General Relativistic Multidimensional Flux-Limited Diffusion Scheme for Neutrino Transport

Ninoy Rahman

Technical University of Munich
H.-Th. Janka
Max Planck Institute for Astrophysics
O. Just
Astrophysical Big Bang Laboratory, RIKEN Cluster for Pioneering Research

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Outline

- Motivation.
- General Relativistic Hydrodynamics code NADA (Montero 2013).
- FLD Neutrino Transport.
- Idealized test for transport.
- 1D CCSN simulation conducted with NADA-FLD code and comparison with M1 based ALCAR code (Just et al. 2015).

Motivation

- Core-collapse supernovae.
- Merger of black hole (BH) and neutron star (NS) and NS and NS.
- Evolution of hypermassive neutron star.

General Relativistic Hydrodynamics-GR

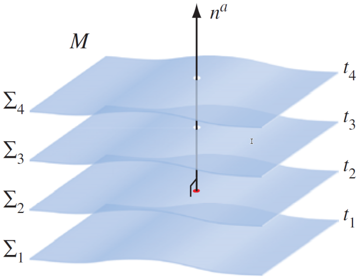


Figure 1: Space-like hyper-surface and time-like normal (Baumgarte and Shapiro 2010).

- Einstein equation is solved dynamically in 3+1 decomposition.

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (1)$$

- Hyperbolic formalism by Baumgarte, Shapiro, Shibata and Nakamura.
- Solved in Spherical polar coordinate.
- Time integration is done by 2nd order partially implicit runge-kutta method (Baumgarte 2012).

General Relativistic Hydrodynamics-Hydro

- Local conservation of matter current density and energy-momentum tensor.

$$\begin{aligned} \nabla_a J^a &= 0 \\ \nabla_a T^{ab} &= S_\nu \\ \nabla_a J^a_e &= Q_\nu . \end{aligned} \quad (2)$$

- Finite difference method (Montero 2013).
- PPM, CENO, MP5 for reconstruction of cell interface values from cell centered values.
- Solved by High resolution shock capturing method of Harten Lax and van Leer.
- Micro-physical equation of state.

Neutrino Transport (Flux Limited Diffusion)

- Co-moving orthonormal frame.
- Solves Flux limited diffusion (FLD) scheme for neutrino transport.
- In FLD, 0th moment of neutrino distribution function is dynamically evolved.
- Neutrino Flux $\mathcal{H}^{\hat{i}} = -\lambda D e^{i\hat{i}} \partial_i J$.
- Flux limiter, λ, smoothly interpolates between diffusive and free-streaming region.
- Flux limiter ensures causality is not violated. Flux: $F_i \leq cJ$.
- Levermore-Pomraning limiter.
- In deriving the FLD flux one assumes $\partial_t F_i = 0, v_i = 0$ and flat metric.
- GR correction to FLD flux:

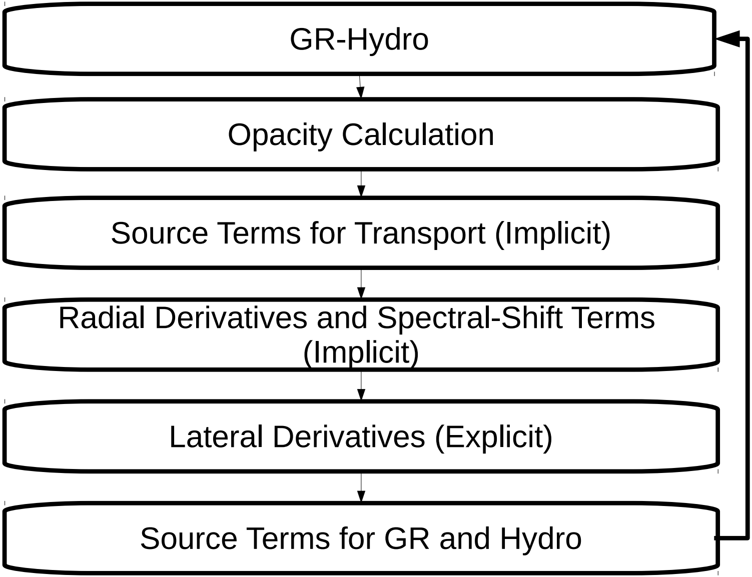
$$\mathcal{H}^{\hat{i},\alpha} = -\lambda D e^{i\hat{i}} \alpha^{-3} \partial_i (\alpha^3 \mathcal{J}) . \tag{3}$$

Neutrino Transport (FLD Equation)

$$\partial_t(\sqrt{\gamma}\mathcal{J}) + \partial_j(\sqrt{\gamma}v^j\mathcal{J}) - \partial_j(\sqrt{\gamma}\lambda D\partial^j\mathcal{J}) - \\ +[R_\epsilon - \frac{\partial}{\partial\epsilon}(\epsilon R_\epsilon)] = \sqrt{\gamma}\kappa_a(\mathcal{J}^{eq} - \mathcal{J}) \tag{4}$$

- 2nd term(lhs): advection.
- 3rd term(lhs): diffusion term.
- 4th term: velocity dependent term responsible for doppler and gravitational redshift.
- 5th term(rhs): absorption and emission source term.

Neutrino Transport (Flowchart)



Neutrino Transport (FLD Equation)

- Apply operator and directional splitting.
- Source terms are solved by iterative Newton-Raphson method along with temperature and electron fraction.
- Terms containing radial derivatives and $[R_\epsilon - \frac{\partial}{\partial \epsilon}(\epsilon R_\epsilon)]$ are solved by 2nd order implicit Crank-Nicolson method.
- Terms containing lateral derivatives are solved by explicit first order Allen-Cheng method (Allen 1970) or second order multi-stage Runge-Kutta-Legendre method (Meyer 2014).

Neutrino Transport (Numerical Scheme for lateral transport)

- Allen Cheng method (AC)
  - Pro:
    - Unconditionally stable for diffusion term.
    - Easily parallelizable and low computational cost.
  - Con:
    - Accuracy drops when integration time scale $\Delta \mathrm{t}$ is above diffusion timescale $\mathrm{t}_{\mathrm{diff}}$.
  - $r_{\mathrm{diff}} = \frac{\Delta t}{t_{\mathrm{diff}}} < 1$ is desirable.
- Runge-Kutta-Legendre method (RKL2)
  - Pro:
    - Conditionally stable for diffusion term, $r_{\mathrm{diff}} \leq s^2/8$, where s=number of stages.
  - Con:
    - Relatively more expansive than AC method.

Transport Test-Hemispheric difference test

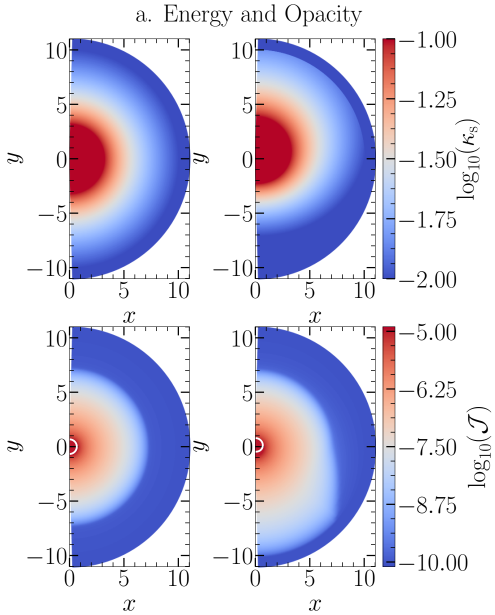
- We consider radiation diffusing out of a static scattering atmosphere in 2D with scattering opacity

$$\kappa_{\rm s}(r) = \frac{1}{r^2}. \tag{5}$$

- Spherical polar coordinates.
- The problem is initialized with a constant value of $\mathcal{J}(r, t = 0) = 10^{-10}$.
- At $r = 0.01$ the "fixed" boundary condition is applied with $\mathcal{J}(r = 0.01, t) = 1$.
- Spherical symmetric opacity version.
- Dipolar version:

$$\kappa_{\rm s}(r, \theta) = \kappa_{\rm s}(r)(1 + 0.5\cos\theta). \tag{6}$$

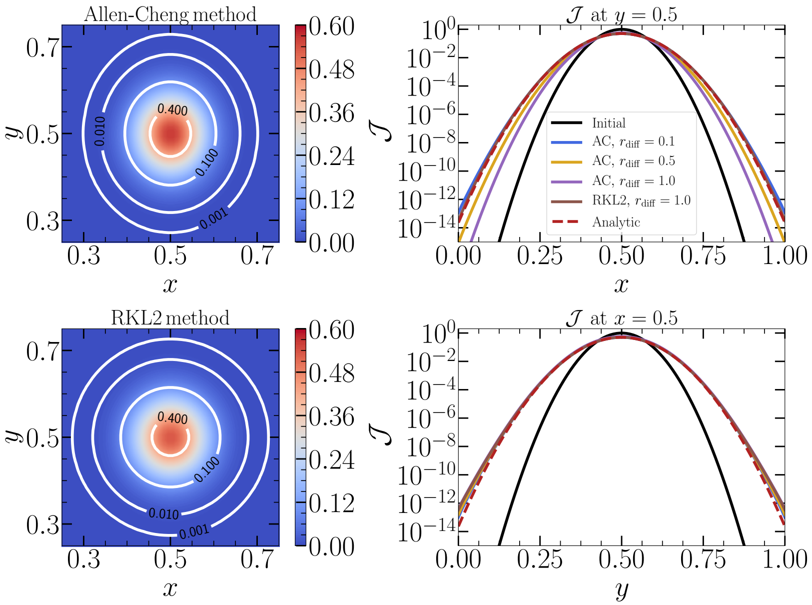
Transport Test-Hemispheric difference test



Transport Test-Diffusion of Gaussian Pulse

- Diffusion of Gaussian Pulse in 2D.
- Analytical solution exists.
- Cartesian coordinates.
- Constant scattering opacity $\kappa_{\rm s} = 10^3$.
- The Allen-Cheng or RKL2 method is applied along the x-direction and the Crank-Nicolson method along the y-direction.
- $r_{\rm diff} = \frac{\Delta t}{t_{\rm diff}} = 0.1, 0.5, 1.0$.

Transport Test-Diffusion of Gaussian Pulse



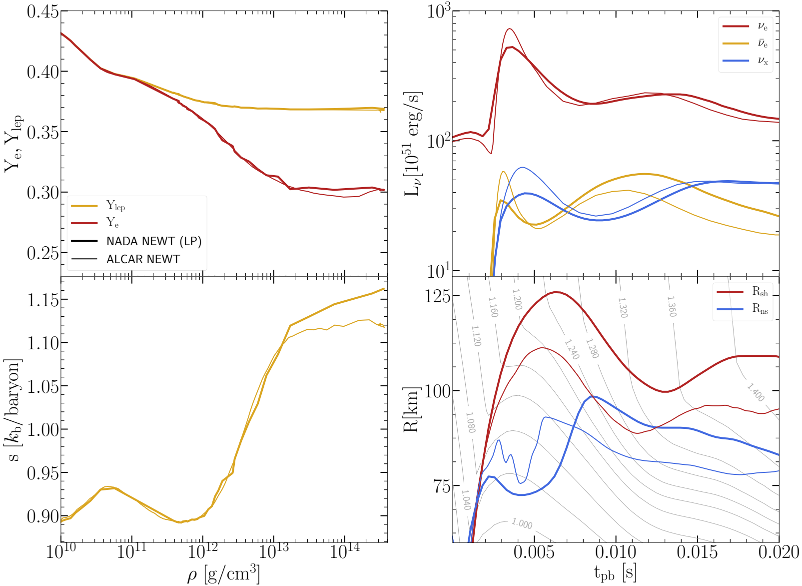
Neutrino Transport Test-CCSN (Setup)

- 1D Radiation-Hydrodynamics simulations with general relativistic and Newtonian gravity.
- $20 M_{\odot}$ ZAMS (Woosley and Heger, 2007).
- Radial grid points = 600
- Nuclear EOS SFHo (Steiner et al. 2012)

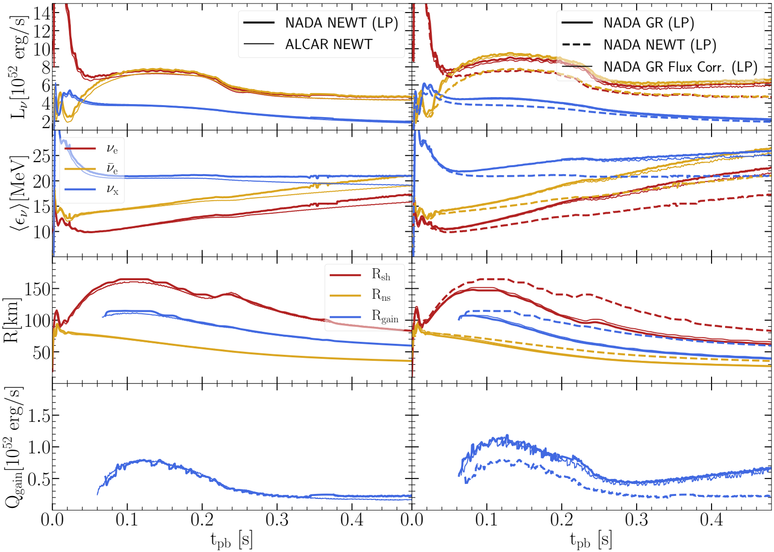
Reaction	Neutrino
$\nu + A \leftrightarrow \nu + A$	$\nu_{\mathrm{e}}, \bar{\nu}_{\mathrm{e}}, \nu_{\mathrm{x}}$
$\nu + N \leftrightarrow \nu + N$	$\nu_{\mathrm{e}}, \bar{\nu}_{\mathrm{e}}, \nu_{\mathrm{x}}$
$\nu_e + A \leftrightarrow e^- + A'$	ν_{e}
$\nu_e + n \leftrightarrow e^- + p$	$\nu_{\mathrm{e}}, \bar{\nu}_e$
$\bar{\nu}_e + p \leftrightarrow e^+ + n$	$\nu_e, \bar{\nu}_e$
$\nu + \bar{\nu} \leftrightarrow e^- + e^+$	ν_x
$\nu + \bar{\nu} + N + N \leftrightarrow N + N$	ν_x

Table 1: Neutrino opacities used for the CCSN simulations. "N" denotes nucleons and "A" and "A' " denote nuclei.

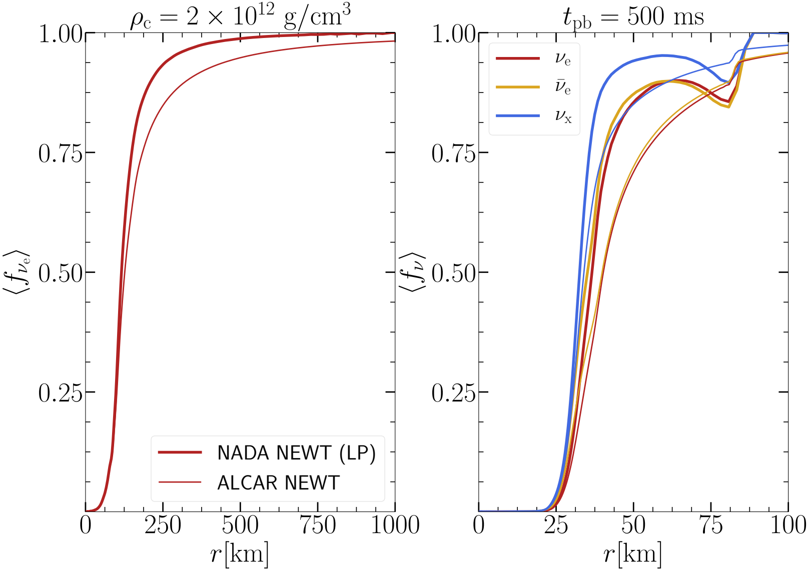
Neutrino Transport Test-CCSN (Code Comparison-Bounce)



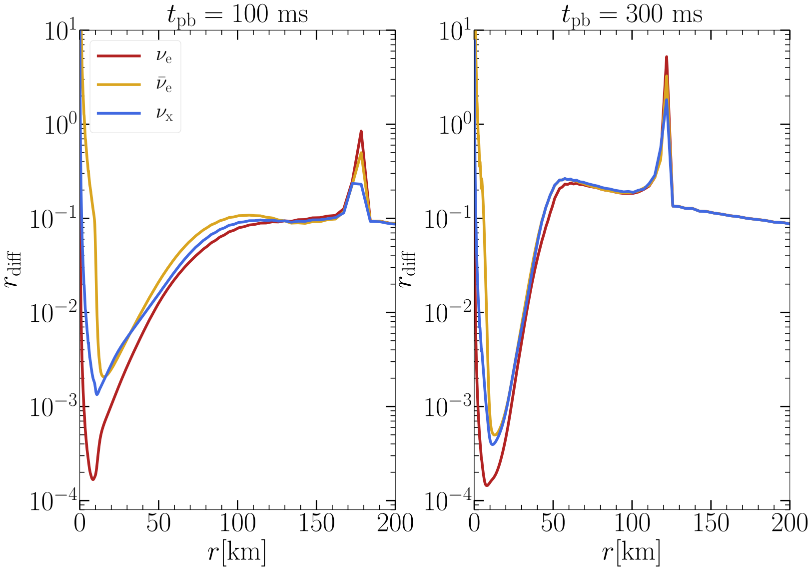
Neutrino Transport Test-CCSN (Code Comparison-Long Term)



Neutrino Transport Test-CCSN (Code Comparison-Flux Factor)



Neutrino Transport Test-CCSN (Code Comparison-Ratio of time step size and lateral diffusion timescale)



Summary

- Multidimensional general relativistic FLD.
- Mixed implicit-explicit methods are used for parallel efficiency and low computational cost.
- Most global properties agree well between FLD based NADA and M1 based ALCAR code.

Backup

$$
\frac{1}{\alpha}\frac{\partial}{\partial t}[W(\hat{\mathcal{J}}+\bar{v}_{\hat{i}}\hat{\mathcal{H}}^{\hat{i}})]+\frac{1}{\alpha}\frac{\partial}{\partial x^j}[\alpha W(v^j-\beta^j/\alpha)\hat{\mathcal{J}}]+\frac{1}{\alpha}\frac{\partial}{\partial x^j}[\alpha e^j_{\hat{i}}\hat{\mathcal{H}}^{\hat{i}}]
+\frac{1}{\alpha}\frac{\partial}{\partial x^j}\left[\alpha W\left(\frac{W}{W+1}v^j-\beta^j/\alpha\right)\bar{v}_{\hat{i}}\hat{\mathcal{H}}^{\hat{i}}\right]+\hat{R}_\epsilon-\frac{\partial}{\partial\epsilon}(\epsilon\hat{R}_\epsilon)
=\sqrt{\gamma}\int C\mathrm{d}\Omega. \quad (7)
$$

$$
R_\epsilon = \Bigg\{W\left[\mathcal{F}_j\frac{\partial v^j}{\partial\tau}+\mathcal{S}_j{}^k\frac{\partial v^j}{\partial x^k}+(\mathcal{F}^j-\mathcal{E}v^j)\frac{\partial\ln\alpha}{\partial x^j}
+v^j\mathcal{F}_k\frac{1}{\alpha}\frac{\partial\beta^k}{\partial x^j}+\mathcal{S}^{jk}(\frac{1}{2}v^l\frac{\partial\gamma_{jk}}{\partial x^l}-K_{jk})\right]
-(\mathcal{E}-v^i\mathcal{F}_i)\frac{\partial W}{\partial\tau}-(\mathcal{F}^j-\mathcal{S}_k{}^jv^k)\frac{\partial W}{\partial x^j}\Bigg\}. \quad (8)
$$