From Random Numbers to the Physics of Hadrons

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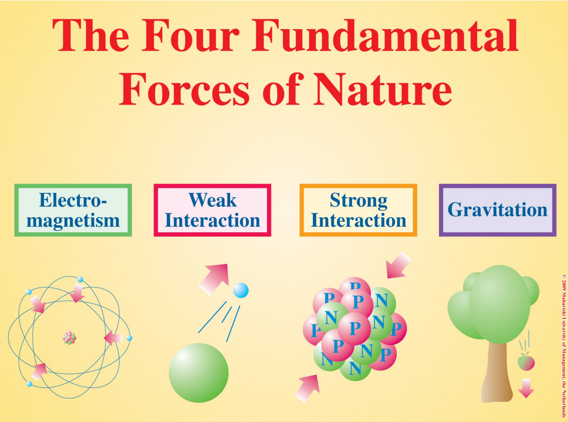
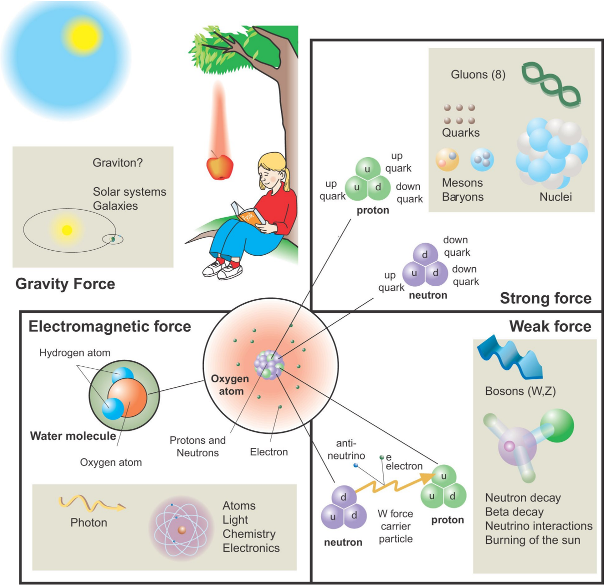
Fundamental forces . . . for managers



Credits: Stichting Maharishi University of Management, the Netherlands

Fundamental forces . . . for physicists

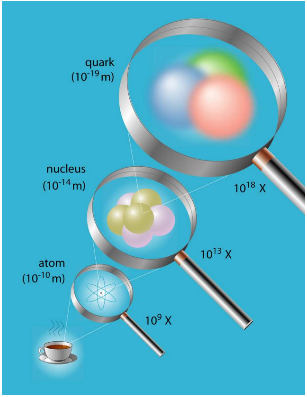


Credits: www.nobelprize.org

What makes us confident about the existence of quarks?

Fun facts:

- Neither quarks nor gluons ever observed as free particles (confinement)
- Only hadrons can be detected (bound states of quarks and gluons)
  - Mesons: $\bar{q}q$-states ($\pi^{\pm 0}$, $K^{\pm}$, ...)
  - Baryons: qqq-states (p, n, ...)
  - Anti-Baryons: $\bar{q}\bar{q}\bar{q}$-states ($\bar{p}$, ...)
- Typical hadron size: ~ 1 fm



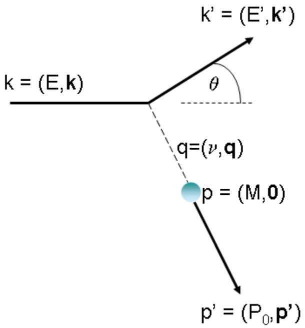
Wrap it up!

- Scattering experiments are our "magnifying-glass"
- Ab initio QCD calculation agree with experimental observations (will show you examples)

Scattering experiments

Elastic $e^- p \to e^- p$ scattering
(Considered as Coulomb scattering)

- both point-like particles
- no recoil of proton ($\vec{p}' = 0$)



Rutherford Scattering
(point-like, spin-less particles, non-relativistic)

$$\frac{d\sigma}{d\Omega}_{Rutherford} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2}$$

Mott-Scattering
(e^- with spin and relativistic)

$$\frac{d\sigma}{d\Omega}_{Mott} \simeq \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2}$$

Dirac-Scattering (both have spin-1/2)

Scattering experiments

Elastic $e^- p \to e^- p$ scattering

- Proton: extended object at rest (charge distributed, no recoil)

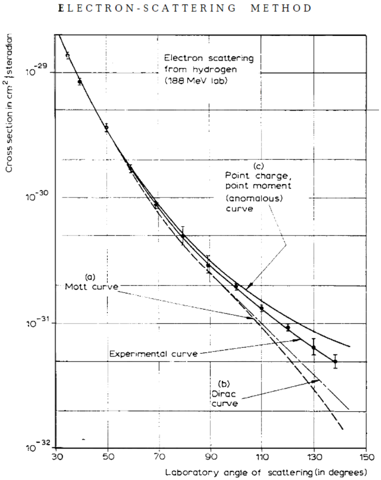
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{Mott} |F(\vec{q}^{\,2})|^2$$

- $F(\vec{q}^{\,2})$ form factor is Fourier transform of charge distribution

$$F(\vec{q}^{\,2}) = \int \rho(\vec{r}) e^{i\vec{q}\vec{r}} d^3 r$$

- At low $\vec{q}^{\,2}$, deviations from 1 measure for proton charge radius

$$F(\vec{q}^{\,2}) \xrightarrow{\vec{q}^2 \simeq 0} 1 - \frac{1}{6}\vec{q}^2 \vec{r}^2 + \ldots$$



Hofstadter (Nobel lecture 1961)

- Experiment by Hofstadter 1954: First evidence proton has a finite size ($\sim 0.74\ fm$)

Scattering experiments

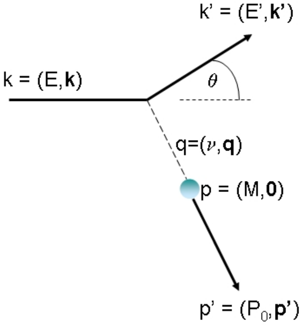
Elastic $e^- p \to e^- p$ scattering Rosenbluth-Formula (Lorentz-invariant)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \frac{E'}{E} \left(\frac{G_E{}^2 + \tau G_M{}^2}{1+\tau} \cos^2 \frac{\theta}{2} + 2\tau G_M{}^2 \sin^2 \frac{\theta}{2} \right) \qquad \tau = -\frac{q^2}{2M_P^2}$$

- Electron ($m_e = 0$) point-like, proton extended recoiling object, both have spin
- Two elastic form factors (functions of q^2)

$$G_E(q^2) \xrightarrow{\tau \ll 1} \int e^{i\vec{q}\vec{r}} \rho(\vec{r}) d^3 r \overset{(q^2=0)}{=} 1$$

$$G_M(q^2) \xrightarrow{\tau \ll 1} \int e^{i\vec{q}\vec{r}} \mu(\vec{r}) d^3 r \overset{(q^2=0)}{=} 2.79$$



- 4-momentum transfer

$$q^2 = (k - k')^2 \overset{(m_e=0)}{\simeq} -4EE' \sin^2 \frac{\theta}{2}$$

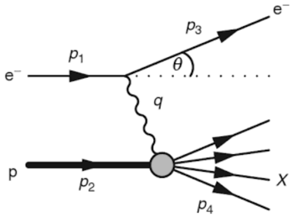
- Anomalous magnetic moment of proton another evidence for not being point-like
- But: does not prove substructure

High-energy scattering experiments

Inelastic scattering

- Collision become inelastic when $E_{e^-} > 1\,\text{GeV}$ (SLAC, 1968: $E_{e^-} = 5 - 20\,\text{GeV}$)
- Invariant mass: $p_4^2 = (p_2 + q)^2$ (virtual photon: $Q^2 = -q^2 > 0$)
- Two degrees of freedom, e.g.,

$$x = -q^2/(2p_2 \cdot q), \qquad y = q \cdot p_2 / p_1 \cdot p_2$$



Deep inelastic scattering ($Q^2 \gg m_p^2 y^2$)

$$\frac{d^2\sigma}{dxdQ^2} \simeq \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]$$

Observation

$$F_i(x,Q^2) \to F_i(x) \qquad \text{Bjorken scaling}$$

$$F_2(x) = 2F_1(x) \qquad \text{Callan-Cross relation}$$

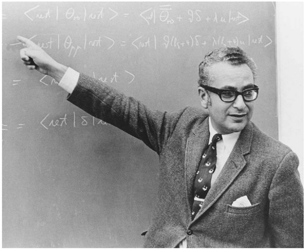
Photon scatters elastically from spin-1/2 constituent particles (partons) within proton

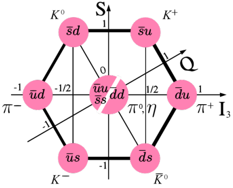
Quark Modell — Eightfold Way of Gell-Mann (1962)

- Many hadrons are known (quantum number, masses)
- Fit to representations of $SU_F(3)$ groups

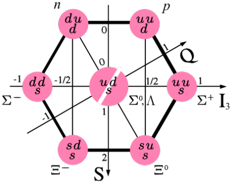
$$3 \otimes \bar{3}\ \text{(mesons)} \qquad 3 \otimes 3 \otimes 3\ \text{(baryons)}$$

- If so, particles (quarks) forming fundamental representation must exists.

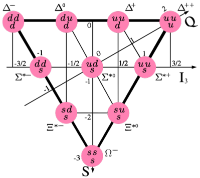




Spin-0 mesons (nonet)



Spin-1/2 baryons (octet)



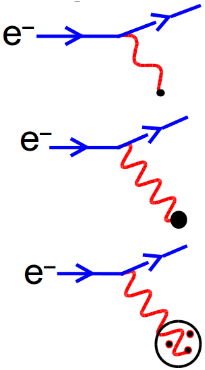
Spin-3/2 baryons (decuplet)

Today understood as flavor symmetry

Scattering experiments: Summary

$e^-p \to e^-p$ scattering: nature of interaction of the virtual γ with proton depends on wavelength (or Q^2)

- Scattering at very low e^- energies ($\lambda \gg r_P$) equivalent to scattering with point-like spin-less particle
- At low e^- energies ($\lambda \sim r_P$) scattering equivalent to that from extended charged object
- At high e^- energies the wavelength ($\lambda < r_P$) Scattering from constituent quarks (partons). Used to resolve sub-structure of proton.

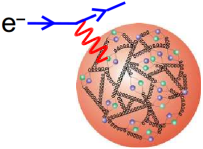


Partonic picture of proton result of SLAC deep inelastic scattering.

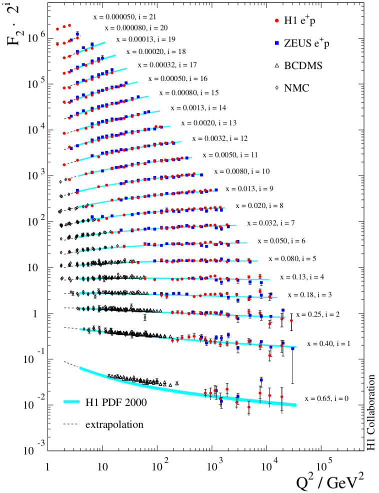
Scattering experiments: Fast-forward many years

At very high energies ($\lambda \ll r_P$):
Proton looks like sea of quarks & gluons.

$$\frac{d^2\sigma}{dxdQ^2} \simeq \frac{4\pi\alpha^2}{2xQ^4}[1+(1-y)^2]F_2(x,Q^2)$$



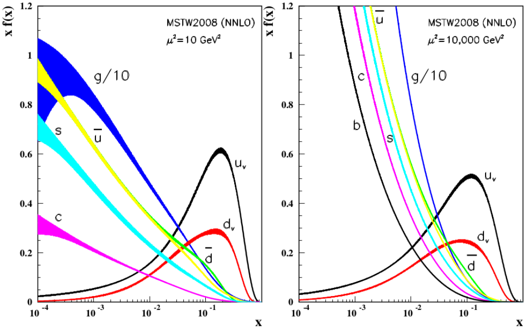
- F_2 not a delta-function (not all quarks have $x = 1/3$)
- Can calculate Q^2-dependence caused by dynamics of quarks and gluons



QCD: quarks have color charge and interact via colored gluons as described by QCD (theory of the strong interaction)

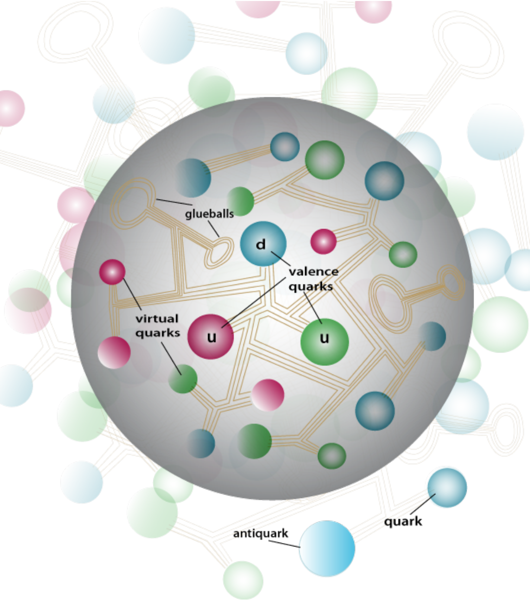
Parton distribution function

Probability distribution of quarks and gluons in (fast moving) proton with momentum fraction x



- Parton model: all quarks have $x = 1/3 \to F_2 =$ delta-function
- Extract $f(x)$ from measurements of proton / neutron structure function
- Cannot predicted them from the theory (nonperturbative), only approximate
- deviations of parton distribution with Q^2 well described by QCD

Proton



glueballs
d
valence
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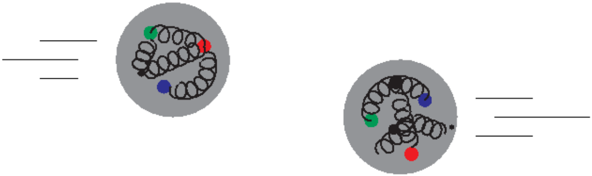
Strong-interaction process

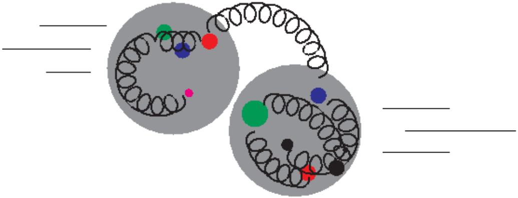
Animation of proton-proton collision



From Richard Ruiz `http://www.quantumdiaries.org/tag/qcd/`

Strong-interaction process

Animation of proton-proton collision



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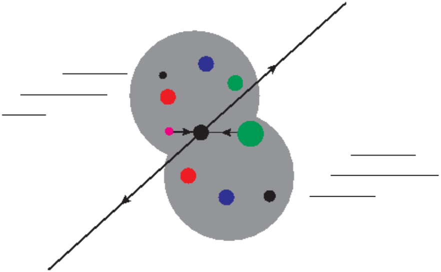

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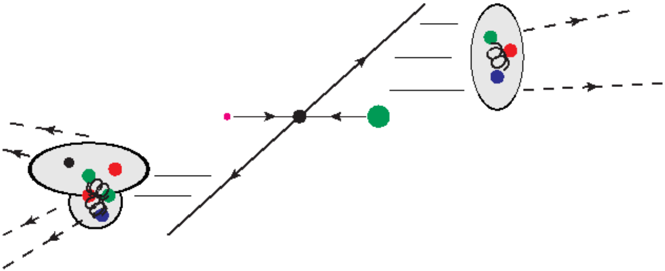

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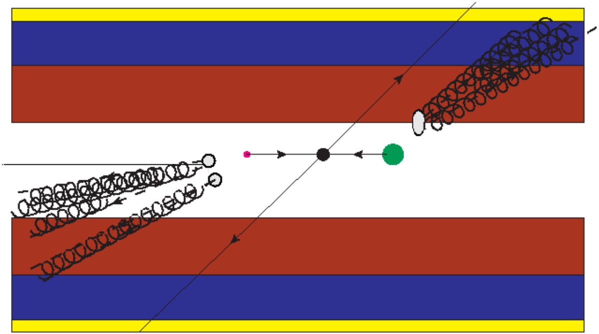

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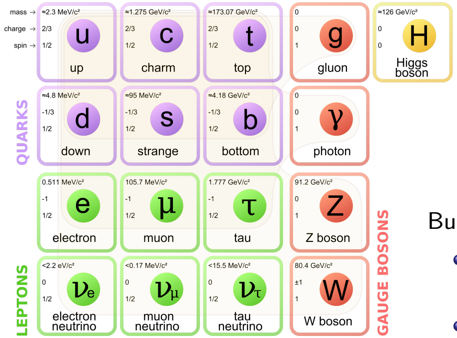
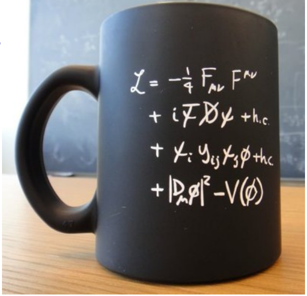
Standard model of elementary particle physics



(Credits: Wikimedia Commons)



(CERN Mug)

Building blocks:

- Quantum Chromodynamic (QCD)
  (strong interaction of quarks and gluons)
- Electroweak interaction
  (electromagnetic + weak interaction)
- Higgs sector
  (gives mass to ℓ, q, Z and $W^{\pm}$)

Quantum Chromodynamic (QCD)

Lagrange density describes dynamics of quarks and gluons (Euclidean)

$$\mathcal{L}_{\rm QCD} = \frac{1}{4} F^a_{\mu\nu}[A] F^{\mu\nu,a}[A] - \bar{\psi}\left(\gamma_\mu D_\mu[A] - m\right)\psi$$

- Field strength tensor: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu$
- Gluons: A^a_μ (massless gauge bosons, color-charge $\rightarrow$ self-int)
- Fermions: $\psi^f_a, \bar{\psi}^f_a$ (quark and anti-quark)
  (Flavour: $f = u, d, s, c, b, t$, Color: $a = r, g, b$)
- invariant under gauge transformations

Parameters (scale & ren-scheme dependent, fixed through experiment)

- Strong coupling constant: g_s
- Quark masses: $m = diag(m_u, m_d, m_s, \ldots)$

Action (Euclidean)

$$\mathcal{S}_{\rm QCD} = \int d^4x \mathcal{L}_{\rm QCD}$$

Path integral formalism

Quantization of a generic field theory

Partition function of a Euclidean QFT as a path integral (formally)

$$Z = \int D\Phi \, e^{-S(\Phi)}$$

- Φ denotes all fields of the theory, $S(\Phi)$ is the action
- "formally" because integral may not be defined a priori
- $D\Phi$ integration over all paths

  $\to$ on a finite lattice with N sites: $\int D\Phi \equiv \int \cdots \int \prod_{i=1}^{N} d\Phi_i$

Expectation value of an observable

$$\langle \mathcal{O} \rangle = \int D\Phi \, \mathcal{O}[\Phi] \, e^{-S(\Phi)}$$

Path integral formalism

Sketch for QCD

Partition function (formally)

$$Z = \int DA_\mu \int D\psi D\bar{\psi}\, e^{-(S_G+S_F)}$$

- $S_G \ldots$ Euclidean gauge action

$$S_G[A] = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}[A]F^a_{\mu\nu}[A],$$

- $S_F \ldots$ fermion action: $\psi^{(f)}$ where $f = \{u, d, s, \ldots\}$

$$S_F = \int dx^4\, \bar{\psi}\, \underbrace{(\gamma_\mu D_\mu + m_0)}_{\text{“fermion matrix“ } M}\, \psi \qquad \text{where} \quad D_\mu = \partial_\mu + ig_0 A_\mu$$

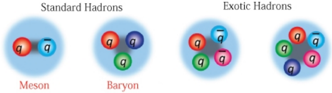
- “formally” because integral over DA not defined a prioi (gauge invariance)
  - have to fix a gauge (perturbation theory) → additional terms to action
  - physical observables independent of gauge
  - often covariant gauges (unique in the context of perturbation theory)

$$S_{gfix} = \frac{1}{2\xi^2}(\partial_\mu A_\mu)^2 + (\partial_\mu \bar{\chi})D_\mu \chi$$

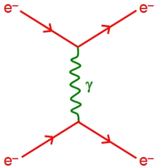
Main difficulties (compared to QED calculations)

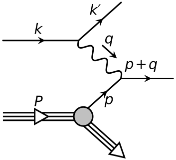
High-energy physics experiments (perturbative regime of QCD)

- Strong coupling much larger ($\alpha_s \sim 0.1$ vs. $\alpha \sim 1/137$)
- α_s constant grows for decreasing energy-momentum transfer
- Loop expansions much more involved (self-interactions of gluons)
- Hadrons are bound states (no free quark nor gluons ever observed)
  1. Mesons: $q\bar{q}$ $(\pi^{\pm 0}, \ldots)$
  2. Baryon: qqq $(P, N, \ldots)$



- quarks and gluons interact strongly, and electro-weak with the other elementary particles





Faces of Strong Interaction physics

Experimental Side

- Hadron properties
- Scattering experiments
- High-Temperature / density
- ...

Theory

- Quantum Chromodynamic (QCD) (QFT of strong interaction)
- Effective Models (Chirale Perturbation, HQET, ...)
- ...

Perturbative Methods

- extensive Loop-Calculations (algebraic, numerical)
- ...

Non-perturbative Numerical Methods

- Lattice QCD calculations (Discretized QCD)
- Numerically solving coupled QCD equations (truncated QCD)
  - Bound-state equations
  - Dyson-Schwinger equations
  - Functional RG
- ...

Lattice regularization of QCD (LQCD)

QCD action

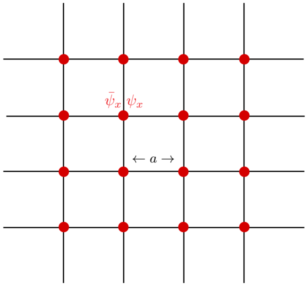
$$\mathcal{S}_{\mathrm{QCD}} = \int d^4x \left[-\frac{1}{2g_0^2} \operatorname{Tr} F^2[A] + \sum_f \bar{\psi}_x^f \left(\not{D}[A] - m_0^f \right) \psi_x^f \right]$$

Discretization of Euclidean space-time

- Introduce 4-dim lattice $L^3 \times T$

$$x = n\,a^4, \quad n \in \mathbf{Z}^4$$

- Quark fields ψ,$\bar{\psi}$ dwell on sites
- But: naive discretization of gluon field $A_\mu(x)$ not gauge-invariant at finite a



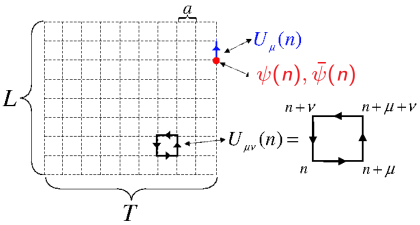
Lattice regularization of QCD (LQCD)

introduced by K. Wilson (1974-77)

Parallel transporter between x and $x + \hat{\mu}$

$$U_\mu(n) = \mathcal{P} e^{iag_0 \int_n^{n+\hat{\mu}} A_\mu(z) dz} \quad \in SU_c(3)$$

Using "links" $U_\mu(n)$ gives a gauge-invariant discretized action for any a.



Example: unimproved Wilson action $(a \equiv 1)$

$$\mathcal{S}^W_{\rm LQCD} = \overbrace{\beta \sum_{n,\mu<\nu} \left(1 - \tfrac{1}{3} \mathfrak{Re}\,\mathrm{Tr}\, \Box_{n,\mu\nu}\right)}^{\text{gauge part}} + \overbrace{\sum_{n,m,f} \bar{\psi}^f_n M^W_{nm}[U, \kappa_f] \psi^f_m}^{\text{fermionic part}} \quad \xrightarrow{a\to 0} \quad S_{QCD}$$

- Fermion matrix: $M^W_{nm}[U, \kappa_f] = \delta_{nm} - \kappa_f \sum_{\pm\mu} \delta_{m,n+\hat{\mu}} (1 + \gamma_\mu) U_\mu(n)$
- QCD Parameters: $\beta \equiv 6/g_0^2, \quad \kappa_f \equiv 1/(2m_0^f - 8)$

Other (improved) discretizations possible, requirement: $S_{\rm LQCD} \xrightarrow{a\to 0} S_{QCD}$

Lattice QCD calculation

Expectation value of an observable via path integral

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{n\mu} dU_{n\mu} \, \underline{d\bar{\psi}_n \, d\psi_n} \, \mathcal{O}[U, \bar{\psi}, \psi] \; e^{-(S_g[U] + S_f[U, \bar{\psi}, \psi])}$$

- No gauge-fixing needed! (compact gauge group)
- Integrate over quark fields exactly $\Longrightarrow$ modified action and integrand (non-local functional of links U)
- Left: **Master integral** for expectation value

$$\langle \mathcal{O} \rangle_{\beta,\kappa,L} = \frac{1}{Z} \int \prod_{n\mu} dU_{n\mu} \; F_{\mathcal{O}}[U] \; e^{-S_{eff}[U,\beta,\kappa]}$$

- Effective (partly integrated) action

$$S_{eff}[U; \beta, \kappa] = S_g[U; \beta] + \log \det M[U; \kappa]$$

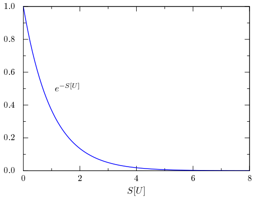
Lattice QCD calculation

Monte-Carlo calculation

Master integral is very-high dimensional integral

$$\langle \mathcal{O} \rangle_{\beta,\kappa,L} = \frac{1}{Z} \int \prod_{n\mu} dU_{n\mu}\ F_{\mathcal{O}}[U]\ e^{-S_{eff}[U,\beta,\kappa]}$$

- Number of integration variables
  ($N_c = 3$, $N_d = 4$)
  $64^3 \times 128 \quad \times (N_c^2 - 1) \times N_d \quad = 1\,073\,741\,824$
  $16^3 \times 32 \quad \times (N_c^2 - 1) \times N_d \quad = 4\,194\,304$
- Estimate integrals stochastically



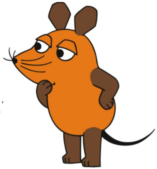
Monte-Carlo integration: Sample links U with probability density ("important sampling")

$$P[U^{(i)}] = \frac{1}{Z} e^{-S_{eff}[U^{(i)}]} \qquad \text{Markov chain:} \quad U^{(1)}, U^{(2)}, \ldots, U^{(N)} \quad \in SU(3)$$

Estimate for expectation value: $$\overline{\mathcal{O}} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{\mathcal{O}}[U^{(i)}] \quad \xrightarrow{N\to\infty} \quad \langle \mathcal{O} \rangle_{\beta,\kappa,L}$$

Monte-Carlo / Stochastic integration: How does it work?

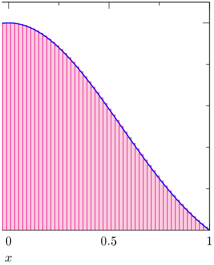
1-dim example



$$f(x) = (1 - x^2) \cdot e^{-x^2}, \quad I = \int_{-1}^{1} f(x)dx = \frac{1}{e} + \frac{\sqrt{\pi}}{2} Erf(1) \simeq 1.1147035$$

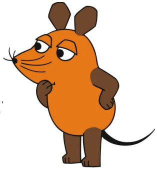
- Numerical integration with rectangular rule

$$I \simeq \sum_{i=1}^{N} f(x_i)\Delta x, \qquad \Delta x = \frac{x_2 - x_1}{N}$$



Monte-Carlo / Stochastic integration: How does it work?

1-dim example



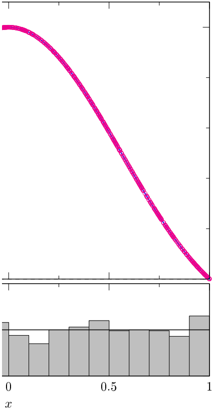
$$f(x) = (1 - x^2) \cdot e^{-x^2}, \quad I = \int_{-1}^{1} f(x) dx = \frac{1}{e} + \frac{\sqrt{\pi}}{2} Erf(1) \simeq 1.1147035$$

- Numerical integration with rectangular rule

$$I \simeq \sum_{i=1}^{N} f(x_i) \Delta x, \qquad \Delta x = \frac{x_2 - x_1}{N}$$

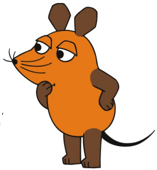
- Stochastic sampling of $x_i \in [-1 : 1]$ with $p(x) \propto 1$

$$I \simeq (x_2 - x_1) \cdot \frac{1}{N} \sum_{i=1}^{N} f(x_i) = (x_2 - x_1) \cdot \langle f \rangle_N$$



Monte-Carlo / Stochastic integration: How does it work?

1-dim example



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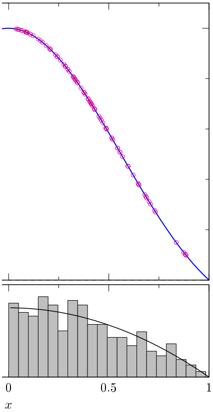
$$I \simeq (x_2 - x_1)\cdot \frac{1}{N}\sum_{i=1}^{N} f(x_i) = (x_2 - x_1)\cdot \langle f \rangle_N$$

- Improved Estimator: sample with $p(x) \propto (1-x^2)$

$$I \simeq (x_2 - x_1)\cdot \frac{1}{N}\sum_{i=1}^{N} p(x_i)\,\frac{f(x_i)}{p(x_i)} = (x_2 - x_1)\cdot \langle f/p \rangle_p$$



Efficiency: stochastic integration wins when dim > 10

Systematics

LQCD comes with the same parameters as QCD
(coupling, quark masses and volume)

$$\langle \mathcal{O} \rangle_{\beta,\kappa,L} = \frac{1}{Z} \int \prod_{n\mu} dU_{x\mu}\, F_{\mathcal{O}}[U]\, e^{-S_{eff}[U,\beta,\kappa]} = \left\langle F_{\mathcal{O}}[U^{(i)}] \right\rangle_{p(\beta,\kappa)}$$

Parameters: $\beta \equiv 6/{g_0}^2$, $\kappa_f \equiv 1/(2m_0^f - 8)$ tune them to "physical point", e.g.,

$$\frac{aM_2}{aM_1}(\beta,\kappa) \xrightarrow{\beta, L \text{ large}} R_{21}(\kappa) \xrightarrow{\kappa^f \to \kappa_c^f} \frac{M_2^{\text{phys}}}{M_1^{\text{phys}}}$$

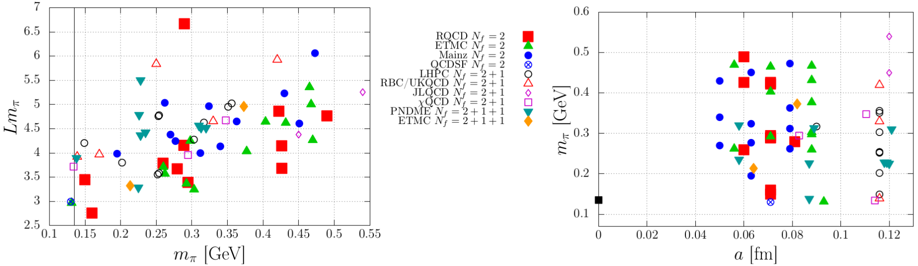
Access to lattice spacing via physical observable $\to a = a(\beta,\kappa) \approx a(\beta)$

Challenge: Control over systematic error when extrapolating to physical point

Landscape of lattice simulations

Status

- $N_f = 2$, $N_f = 2{+}1$, $N_f = 2{+}1{+}1$
- Lattice spacings: $a \geq 0.04$ fm, Volumes: $L \leq 6$ fm



compiled by S. Collins (Regensburg)

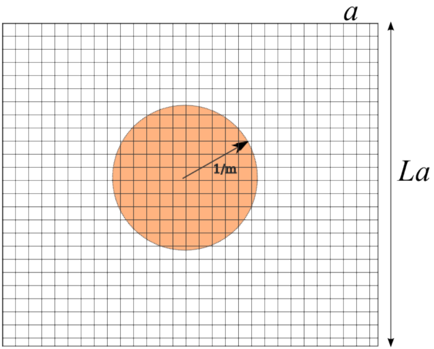
- Much progress in last ten years (algorithmic and computationally)
- Many lattice collaborations reach physical point (hadron masses)
- Safe extrapolation to continuum limit often remains (no always an issue)

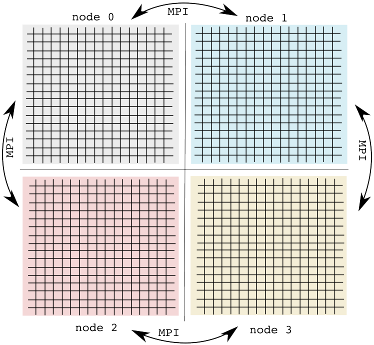
Calculations close to physical point very expensive

Monte-Carlo chain needs numerical inversion for every step: $[\not{D}(U) + m_f]^{-1}$

Require

- Volume large enough: Lightest hadron is pion with $m_\pi^2 \propto (m_u + m_d)$
- Lattice fine enough: discretization effects small





Rule of thumb: $Lm_\pi \geq 4$

~~Proton-Proton~~ "Lattice QCD" accelerator

Lattice QCD often requires supercomputers and highly efficient codes

Challenge and Progress in recent years

— Big computers are only one part of the success —

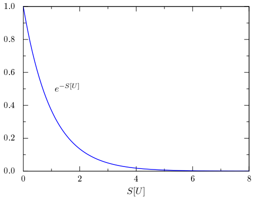
Lattice QCD calculation

Monte-Carlo calculation

Master integral is very-high dimensional integral

$$\langle \mathcal{O} \rangle_{\beta,\kappa,L} = \frac{1}{Z} \int \prod_{n\mu} dU_{n\mu}\; F_{\mathcal{O}}[U]\; e^{-S_{eff}[U,\beta,\kappa]}$$

- Number of integration variables
  ($N_c = 3$, $N_d = 4$)
  $64^3 \times 128 \quad \times (N_c^2 - 1) \times N_d \quad = 1\,073\,741\,824$
  $16^3 \times 32 \quad \times (N_c^2 - 1) \times N_d \quad = 4\,194\,304$
- Estimate integrals stochastically



Monte-Carlo integration: Sample links U with probability density ("important sampling")

$$P[U^{(i)}] = \frac{1}{Z} e^{-S_{eff}[U^{(i)}]} \qquad \text{Markov chain:} \quad U^{(1)}, U^{(2)}, \ldots, U^{(N)} \quad \in SU(3)$$

Estimate for expectation value: $\overline{\mathcal{O}} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{\mathcal{O}}[U^{(i)}] \xrightarrow{N\to\infty} \langle \mathcal{O} \rangle_{\beta,\kappa,L}$

Main numerical challenge

Solving a linear system

For each U (varies strongly) solve **linear system** again, again and again

$$D(U) \cdot \vec{\psi} = \vec{b} \qquad \rightarrow \qquad \vec{\psi} = D^{-1}(U) \cdot \vec{b}$$

$D(U)$: sparse matrix, $\mathbb{C}^{12V \times 12V}$ elements, mostly zero

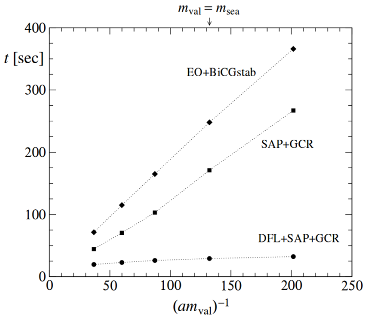
$\vec{b}$: $\mathbb{C}^{12V}$ elements, for $V = 64^3 \times 128 \mathrel{\hat{=}} 402\,653\,184$ elements

Numerical solution needed for:

- Monte-Carlo chain
- many observable
  (whenever quarks are part of the operator)

Problem:

- Simulation parameters β, κ direcly connected to phys. parameters
- Physical limit very expensive
  ("Critical slowing down")
- Reason: eigen vectors for smallest eigenvalues



Standard-Trick: Präkonditionierung

Solver

- typical: Krylov-space methods
- Number of iterations grows $\propto 1/\lambda_0$

Präconditioning

$$A \cdot \psi = b$$
$$LA \cdot R\psi = Lb$$

- if $LAR \simeq \mathbb{I}$, then $LA \cdot \tilde{\psi} = \tilde{b}$ found fast and so $\psi = R^{-1}\bar{\psi}$
- additional numerical costs vs. reduction of number of iterations

Typical methods used for lattice QCD

- Domain decomp. (checkerboard, DD, Hasenbusch trick)
- Multigrid (MG)
- Helps but can not to avoid "critical slowing down"

Deflation

- Determination of Eigen space for smallest EV $\lambda_0, \lambda_1, \ldots$

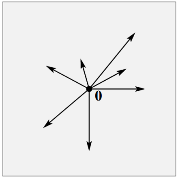
$$A\psi^{(n)} = \lambda_n b$$

- Find solution in remaining space (orthogonal to eigen space)
- helps a lot but similar expensive as original system

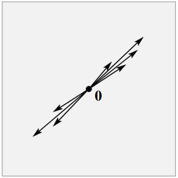
Empirical fact

Lowest EV for fermion matrix

- if reduced to small space region (4^4, 6^4), build lower dimensional eigen space
- EV are local coherent, reason unknown, but seen empirically
  (maybe connected to χ-Symmetry breaking in QCD)
- i.e. approximation of eigen space sufficient (inexact deflation)
- Combined with Domain-Decomposition (DD) or MG



incoherent



coherent

- Used in recent years in several algorithms, in particular AMG
- allows lattice QCD calculations at physical point (expensive but possible)

Examples for typical lattice QCD calculations

Extracting physics from LQCD calculations

Typical observables

- Plaquette or other Wilson loops
- Chiral condensates
- Hadronic properties
  (e.g., masses, decay constants, form factors, structure functions. . .)
- Fundamental parameters of QCD (strong coupling, quark masses)
- . . .

Will illustrate calculation of

1. Wilson loop: quark confinement
2. 2-point functions: Hadron masses
3. 3-point functions: Hadron structure

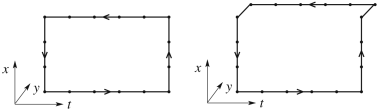
Extracting Wilson loops

Wilson loops

- Allow to estimate potential between (two) static color sources.
- Prototype of gauge-invariant observable (purely gluonic)
- Also used as operators for gluonic bound states (glueballs) (from exponential decay of correlation function)

Wilson Loop on gauge field configuration U

$$W[U] = \mathrm{Tr} \prod_{x,\mu \in \mathrm{loop(R,T)}} U_{x\mu}$$



Expectation value

$$\langle W \rangle_{R,T} = \frac{1}{Z} \int DU\, W[U] e^{-S_{\mathrm{eff}}[U]} = \frac{1}{N} \sum_{i=1}^{N} W[U^{(i)}] + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\propto \; e^{-tV(r)} \left(1 + O(e^{-t\Delta E})\right)$$

Static quark potential from expectation values of Wilson loops

Static quark potential

$$V(r) = A + \frac{B}{r} + \sigma r$$

- Force between quarks is dV/dr
- A irrelevant
- 2^{nd} term: Coulomb term with strength B
- 3^{rd} term: lin. rising potential
  σ is string tension ($\approx 900\,\mathrm{MeV/fm}$)

Linear rising potential:
energy rises the further quark and antiquark are pulled apart
(quark confinement)

Lattice Data for quen. approximation
(no sea quarks)

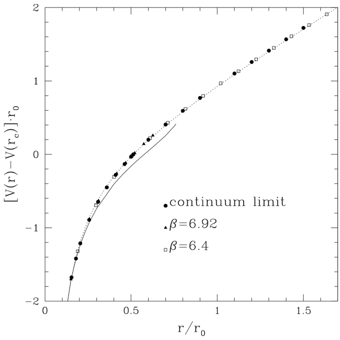


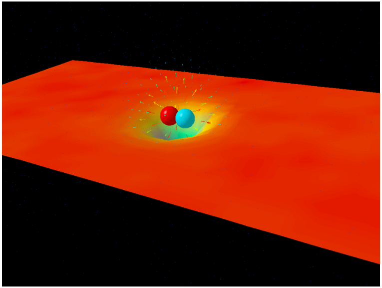
Figure : [Necco/Sommer, NPB622(2002)328] dashed line: bosonic string model, solid line pert. theory

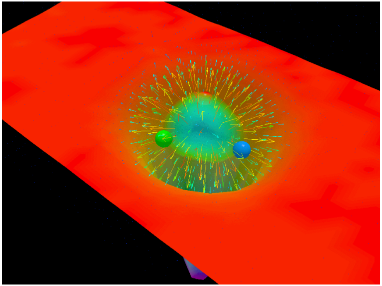
String formation

Meson and baryon fluxtubes (by Derek Leinweber)

See CSSM, Adelaide for animations

www.physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel





- Meson fluxtube (left)

  www.physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel/FluxTubeAnim2.gif

- Baryon fluxtube (right)

  www.physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel/VacuumRespAction16t32_Yshape8.gif

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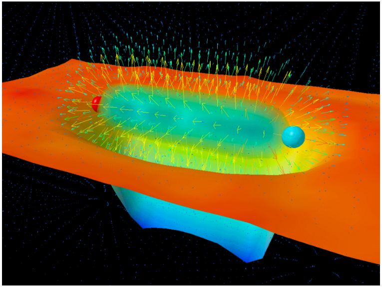

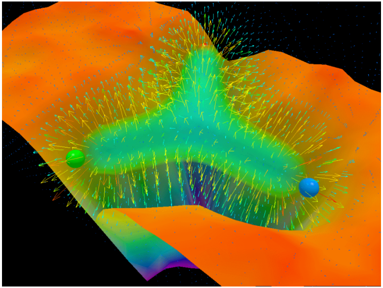

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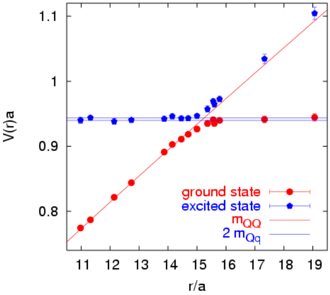
String formation and breaking

Full QCD: If QCD vacuum contains "sea quarks" (unquenched)

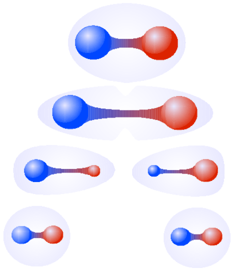
- creation of sea quark-antiquark pairs at sufficient energy
- flux tube breaks for large enough *r*

String-Breaking from Lattice QCD



SESAM-Collaboration: Bali et al. (2005)

Illustration:



Extracting physics from LQCD calculations

Typical observables

- Plaquette or other Wilson loops
- Chiral condensates
- Hadronic properties
  (e.g., masses, decay constants, form factors, structure functions. . .)
- . . .

Will illustrate calculation of

1. Wilson loop: quark confinement
2. 2-point functions: Hadron masses, decay constants
3. 3-point functions: hadron structure, renormalization etc.

Hadronic two- and three-point functions on the lattice

1. 2-point correlation functions

$$C^h_{2pt}(\vec{p},t) = \frac{1}{\sqrt{V_S}} \sum_{\vec{x}} e^{i\vec{x}\cdot\vec{p}} \left\langle \bar{h}(\vec{x},t)\, h(\vec{0},0) \right\rangle_U \overset{(\vec{p}=0)}{=} |Z_0|^2 e^{-m_0 t} + |Z_1|^2 e^{-m_1 t} + \ldots$$

2. 3-point correlation functions

$$C^{\mathcal{O}}_{3pt}(\tau,t,\vec{p},\vec{p}\,') = \frac{1}{\sqrt{V_S}} \sum_{\vec{x}} e^{i\vec{x}\cdot\vec{p}} e^{i\vec{z}\cdot\vec{\Delta}P} \left\langle h(t,\vec{x}) \,|\, \mathcal{O}(\tau,z) \,|\, \bar{h}(0,\vec{0}) \right\rangle_U$$

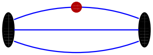
$$= prefactor(t,\tau,\vec{p},\vec{p}\,') \otimes \langle H|\mathcal{O}|H\rangle + \ldots$$

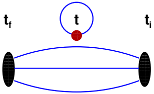
- Interpolation operators h have quantum number of hadron
  - Proton: $h_{p^+}(x) = \varepsilon^{abc} \left\{ u^{aT}(x) C\gamma_5 d^b(x) \right\} u^c(x)$
  - Pion: $h_{\pi^+}(x) = \bar{d}(x)\,\gamma_5\, u(x)$, $\qquad x = (\vec{x},t)$
- Insertion operator $\mathcal{O}$ chosen wrt. desired matrix element
  - Vector current: $\bar{\psi}\gamma_\mu\psi$
  - Axial-Vector current: $\bar{\psi}\gamma_\mu\gamma_5\psi$

Illustration: Measurement of a three-point function on the lattice

Connected and disconnected contribution

Diagrams





On the Lattice

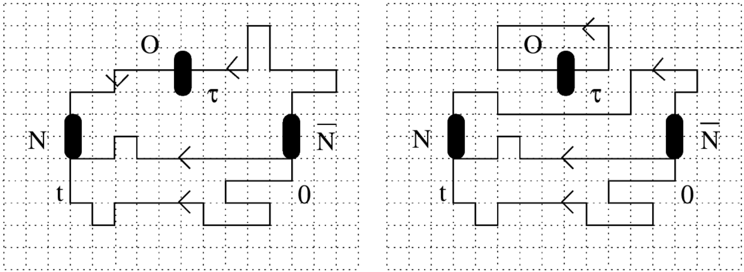


Figure : from R. Horsley (1999)

Hadron masses from the lattice

QCD works!

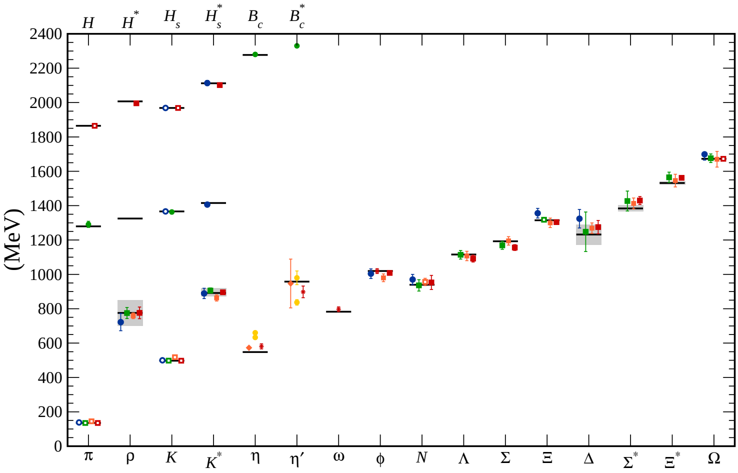
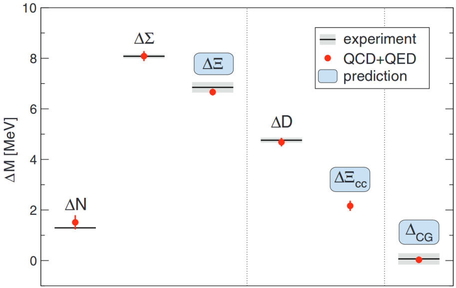


Figure : Hadron masses from lattice QCD: experimental vs. lattice QCD results (MILC, PACS-CS, BMW, QCDSF und RBC&UKQCD). From [Kronfeld 1203.1204].

Hadron masses splitting from the lattice

Lattice QCD works also when adding QED effects!



	Mass splitting [MeV]	QCD [MeV]	QED [MeV]
$\Delta N = n - p$	1.51(16)(23)	2.52(17)(24)	−1.00(07)(14)
$\Delta\Sigma = \Sigma^- - \Sigma^+$	8.09(16)(11)	8.09(16)(11)	0
$\Delta\Xi = \Xi^- - \Xi^0$	6.66(11)(09)	5.53(17)(17)	1.14(16)(09)
$\Delta D = D^\pm - D^0$	4.68(10)(13)	2.54(08)(10)	2.14(11)(07)
$\Delta\Xi_{cc} = \Xi_{cc}^{++} - \Xi_{cc}^{+}$	2.16(11)(17)	−2.53 (11)(06)	4.69(10)(17)
$\Delta_{CG} = \Delta N - \Delta\Sigma + \Delta\Xi$	0.00(11)(06)	−0.00 (13)(05)	0.00(06)(02)

Figure : Borsanyi et al., Science 347 (2015) 1452.

Extracting physics from LQCD calculations

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Will illustrate calculation of

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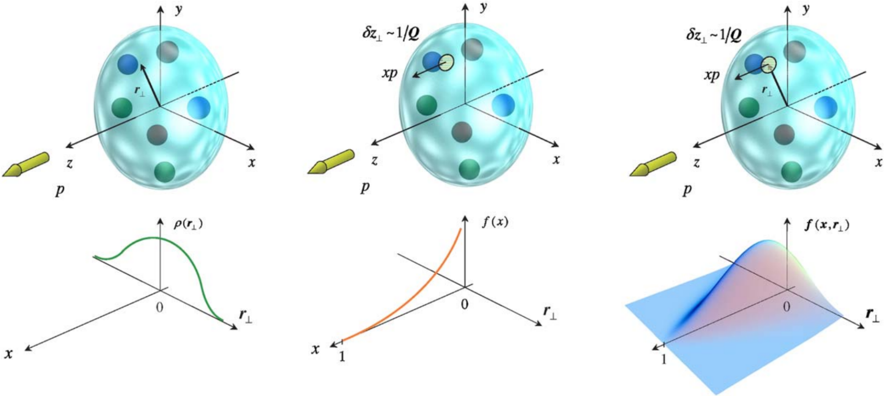
Hadron structure functions



Form factors: Spatial distribution of charge and magnetization

Partondistribution: Distribution of momentum and spin

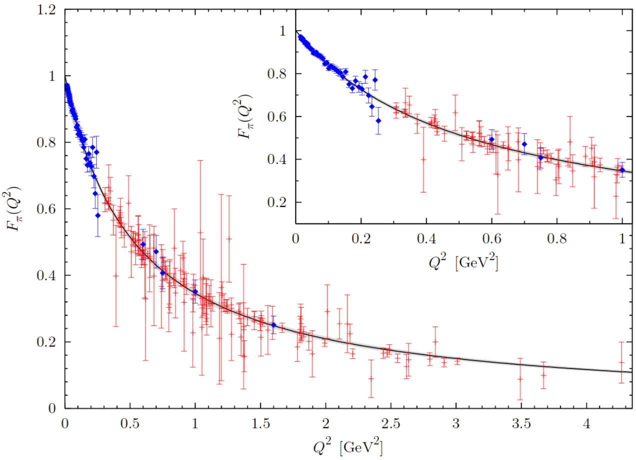
Generalized parton distribution (GPDs): correlated distribution of momentum, spin, charge

Goal: Determination of non-pertubative functions directly from QCD

- Form factors possible from lattice QCD
- GPDs, PDFs: only moments accessible so far (Ji'13: quasi-PDFs possible)

Pion form factor

Experiment vs. Lattice



Moments of Parton Distributions (PDFs)

Nucleon parton distribution function

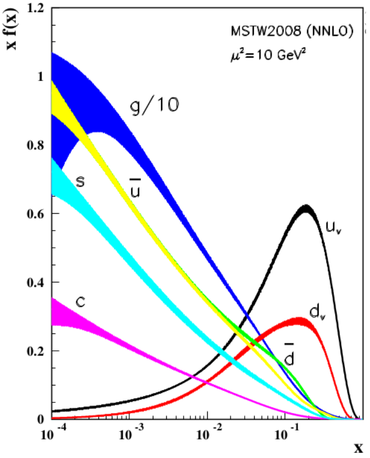
- Important for phenomenology
- Known for large range of $x = [0, 1]$ (long. mom. fraction)
- Good knowledge for u and d
- For other quarks not as good

Lattice QCD

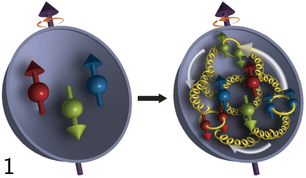
- Moments of nucleon PDF

$$\langle x^n \rangle_{(q)} = \int_0^1 dx\, x^n\, [q(x) + \bar{q}(x)]$$

- **New:** indirect access to $q(x)$ itself



Spin structure of the Nucleon



Naive Parton model

- Nucleon spin = sum of valence quark spins

$$\frac{1}{2} = \frac{1}{2}\overbrace{(\Delta u_v + \Delta d_v)}^{\Delta\Sigma=1} \quad \text{where} \quad \Delta u_v = \frac{4}{3}, \Delta d_v = -\frac{1}{3}$$

- "Spin puzzle/crisis": Eμ-Collaboration (1989) found: $\Delta\Sigma \approx 0.120$

Modern view

- Valence quarks contribute only a fraction
- Significant contribution also from gluons and sea quarks
- Also important: Orbital angular momentum L of quarks and gluons
- Total spin:

$$\frac{1}{2} = \frac{1}{2}\underbrace{(\Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s})}_{\Delta\Sigma} + L_q + \Delta g + L_g$$

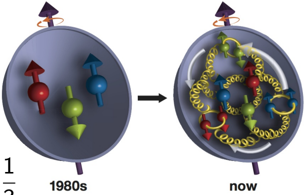
- Individual contributions can be calculated using lattice QCD

Bali et al., PRL108 (2012) 222001

$$\Delta s + \Delta\bar{s} = -0.020(10)(4)$$

$$\Delta\Sigma = \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s} = 0.45(4)(9)$$

Spin structure of the Nucleon



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- Moments of GPDs deliver total quark momentum

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L_q + \Delta g + L_g$$

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2. 2-point functions: Hadron masses
3. 3-point functions: Hadron structure

Quantum Chromodynamic $\mathcal{L}_{\text{QCD}} = \frac{1}{4}F^2[A, g] - \bar{\psi}\,(\gamma_\mu D_\mu[A, g] - m)\,\psi$

Regularisierung & Renormierung

QCD parameters g and $m = \text{diag}(m_u, m_d, \ldots)$ are scale- and scheme-dependent

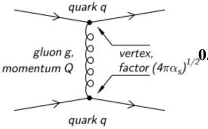
$\alpha_s = g^2/4\pi$

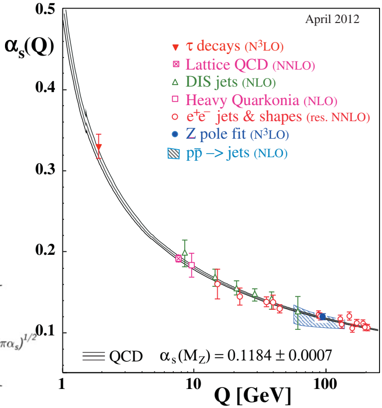
Strong coupling

- Not a constant, depends on scale μ and renormalization scheme S

$$\mu^2 \frac{\partial g(\mu^2)}{\partial \mu^2} = \beta_S[g(\mu^2)]$$

- $g_S(\mu) \xrightarrow{(\mu\to\infty)} 0$ "asymptotic freedom"
- Precise value essential (QCD phenomenology)
- Convention: $\alpha_{\overline{MS}}^{N_f=5}(M_Z)$

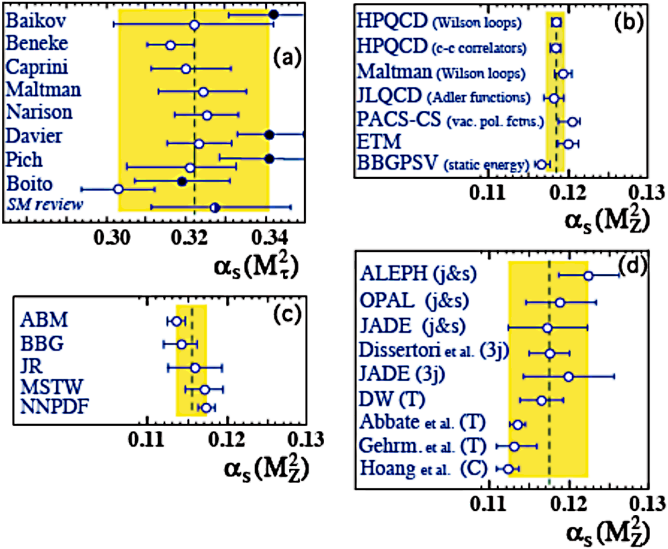




Access Experiment (indirect), Lattice QCD (direct)

Strong coupling from QCD phenomenology and lattice QCD

Bethke (2015)



Wrapping up

Lattice QCD

- Provides numerical access to many quantities of strong interaction physics
- Systematically improvable, becomes QCD in the respective limits
- Needs usage of supercomputers and parallel programming
- Success nicely demonstrates correctness of QCD

— Thank you for your attention! —