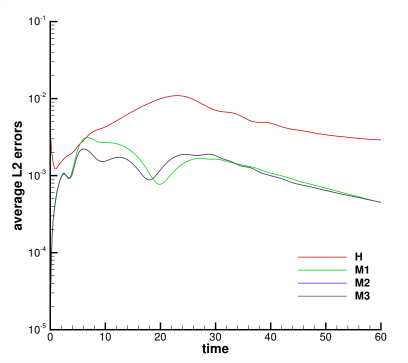


Constraints violations as function of time

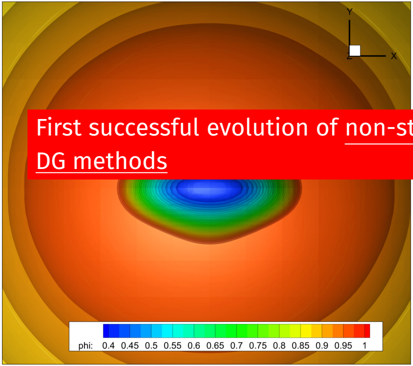
$$\bar{L}_2 = \sqrt{\frac{\int_\Omega \epsilon^2 dV}{\int_\Omega dV}}$$

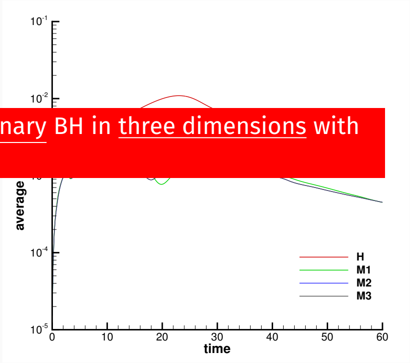


Solution profile of the conformal factor



Constraints violations as function of time





First successful evolution of non-stationary BH in three dimensions with DG methods

Solution profile of the conformal factor

Constraints violations as function of time

Conclusions

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The ADER(-DG) method is a promising framework, which can deliver

- stable evoutions (also against *e.g.* stiffness)
- high-order, accurate results
- performance improvement opportunities

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Testing of a FO-CCZ4/ADER-DG scheme yielded excellent results. Next step: include matter and microphysics.