UNIVERSITÄT
LEIPZIG

based on arXiv:2206.05073

Hadamard property of the Unruh state on Kerr-de Sitter spacetime

October 4, 2022

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Outline

The Unruh state

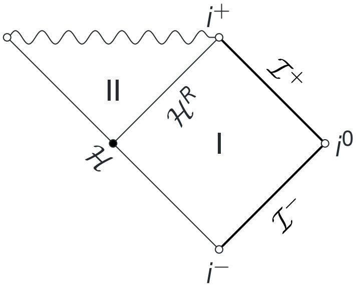
The Kerr-de Sitter spacetime

The Unruh state on Kerr-de Sitter

Hadamard Property

THE UNRUH STATE

States on Schwarzschild spacetime



- Boulware: vacuum in exterior I
- Hartle-Hawking: KMS-state at Hawking temperature
- Unruh: non-equilibrium stationary state. Empty at $\mathcal{I}^-$ with thermal energy flux at $\mathcal{I}^+$

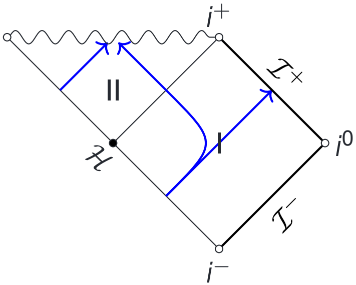
⇒ captures late-time behaviour in collapse

⇒ extends naturally to Kerr-de Sitter

Rigorously constructed states

- Hartle-Hawking state in spacetimes with
  - static bifurcate Killing horizons [Sanders: 2013]
  - stationary bifurcate Killing horizon [Gérard: 2021]
- Unruh state on
  - Schwarzschild spacetimes [Dappiaggi, Moretti, Pinamonti: 2009]
  - Schwarzschild-de Sitter spacetimes [Brum, Jorás: 2014]
  - Reissner-Nordström-de Sitter spacetimes [Hollands, Wald, Zahn: 2019]
- Unruh state for massless free fermions on Kerr [Gérard, Häfner, Wrochna: 2020]

The Unruh state



- Scalar field: $\mathcal{K}\phi = \Box_g \phi = 0$
- Quantization: expansion in modes $\psi_{\omega J}$
  $$\phi(x) = \sum_J \int_0^\infty d\omega \psi_{\omega J}(x) a_{\omega J} + \overline{\psi_{\omega J}}(x) a^\dagger_{\omega J}$$
- $\psi_{\omega J}$: Positive frequency modes w.r.t. affine parameter of null geodesics on $\mathcal{H}$ and $\mathcal{I}^-$
- Unruh vacuum: $a_{\omega J}|0\rangle_U = 0$

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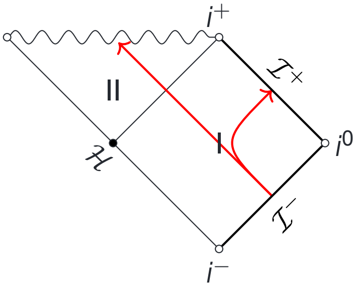

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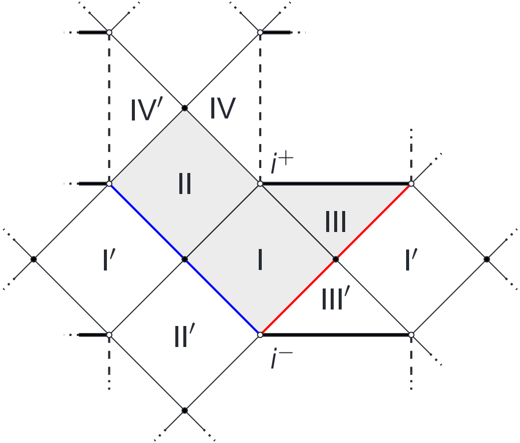
THE KERR-DE SITTER SPACETIME

The Kerr-de Sitter spacetime



$$g = \frac{\Delta_\theta \sin^2\theta}{\rho^2\chi^2}(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\varphi)^2 + \frac{-\Delta_r}{\chi^2\rho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 + \rho^2\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right),$$

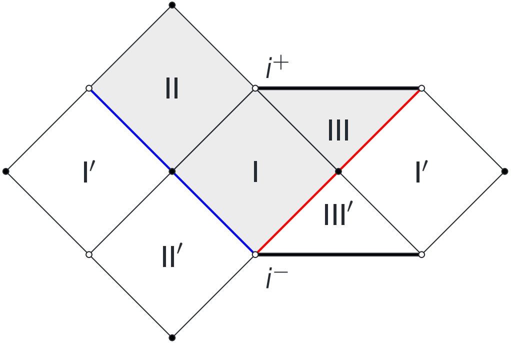
$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\chi = 1 + a^2\Lambda/3,$$

$$\Delta_\theta = 1 + a^2\Lambda/3\cos^2\theta,$$

$$\Delta_r = (1-\Lambda/3r^2)(r^2+a^2) - 2Mr$$

The Kerr-de Sitter spacetime - coordinates



- Horizons: $r_- \sim \mathcal{CH}$, $r_+ \sim \mathcal{H}$ and $r_c \sim \mathcal{H}_c$

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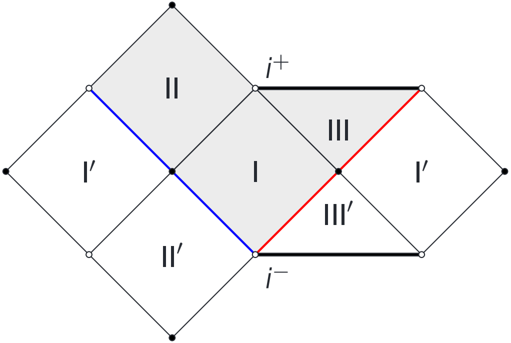

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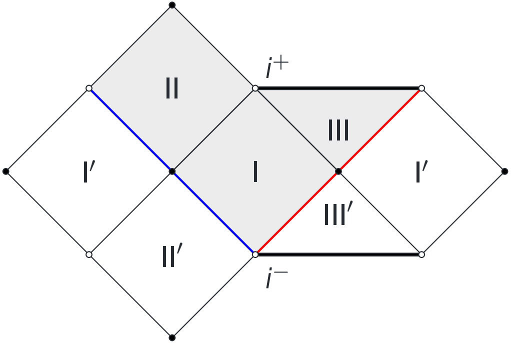

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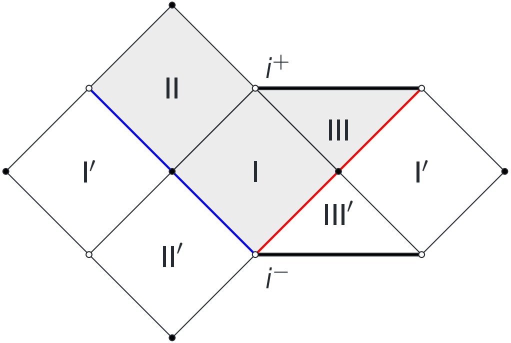

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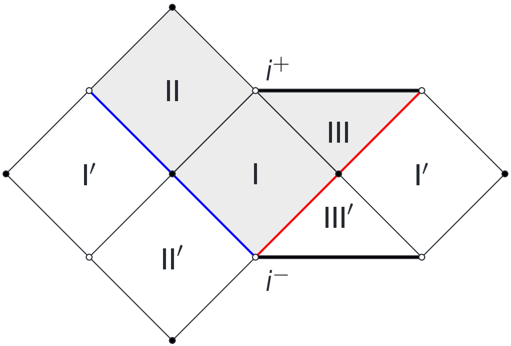
- $v = t + r_*$, $u = t - r_*$,

  with $dr_* = \frac{\chi(r^2+a^2)}{\Delta_r} dr$
- $U_+ = -e^{-\kappa_+ u}$, $V_+ = e^{\kappa_+ v}$,

  $U_c = e^{\kappa_c u}$ and $V_c = -e^{-\kappa_c v}$,

  with $\kappa_i = \frac{|\partial_r \Delta_r|_{r=r_i}}{2\chi(r_i^2+a^2)}$

The Kerr-de Sitter spacetime



Spacetime M (gray) and its extension $\tilde{M}$

Lemma

M and $\tilde{M}$ are globally hyperbolic.

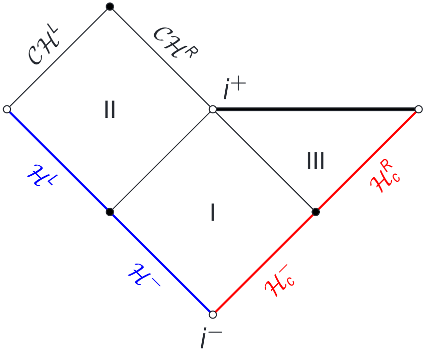
Lemma

If a, Λ sufficiently small, inextendible null geodesic not crossing $\mathcal{H}$ or $\mathcal{H}_c$ pass through region where $\partial_t + \Omega_+\partial_\varphi$ and $\partial_t + \Omega_c\partial_\varphi$ are both timelike.

[Gérard, Häfner, Wrochna: 2020]

THE UNRUH STATE ON KERR-DE SITTER

The Unruh state - idea



- Scalar field: $\mathcal{K}\phi = (\Box_g - m^2)\phi = 0$
- Expand:

$$\phi(x) = \sum_J \int_0^\infty d\omega \, \psi_{\omega J}(x) a_{\omega J} + \overline{\psi}_{\omega J}(x) a^\dagger_{\omega J}$$

$\Rightarrow$ Unruh state: $a_{\omega J}|0\rangle = 0$

$\Rightarrow$ Two-point function:

$$\langle \phi(x)\phi(y)\rangle_U = \sum_J \int_0^\infty d\omega \psi_{\omega J}(x)\overline{\psi}_{\omega J}(y)$$

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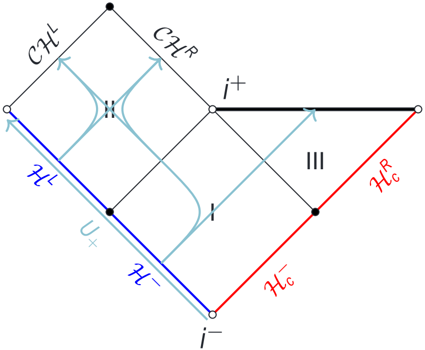

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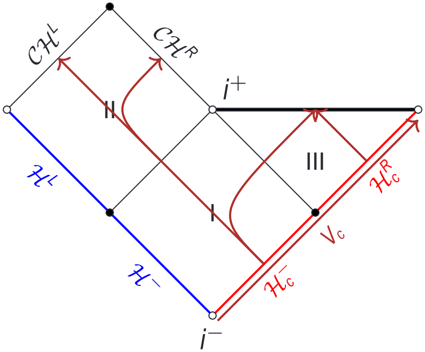

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Quasi-free states

Quasi-free state: determined by 2-point function $w(x, y) = \langle\phi(x)\phi(y)\rangle$ with

- $w \in \mathcal{D}'(M \times M)$
- Positivity: $w(\bar{f}, f) \geq 0$ for all $f \in C_0^\infty(M)$
- Bi-solution: $w(\mathcal{K}f, h) = w(f, \mathcal{K}h) = 0$ for all $f, h \in C_0^\infty(M)$
- Commutator property: $w(f, h) - w(h, f) = iE(f, h)$ for all $f, h \in C_0^\infty(M)$

The Unruh state

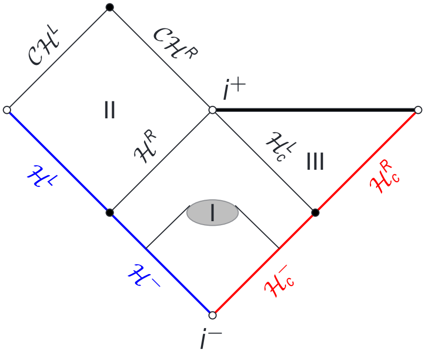
- Idea: choose positive frequency modes w.r.t. U_+ on $\mathcal{H}$ and w.r.t. V_c on $\mathcal{H}_c$
- Construction: use Kay-Wald two-point function at $\mathcal{H}$ and $\mathcal{H}_c$ [Kay, Wald: 1988]

$$\Rightarrow w(f,h) = \sum_{i=+,c} A_i \left(E(f)|_{\mathcal{H}_i}, E(h)|_{\mathcal{H}_i}\right)$$

$$= -\lim_{\epsilon\to 0^+} \frac{r_+^2+a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}}(U_+,\Omega_+)E(h)|_{\mathcal{H}}(U'_+,\Omega_+)}{(U_+-U'_+-i\epsilon)^2} \mathrm{d}U_+\mathrm{d}U'_+\mathrm{d}^2\Omega_+$$

$$-\lim_{\epsilon\to 0^+} \frac{r_c^2+a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}_c}(V_c,\Omega_c)E(h)|_{\mathcal{H}_c}(V'_c,\Omega_c)}{(V_c-V'_c-i\epsilon)^2} \mathrm{d}V_c\mathrm{d}V'_c\mathrm{d}^2\Omega_c$$

Well-definedness, Commutator property, Positivity



- Near i^-: $|E(f)|_{\mathcal{H}}(U_+, \Omega_+)| \lesssim C|U_+|^{-\frac{\alpha}{\kappa_+}}$

  [Hintz, Vasy: 2015]

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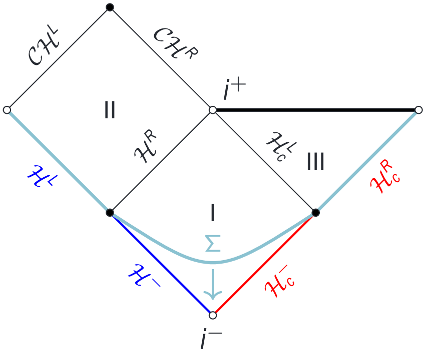

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- Commutator: $E(f,h) = \int_\Sigma E(f) \overleftrightarrow{\nabla}_n E(h) d\Sigma$

$\Rightarrow$ Take limit $\Sigma \to \mathcal{H} \cup \mathcal{H}_c$

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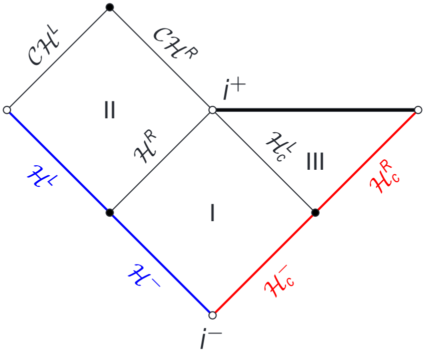

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$\Rightarrow$ Take limit $\Sigma \to \mathcal{H} \cup \mathcal{H}_c$

- $w(f,h) = \sum_{i=+,c} \left\langle K_i(\bar{f}), K_i(h) \right\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2;\nu_i)}$

$\Rightarrow$ $\nu_i = 2\eta(r_i^2 + a^2)\chi^{-1}\mathrm{d}\eta\mathrm{d}\Omega_i$

$\Rightarrow$ $K_i(f)(\eta,\Omega_i) = \mathcal{F}_i(E(f)|_{\mathcal{H}_i})|_{\eta\geq 0}(\eta,\Omega_i)$

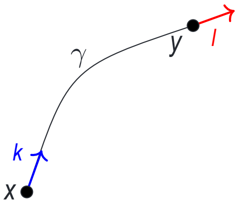
[Dappiaggi, Moretti, Pinamonti: 2009]

HADAMARD PROPERTY

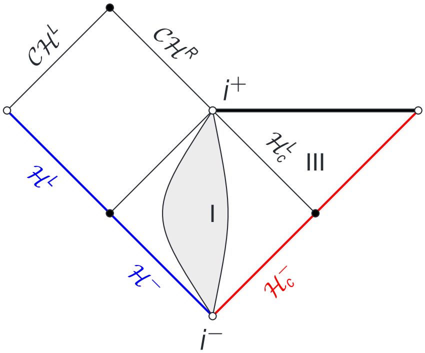
Definition

- Motivation: Definition of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ [Wald:1977]
- Idea: $\lim_{y \to x} w(x, y) \sim$ Minkowski vacuum [Kay, Wald:1991]
- Hadamard property: [Radzikowski:1996]

  $\mathrm{WF}(w) = \{(x, k; y, -l) \in T^*(M \times M) \setminus o : (x, k) \sim (y, l)$ and k future pointing$\}$



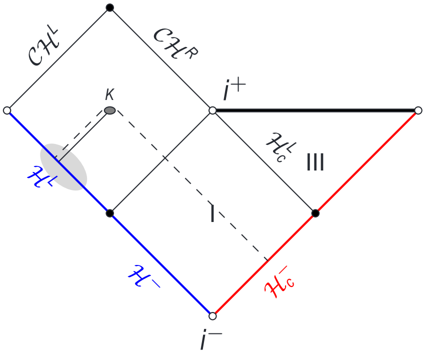
Hadamard property



- Propagation of singularities
- In $\mathcal{O}$, $\partial_t + \Omega_{+/c}\partial_\varphi$ both timelike

$\Rightarrow$ Apply proof for passive states [Sahlmann, Verch: 2001]

Hadamard property



- Propagation of singularities
- In $\mathcal{O}$, $\partial_t + \Omega_{+/c}\partial_\varphi$ both timelike

⇒ Apply proof for passive states [Sahlmann, Verch: 2001]

- For geodesics intersecting horizon, split into relevant term + rest term

⇒ Direct computation for relevant term

⇒ Show that rest terms don't contribute

Summary

Unruh state on Kerr-de Sitter:

- non-equilibrium stationary state
- natural extension of the Unruh state in Schwarzschild
- Hadamard in the whole spacetime M
- not Hadamard across $\mathcal{H}$ or $\mathcal{H}_c$
- restricted to region I, in-/outgoing modes thermally populated with temperature $T_{in/out} = \kappa_{c/+}(2\pi)^{-1}$ w.r.t. $\partial_t + \Omega_{c/+}\partial_\varphi$