Is Gravitational Entanglement Evidence for the Quantization of the Spacetime?

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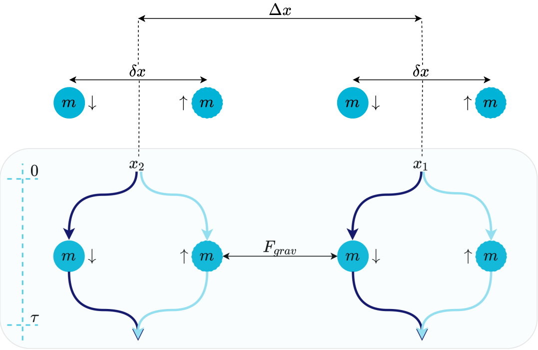
Introduction

Two Equal Mass Particles in Double Stern-Gerlach Experiment



We present a model as a counterexample to the claim that gravitational entanglement is evidence against semiclassical theories[a]; one that makes use of the trajectory[b] as an additional "hidden" variable in the de Broglie–Bohm theory, and is closer in spirit to the mean-field approach.

[a] S. Bose et al. (2017), C. Marletto, V. Vedral, (2017) [1, 2]
[b] see Hall, Reginatto [3] and by Andersen [5]

How does one model the effect of quantum matter on spacetime curvature?

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle \tag{1}$$

for nonrelativistic case potential can be described as:

$$V_{\mathrm{sc}}(t, \mathbf{r}_1, \mathbf{r}_2) = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \langle \mathbf{r}_2 \rangle|} - \frac{G m_1 m_2}{|\langle \mathbf{r}_1 \rangle - \mathbf{r}_2|} + V_{\mathrm{self}}^{(1)}(t, \mathbf{r}_1) + V_{\mathrm{self}}^{(2)}(t, \mathbf{r}_2) \tag{2}$$

on the other hand, by perturbative quantized gravity we have:

$$V_{\mathrm{qg}}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\,. \tag{3}$$

Gravity Sourced Along Bohmian Trajectories

Gravity Source Along Bohmian Trajectories

Using the tools of, de Broglie-Bohm Theory, especially the guiding equation,

$$\frac{\mathrm{d}\mathbf{q}_i(t)}{\mathrm{d}t} = \frac{\hbar}{m_i} \operatorname{Im}\left(\frac{\nabla_i \Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N)}{\Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N)}\right)\Bigg|_{\mathbf{r}_1=\mathbf{q}_1(t),\ldots,\mathbf{r}_N=\mathbf{q}_N(t)} \tag{4}$$

with potential term:

$$V_{\mathrm{bb}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{q}_1,\mathbf{q}_2) = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{q}_2|} - \frac{Gm_1m_2}{|\mathbf{q}_1-\mathbf{r}_2|} + \gamma_0(\mathbf{q}_1,\mathbf{q}_2) \tag{5}$$

Choosing $\gamma_0(q_1,q_2) = Gm_1m_2|q_1-q_2|^{-1}$ and together with some arrangements we can obtain the same phases: $\Phi^{++}_{grav}, \Phi^{+-}_{grav}, \Phi^{-+}_{grav}, \Phi^{--}_{grav}$ as V_{qg} (3).

Hence we concluded that V_{bb} (5) and V_{qg} (3) make the same predictions for both classical trajectories and gravitational phase shifts.

Mean-Field Trajectory Hybrid Model (M-FTH Model)

Mean-Field Trajectory Hybrid Model

We have shown a model, based on Broglie-Bohm trajectories. However, we also need to demonstrate its connection to mean-field semiclassical gravity.

$$\mathbf{i}\hbar\frac{\partial\Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N)}{\partial t} = \left(-\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{\nabla_i^2}{m_i} + V(t;\mathbf{r}_1,\ldots,\mathbf{r}_N;\mathbf{q}_1,\ldots,\mathbf{q}_N;\Psi)\right)\Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N) \tag{6}$$

by introducing the parameter $R\in(0,\infty)$, for potential term, we gathered a hybrid model that is able to interpolate between $V_{bb}(V_{qg})$ and V_{sc}. Therefore, as a result we defined V_R which can mimic $V_{bb}(V_{qg})$ in limit $R\to 0$ and V_{sc} in limit $R\to\infty$:

$$V_R(\mathbf{r}_1,\ldots,\mathbf{r}_N;\mathbf{q}_1,\ldots,\mathbf{q}_N;\Psi) = -G\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j \frac{1-\delta_{ij} f_{\text{reg}}(R)}{N_j(t,R)} \times \int \mathrm{d}^3 r\, \frac{\chi(R,\mathbf{q}_j-\mathbf{r})P_j(t,\mathbf{r})}{|\mathbf{r}_i-\mathbf{r}|} + \gamma_R(\mathbf{q}_1,\ldots,\mathbf{q}_N) \tag{7}$$

Mean-Field Trajectory Hybrid Model

Functions that are presented for V_R in the equation (7):

$$\chi(R,\mathbf{r}) = \begin{cases} 1 & \text{for } |\mathbf{r}| \leq R \\ 0 & \text{for } |\mathbf{r}| > R \end{cases} \tag{8a}$$

$$P_j(t,\mathbf{r}) = \int \mathrm{d}^3 r_1 \cdots \int \mathrm{d}^3 r_{j-1} \int \mathrm{d}^3 r_{j+1} \cdots \int \mathrm{d}^3 r_N \times \left|\Psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_{j-1},\mathbf{r},\mathbf{r}_{j+1},\ldots,\mathbf{r}_N)\right|^2 \tag{8b}$$

$$N_j(t,R) = \int \mathrm{d}^3 r \, P_j(t,\mathbf{r}) \chi(R,\mathbf{q}_j - \mathbf{r}), \tag{8c}$$

a regularization function $f_{\text{reg}} : \mathbb{R}_+ \to [0,1]$ with

$$f_{\text{reg}}(R) \to \begin{cases} 0 & \text{for } R \to \infty \\ 1 & \text{for } R \to 0 \quad \text{“sufficiently fast”} \end{cases} \tag{8d}$$

γ_R for now an arbitrary function of $\mathbf{q}_i$, with $\gamma_R \to 0$ for $R \to \infty$.

M-FTH Model: Classical and Quantum Limit

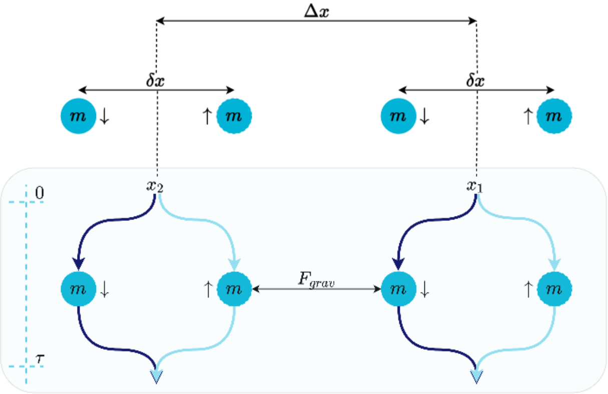
Why "semiclassical limit?" This is just Ehrenfest's theorem, which is an exact equality for the expectation values:

$$\begin{aligned}\partial_t^2 \langle \mathbf{r}_i \rangle &= -\frac{1}{m_i} \langle \nabla_i V_R(t; \mathbf{r}_1, \dots, \mathbf{r}_N; \mathbf{q}_1, \dots, \mathbf{q}_N; \Psi) \rangle \\ &= G \sum_{j=1}^{N} m_j \frac{1 - \delta_{ij} f_{\text{reg}}(R)}{N_j(t, R)} \int \mathrm{d}^3 r \, \chi(R, \mathbf{q}_j - \mathbf{r}) P_j(t, \mathbf{r}) \left\langle \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r}_i - \mathbf{r}|^3} \right\rangle \end{aligned} \tag{9}$$

With appropriate approximations this results in the Newtonian equations of motion:

$$\partial_t^2 \mathbf{q}_i \approx G \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \frac{\mathbf{q}_j - \mathbf{q}_i}{|\mathbf{q}_j - \mathbf{q}_i|^3} \, . \tag{10}$$

M-FTH Model: Two Equal Mass Particles in Double Stern-Gerlach Experiment



Schematic depiction of the double Stern–Gerlach experiment as proposed by Bose et al. [1]. Two spin-$\frac{1}{2}$ particles are each brought into superposition of two possible trajectories in an inhomogeneous magnetic field, resulting in a total of four possible trajectory combinations located around x coordinates $u_{1,2}^{s_{1,2}} = \pm(\Delta x/2) + s_{1,2}(\delta x/2)$ depending on the spin eigenvalues $s_{1,2} = \pm 1$.

M-FTH Model: Two Equal Mass Particles in Double Stern-Gerlach Experiment

Ignoring the self-gravitational terms, that affect only each particle at its site but not both together, and assuming same mass, m, for both particles we defined:

$$
\begin{aligned}
V_R(\mathbf{r}_1,\mathbf{r}_2;\mathbf{q}_1,\mathbf{q}_2;\Psi) \approx & -\frac{Gm^2}{N_1(t,R)}\int \mathrm{d}^3 r\,\frac{\chi(R,\mathbf{q}_1-\mathbf{r})P_1(t,\mathbf{r})}{|\mathbf{r}_2-\mathbf{r}|} \\
& -\frac{Gm^2}{N_2(t,R)}\int \mathrm{d}^3 r\,\frac{\chi(R,\mathbf{q}_2-\mathbf{r})P_2(t,\mathbf{r})}{|\mathbf{r}_1-\mathbf{r}|} \\
& +\gamma_R(\mathbf{q}_1,\mathbf{q}_2)
\end{aligned}
\tag{11}
$$

Assuming, $\tau \gg \tau_a$, classical trajectories $\mathbf{u}_{1,2}^{s_{1,2}} = (u_{1,2}^{s_{1,2}}, 0, vt)$ for $t \in [0,\tau]$, with $u_{1,2}^{s_{1,2}} = \pm(\Delta x/2) + s_{1,2}(\delta x/2)$ and $s_{1,2} = \pm 1$. After the integration of V_R:

$$
\phi_R^{s_1 s_2} \approx -\frac{1}{\hbar}\int_0^\tau \mathrm{d}t\, V_R(\mathbf{u}_1^{s_1}(t),\mathbf{u}_2^{s_2}(t);\mathbf{q}_1^{s_1}(t),\mathbf{q}_2^{s_2}(t);\Psi(t;\mathbf{u}_1^{s_1}(t),\mathbf{u}_2^{s_2}(t)))
\tag{12}
$$

M-FTH Model: Two Equal Mass Particles in Double Stern-Gerlach Experiment

By making some choices; $m\sigma^2 \gg \hbar\tau$ (σ initial width of wave function), and looking at quasi-classical trajectories; $\mathbf{q}_i(t) \approx \mathbf{u}_i^{s_i}(t)$ ($\Gamma = Gm^2\tau/\hbar$):

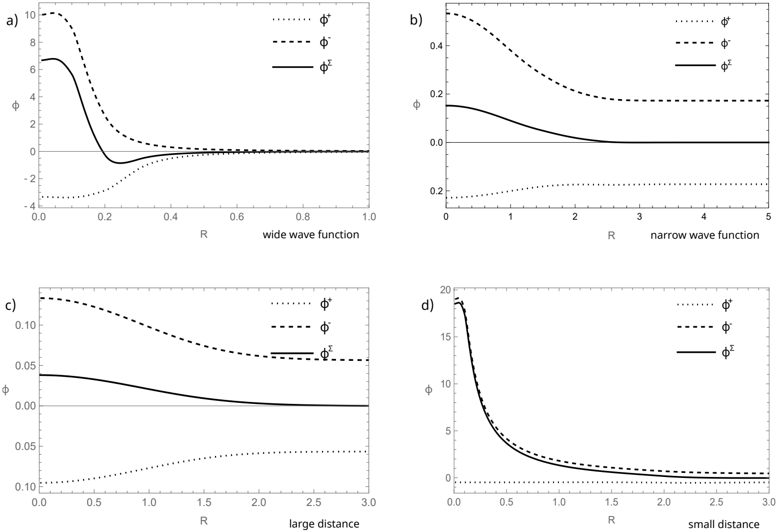
$$\phi_R^{s_1 s_2} \approx \Gamma \int \mathrm{d}^3 r \left(\frac{\chi(R, \mathbf{u}_1^{s_1} - \mathbf{r}) P_1(\mathbf{r})}{N_1(R)\, |\mathbf{u}_2^{s_2} - \mathbf{r}|} + \frac{\chi(R, \mathbf{u}_2^{s_2} - \mathbf{r}) P_2(\mathbf{r})}{N_2(R)\, |\mathbf{u}_1^{s_1} - \mathbf{r}|} \right) + \phi_\gamma^{s_1 s_2} \tag{13}$$

Considering the four spin combinations independently, and defining global phase as $\Phi_R = \phi_R^{++} = \phi_R^{--}$ (with $\phi_\gamma^{s_1 s_2}$ also contributing only to Φ_R), the relative phases:

$$\phi_R^{\pm} = \phi_R^{\pm\mp} - \Phi_R \tag{14}$$

and their average

$$\phi_R^{\Sigma} = \frac{\phi_R^{+} + \phi_R^{-}}{2} = \frac{\phi_R^{+-} + \phi_R^{-+}}{2} - \Phi_R \tag{15}$$



Change of phases ϕ_R^+, ϕ_R^-, ϕ_R^Σ as functions of R; a) wide wave function: $\Delta x = 0.25$, $\delta x = 0.1$, b) narrow wave function: $\Delta x = 2.5$, $\delta x = 1$, c) large distance: $\Delta x = 3$, $\delta x = 0.5$, d) small distance:$\Delta x = 2$, $\delta x = 1.9$, according to equations (14) and (15).

Witness for Spin Entanglement

For these derived phases we can now check experimentally the witnessing of the entanglement that is induced via gravitation. The wave function of such system:

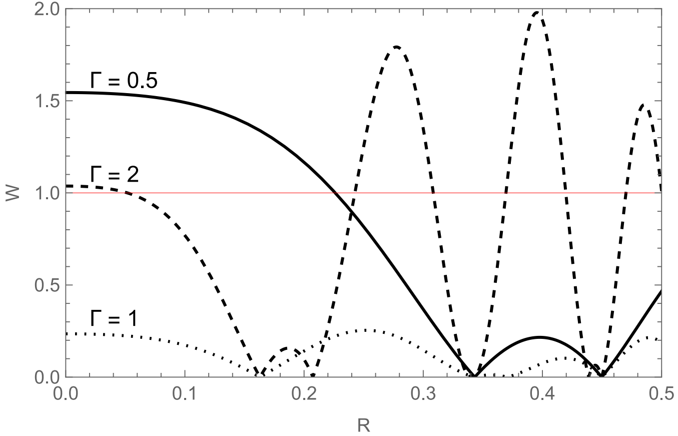
$$|\psi\rangle = \frac{e^{i\Phi_R}}{2}\left(|\uparrow\uparrow\rangle + e^{i\phi_R^+}|\uparrow\downarrow\rangle + e^{i\phi_R^-}|\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \tag{16}$$

and the proposed entanglement witness function, W,Bose et al. [1]:

$$\begin{aligned} W &= \left|\langle\sigma_x^{(1)}\otimes\sigma_z^{(2)}\rangle + \langle\sigma_y^{(1)}\otimes\sigma_y^{(2)}\rangle\right| \\ &= \frac{1}{2}\left|\cos\left(\phi_R^-\right) + \cos\left(\phi_R^- - \phi_R^+\right) - \cos\left(\phi_R^+\right) - 1\right| = \left|\sin\phi_R^\Delta\left(\sin\phi_R^\Delta - \sin\phi_R^\Sigma\right)\right| \end{aligned} \tag{17}$$

In the limit $R\to\infty$ of mean-field semiclassical gravity one has $\phi_\infty^+ = -\phi_\infty^-$, with the symmetry of the cosine, hence one can see the witness function can only take values $0 \leq W \leq 1$, depending on the phase difference ϕ_∞^Δ $(2\phi_R^\Delta = \phi_R^+ - \phi_R^-)$.

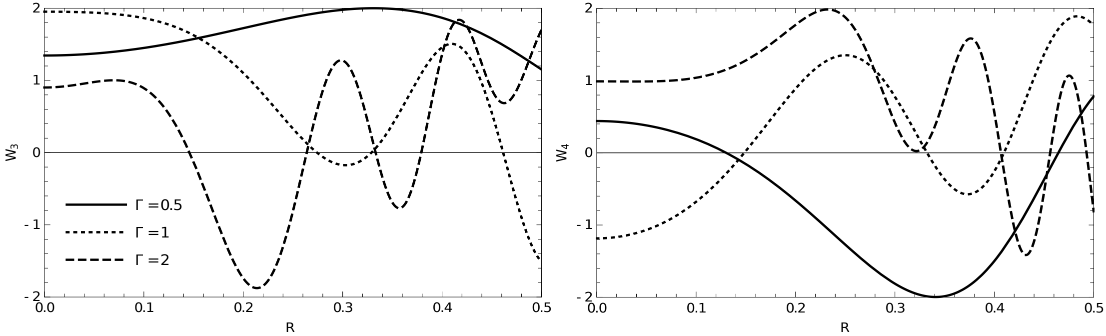
For any separable spin wave function $W \leq 1$, witnesses entanglement for any value $W > 1$.



Entanglement witness W defined in equation (17) as a function of R for the wide wave function case $\Delta x = 0.25$, $\delta x = 0.1$, Γ of $0.5, 1, 2$

Witness for Spin Entanglement

For a more general treatment [1]: $W_G(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)\sin^2\phi_R^{\Delta} + \sin\theta\left(\sin\phi_R^{-} + \sin\phi_R^{+}\right)$
(parameterized by $|\theta\rangle = \frac{1}{\sqrt{2}}\left(|\psi(0,0)\rangle + \mathrm{e}^{\mathrm{i}\theta}|\psi(\pi,\pi)\rangle\right)$)



Change of special cases: $W_3 = W_G(\frac{3\pi}{2})$; small and $W_4 = W_G(\frac{\pi}{2})$; maximally entangled cases of the system, Γ; 0.5,1,2 and $R_{max} = 0.5$

[1] T. Guff, N. Boulle, I. Pikovski (2021) [8]

Conclusions

Conclusion

- We have presented a model for nonrelativistic quantum system on a classical spacetime, where the curvature of spacetime is sourced by the wave function and the particle trajectory. (except $R \to \infty$)
- We have shown that this model results in a gravitational phase shift for which the presented witnesses can find entanglement.
- This is an explicit counterexample that gravity can be unquantized and still result in entanglement between two particles.

Thank You

Questions?

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DOI 10.1007/s11229-014-0577-0

T. Guff, N. Boulle, I. Pikovski, arXiv:2112.08564 [quant-ph] (2021)

However, this claim (idea) has been also criticized by many opponents, and shown that it is only valid for certain classical interactions, and discussed that for hybrid models quantum-classical ensembles entanglement can increased[2].

With similar intentions we have constructed a different model, based on de Broglie-Bohm theory, which takes trajectories[3] as additional "hidden" variables and use them to construct the necessary connection for entanglement between subsystem.

By its dependence on both the trajectories and the wave function, this model can interpolate between maximally entangling model and non-entangling model (semiclassical potential, V_{sc},).

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[2] M.J.W. Hall, M. Reginatto; (2018), (2005) [3],[4]

[3] This "trajectory approach" is briefly mentioned also by Hall, Reginatto [3] and by Andersen [5]

The concept of Norsen (TELB),[4] the conditional wave functions:

$$\begin{aligned} \psi_1(t,x) &\equiv \Psi(t;x,q_2(t)) \\ \psi_2(t,x) &\equiv \Psi(t;q_1(t),x) \end{aligned} \tag{18}$$

$$i\hbar\frac{\partial}{\partial t}\psi_i(t,x) = -\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial x^2}\psi_i(t,x) + V_i^{\text{eff}}(t,x)\psi_i(t,x) \tag{19}$$

$$V_{1,2}^{\text{eff}} = \gamma_0(q_1,q_2) - \frac{Gm_1m_2}{|q_1-q_2|} - \frac{Gm_1m_2}{|x-q_{2,1}|} + i\hbar\frac{\mathrm{d}q_{2,1}}{\mathrm{d}t}\Pi_{1,2}^{(1)} - \frac{\hbar^2}{2m_{2,1}}\Pi_{1,2}^{(2)} \tag{20}$$

$$\begin{aligned} \Pi_1^{(n)}(t,x) &\equiv \frac{1}{\psi_1(t,x)}\left.\frac{\partial^n\Psi(x,x_2,t)}{\partial x_2^n}\right|_{x_2=q_2(t)} \\ \Pi_2^{(n)}(t,x) &\equiv \frac{1}{\psi_2(t,x)}\left.\frac{\partial^n\Psi(x_1,x,t)}{\partial x_1^n}\right|_{x_1=q_1(t)} \end{aligned} \tag{21}$$

[4] T. Norsen (2010), T. Norsen, D. Marian, X. Oriols (2015) [6, 7]

Gaussian wave function by the trajectories:

$$\Psi(x_1,r_1,x_2,r_2)=\sum_{s_1}\sum_{s_2}\frac{\exp\left[-\frac{1}{2}\left(r_1^2+r_2^2+(x_1-u_1^{s_1})^2+(x_2-u_2^{s_2})^2\right)\right]}{2\sqrt{\pi^3}\left(1+\exp\left(-\frac{\delta x^2}{4}\right)\right)} \tag{22}$$

using cylindrical coordinates, $(x,y,z)=(x,r\cos\theta,r\sin\theta)$ and we have:

$$P_i(x,r)=\frac{\mathrm{e}^{-r^2}}{2\sqrt{\pi^3}}Q(x-u_i^+,x-u_i^-),\qquad Q(p,q)=\frac{\mathrm{e}^{-p^2}+\mathrm{e}^{-q^2}}{1+\mathrm{e}^{-\frac{\delta x^2}{4}}}+\frac{2\mathrm{e}^{-\left(\frac{p+q}{2}\right)^2}}{1+\mathrm{e}^{\frac{\delta x^2}{4}}} \tag{23}$$

normalization (where $N_1(R)=N_2(R)=N(R)$ regardless of the trajectory):

$$N(R)=\left(1-\mathrm{e}^{-R^2}\right)\left[\frac{\mathrm{erf}\left(R+\frac{\delta x}{2}\right)+\mathrm{erf}\left(R-\frac{\delta x}{2}\right)}{2\left(1+\mathrm{e}^{\frac{\delta x^2}{4}}\right)}\right.$$
$$\left.+\frac{\mathrm{erf}(R+\delta x)+\mathrm{erf}(R-\delta x)+2\mathrm{erf}(R)}{4\left(1+\mathrm{e}^{-\frac{\delta x^2}{4}}\right)}\right] \tag{24a}$$

$$J_R(\xi) = \frac{2}{\sqrt{\pi}} \int_0^R \mathrm{d}r \, \frac{r\,\mathrm{e}^{-r^2}}{\sqrt{r^2+\xi^2}} = \mathrm{e}^{\xi^2}\left[\mathrm{erf}\left(\sqrt{R^2+\xi^2}\right) - \mathrm{erf}\left(\sqrt{\xi^2}\right)\right] \tag{25}$$

and writing $\Delta u^{s_1 s_2} = u_1^{s_1} - u_2^{s_2}$, using $Q(p,q) = Q(q,p) = Q(-p,-q)$, we find:

$$\begin{aligned}\phi_R^{s_1 s_2} &\approx \frac{\Gamma}{N(R)} \int_{-R}^{R} \mathrm{d}x \int_0^R \mathrm{d}r \left(\frac{2\pi r\, P_1(x+u_1^{s_1}, r)}{\sqrt{(x+\Delta u^{s_1 s_2})^2 + r^2}} + \frac{2\pi r\, P_2(x+u_2^{s_2}, r)}{\sqrt{(x-\Delta u^{s_1 s_2})^2 + r^2}} \right) \\ &= \frac{\Gamma}{2N(R)} \int_{-R}^{R} \mathrm{d}x \Bigg(Q(x+u_1^{s_1}-u_1^{+}, x+u_1^{s_1}-u_1^{-}) \\ &\qquad + Q(x-u_2^{s_2}+u_2^{+}, x-u_2^{s_2}+u_2^{-}) \Bigg) J_R(x+\Delta u^{s_1 s_2})\end{aligned} \tag{26}$$

$$\phi_R^{\pm} \approx \frac{\Gamma}{N(R)} \int_{-R}^{R} \mathrm{d}x \, Q(x, x+\delta x) \left(J_R(x \pm \Delta x + \delta x) - \frac{J_R(x+\Delta x) + J_R(x-\Delta x)}{2} \right) \tag{27}$$

In the limit $R \to 0$, both the integral from $-R$ to R and the normalization function $N(R)$ tend to zero like R^3.

$$\phi_0^{\pm} = \frac{2\Gamma}{|\Delta x \pm \delta x|} - \frac{2\Gamma}{|\Delta x|} + \phi_\gamma^{\pm} \tag{28}$$

expansion around small $R \ll 1$, by approximating up to and including $\mathcal{O}\left(R^5\right)$:

$$J_R(\xi) \approx \frac{R^2}{\sqrt{\pi\xi^2}}\left[1 - \frac{R^2}{2}\left(1 + \frac{1}{2\xi^2}\right)\right] \tag{29}$$

$$N(R) \approx \frac{R^3\,\mathrm{e}^{-\delta x^2}}{\sqrt{\pi}\left(1+\mathrm{e}^{-\frac{\delta x^2}{4}}\right)}\left[\left(1+\mathrm{e}^{\frac{\delta x^2}{2}}\right)^2\left(1 - \frac{5R^2}{6}\right) + \frac{R^2\,\delta x^2}{3}\left(2+\mathrm{e}^{\frac{\delta x^2}{2}}\right)\right] \tag{30}$$

$$\frac{J_R(\xi)}{N(R)} \approx \frac{\left(1+\mathrm{e}^{-\frac{\delta x^2}{4}}\right)}{\left(1+\mathrm{e}^{-\frac{\delta x^2}{2}}\right)^2}\left[\frac{1}{R|\xi|} + \frac{R}{3|\xi|}\left(1 - \frac{3}{4\xi^2} - \delta x^2\frac{2+\mathrm{e}^{\frac{\delta x^2}{2}}}{\left(1+\mathrm{e}^{\frac{\delta x^2}{2}}\right)^2}\right)\right] \tag{31}$$

Backup slide

For arbitrary functions $q(x)$, $f(x)$, $g(x)$, we have to cubic order in R:

$$\int_{-R}^{R} q(x)\left(\frac{f(x+\xi)}{R}+Rg(x+\xi)\right)\mathrm{d}x \approx 2q(0)f(\xi) + \frac{R^2}{3}\left(q(0)\left(6g(\xi)+f''(\xi)\right)+2q'(0)f'(\xi)+q''(0)f(\xi)\right), \quad (32)$$

and hence, with $q(x) = Q(x, x+\delta x)$,

$$I_R(\xi) = \frac{\Gamma}{2}\int_{-R}^{R}\mathrm{d}x\, q(x)\frac{J_R(x+\xi)}{N(R)} \approx \frac{\Gamma}{|\xi|}\left[1+\frac{R^2}{12\xi^2}\left(1+\frac{8\xi\,\delta x}{1+\mathrm{e}^{\frac{\delta x^2}{2}}}\right)\right] \quad (33)$$

The phases are then

$$\phi_R^{\pm} = 2I_R(\delta x \pm \Delta x) - I_R(\Delta x) - I_R(-\Delta x) + \phi_\gamma^{\pm}\,. \quad (34)$$

To lowest order, we again obtain the phases (28). As expected, this is the phase obtained from the Bohmian potential (5).

Backup slide

In the limit $R \to \infty$, using the asymptotic expansion of error function and $N(R) \to 1$, we find:

$$\frac{J_R(\xi)}{N(R)} \approx J_\infty(\xi)\left(1+\frac{\mathrm{e}^{-R^2}R}{2\sqrt{\pi}}\delta x^2\frac{1+2\mathrm{e}^{\delta x^2/4}}{1+\mathrm{e}^{\delta x^2/4}}\right)-\frac{\mathrm{e}^{-R^2}}{\sqrt{\pi}R}\left(1-\frac{\xi^2}{2R^2}\right), \tag{35}$$

assuming $\phi_\gamma^\pm \to 0$ sufficiently fast,

$$\phi_R^\pm \approx \left(1+\frac{\mathrm{e}^{-R^2}R}{2\sqrt{\pi}}\delta x^2\frac{1+2\mathrm{e}^{\delta x^2/4}}{1+\mathrm{e}^{\delta x^2/4}}\right)\phi_\infty^\pm+\frac{\Gamma\mathrm{e}^{-R^2}}{\sqrt{\pi}R^3}\left[\frac{\delta x}{1+\mathrm{e}^{-\delta x^2/4}}\pm\Delta x\left(\frac{1}{1+\mathrm{e}^{-\delta x^2/4}}+\sqrt{\pi}\,\delta x\right)\right] \tag{36}$$

Since $\phi_\infty^\Sigma = 0$, we find

$$\phi_R^\Sigma \approx \frac{\Gamma\mathrm{e}^{-R^2}\delta x}{\sqrt{\pi}R^3\left(1+\mathrm{e}^{-\delta x^2/4}\right)}. \tag{37}$$